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Determination and analysis of the thermoelastic state of layered orthotropic cylindrical shells

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The fundamental relations of the quasi-static problem of thermoelasticity are written for a finite layered orthotropic cylindrical shell of an antisymmetric structure. Under convective heat transfer on the surfaces of this shell and under a linear dependence of temperature on the transverse coordinate, the basic system of equations for the integral characteristics of temperature is given. The method is proposed for solving the formulated problems of thermoelasticity and thermal conductivity, using the double finite integral Fourier transform with respect to the corresponding coordinates of the transformation and Laplace transform with respect to the time. The results of a numerical analysis of temperature, deflections, and stresses for the considered two-layer shell hinged at the edges under local heating by the initially specified temperature field are presented.

Keywords: *orthotropic; layered; cylindrical shell; temperature; thermally stressed state; heat transfer.*

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1. Introduction

Cylindrical shells of a layered structure are widely used in many branches of modern technology, in particular, in aircraft and space constructions, to increase the strength and rigidity of structures and protect them from low-temperature or high-temperature thermal effects. Therefore, estimating of temperature stresses in such structures is a significant engineering task.

Elements of layered structures have been studied by many scientists [1–4]. There are developed refined models taking into account the characteristic features of composite materials, in particular, high anisotropy in the transverse direction [3–5]. The exact solutions of thermoelasticity problems for layered shells are constructed on the base of three-dimensional equations in [6,7]. Using the equations of classical and various refined theories, the analytical solutions are obtained in [8–10]. Using the equation of interrelated thermoelasticity, the influence of the coupling coefficient on the dynamic behavior of composite shells is analyzed in [11]. The method of finite elements for studying thermoelastic processes in shells of a layered structure was used in [12]. In [13], the focus was on the thermoelectromechanical analysis of multilayer piezoelectric cylindrical shells of an open profile. The thermoelastic properties of a functional-gradient isotropic cylindrical shell locally heated by heat sources are considered in [14]. The stress-strain state of a layered cylindrical surface under its local convective heating is investigated in [15]. Detailed overviews of various models and methods are given in [1–3].

The aim of this article is to investigate the change in temperature, deflection, and stresses of a two-layer circular cylindrical shell of a regular antisymmetric structure under its local heating by an initially specified temperature field based on the equations of thermoelasticity and heat conduction equation of the six-modal theory of layered shells.

2. Formulation of the problem and system of basic equations

Consider an inhomogeneous orthotropic circular cylindrical shell with the constant thickness $2h$ and a finite length l. We refer the points of the shell space to the cylindrical coordinate system (x, θ, z)

denoting the axial, circular, and radial coordinates, respectively. We place the origin of the coordinates in the middle surface of the shell with the radius R. Hereafter, the indices 1, 2, 3 correspond to these coordinates.

Let the shell be under the external force action, and let it be heated by heat sources and environment through convective heat exchange. To study the thermoelastic behavior of such a shell, let us use a mathematical model with six degrees of freedom, which is based on assumptions about the linear distribution of the displacement vector $U_i(x,\theta,z,\tau)$, $i=1,2,3$, and the temperature $t(x,\theta,z,\tau)$ in the shell thickness

$$U_i(x,\theta,z,\tau) = u_i(x,\theta,\tau) + z\gamma_i(x,\theta,\tau), \tag{1}$$

$$t(x,\theta,z,\tau) = T_1(x,\theta,\tau) + \frac{z}{h}T_2(x,\theta,\tau), \tag{2}$$

where u_i are components of the mid-surface points displacement vector; γ_i are components of the vector of normal rotation angles; $T_n = \frac{2n-1}{2h^n}\int_{-h}^{h} t\, z^{n-1}\, dz$, $n=1,2$, are integral characteristics of the temperature.

In the general case, this model consists of interrelated systems of thermoelasticity equations and heat conduction equations. If the effect of deformation on the temperature field change is neglected, these systems are independent.

3. Divergence measure for FFSs

3.1. System of thermoelasticity equations

The kinematic relations for the components e_{ij} of the deformation tensor at an arbitrary point of the shell have the following form

$$\begin{gathered} e_{11} = \varepsilon_{11} + z\varkappa_{11}, \quad e_{22} = (\varepsilon_{22} + z\varkappa_{22})/(1+z/R), \quad e_{33} = \varepsilon_{33}, \\ e_{12} = (\varepsilon_{12} + z\varkappa_{12} + z^2\omega_{12})/(1+z/R), \\ e_{13} = \varepsilon_{13} + z\varkappa_{13}, \quad e_{23} = (\varepsilon_{23} + z\varkappa_{23})/(1+z/R). \end{gathered} \tag{3}$$

Here, the components ε_{ij}, $\varkappa_{ij}$ of the deformation tensor of the mid-surface in terms of generalized displacements u_i, γ_i are expressed by formulae

$$\begin{gathered} \varepsilon_{11} = \partial_1 u_1, \quad \varepsilon_{22} = (u_3 + \partial_2 u_2)/R, \quad \varepsilon_{33} = \gamma_3, \\ \varepsilon_{12} = \partial_2 u_1/R + \partial_1 u_2, \quad \varepsilon_{23} = \gamma_2 + (\partial_2 u_3 - u_2)/R, \\ \varepsilon_{13} = \gamma_1 + \partial_1 u_3, \quad \omega_{12} = \partial_1\gamma_2/R, \\ \varkappa_{11} = \partial_1\gamma_1, \quad \varkappa_{22} = (\gamma_3 + \partial_2\gamma_2)/R, \quad \varkappa_{13} = \partial_1\gamma_3, \\ \varkappa_{12} = \partial_1\gamma_2 + \partial_2\gamma_1/R + \partial_1 u_2/R, \quad \varkappa_{23} = \partial_2\gamma_3/R. \end{gathered} \tag{4}$$

Physical equations for stresses and deformations can be written as follows

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} & c_{13} & \\ c_{12} & c_{22} & c_{23} & \\ c_{13} & c_{23} & c_{33} & \\ & & & c_{66} \end{pmatrix} \begin{pmatrix} e_{11} \\ e_{22} \\ e_{33} \\ e_{12} \end{pmatrix} - \begin{pmatrix} \beta_{11}^t \\ \beta_{22}^t \\ \beta_{33}^t \end{pmatrix} t, \tag{5}$$

$$\begin{pmatrix} \sigma_{13} \\ \sigma_{23} \end{pmatrix} = \begin{pmatrix} c_{44} & \\ & c_{55} \end{pmatrix} \begin{pmatrix} e_{13} \\ e_{23} \end{pmatrix}.$$

Here $c_{ij}(z)$ are the elasticity coefficients; $\beta_{ii}^t(z) = c_{i1}\alpha_{11}^t + c_{i2}\alpha_{22}^t + c_{i3}\alpha_{33}^t$ are coefficients of thermal elasticity; $\alpha_{ij}^t(z)$ are coefficients of linear thermal expansion.

Physical equations for internal forces N_{ij} and moments M_{ij} are obtained from the relations

$$\{N_{11}, N_{12}, N_{13}\} = \int_{-h}^{h} \{\sigma_{11}, \sigma_{12}, \sigma_{13}\}(1+z/R)\, dz, \qquad \{N_{22}, N_{21}, N_{23}\} = \int_{-h}^{h} \{\sigma_{22}, \sigma_{12}, \sigma_{23}\}\, dz,$$

$$\{M_{11}, M_{12}, M_{13}\} = \int_{-h}^{h} \{\sigma_{11}, \sigma_{12}, \sigma_{13}\}(1+z/R)\, z\, dz, \quad \{M_{22}, M_{21}, M_{23}\} = \int_{-h}^{h} \{\sigma_{22}, \sigma_{12}, \sigma_{23}\}\, z\, dz,$$

$$N_{33}=\int_{-h}^{h}\sigma_{33}(1+z/R)\,dz. \tag{6}$$

The equilibrium equations:

$$\begin{gathered}\partial_1 N_{11}+\partial_2 N_{21}/R=-q_1,\\ \partial_1 N_{12}+\partial_2 N_{22}/R+N_{23}/R=-q_2,\\ \partial_1 N_{13}+\partial_2 N_{23}/R-N_{22}/R=-q_3,\\ \partial_1 M_{11}+\partial_2 M_{21}/R-N_{13}=-m_1,\\ \partial_1 M_{12}+\partial_2 M_{22}/R-N_{23}=-m_2,\\ \partial_1 M_{13}+\partial_2 M_{23}/R-M_{22}/R-N_{33}=-m_3,\end{gathered} \tag{7}$$

where q_i, m_i denote the external load, $\partial_1=\frac{\partial}{\partial x}$, $\partial_2=\frac{\partial}{\partial\theta}$.

Using the above relations, we write the system of equilibrium equations (7) in terms of generalized displacements in the form

$$\sum_{k}^{6} L_{rk}y_k=b_r \quad (r,k=1,2,\dots,6)\,. \tag{8}$$

Here $y_i=u_i$, $y_{3+i}=\gamma_i$ $(i=1,2,3)$. Differential operators L_{rk} $(L_{rk}=L_{kr})$ and absolute terms b_r are described by the expressions:

$$\begin{gathered}L_{11}=A_{11}\partial_{11}^2+A_{66}/R^2\partial_{22}^2,\quad L_{12}=(A_{12}+A_{66})/R\,\partial_{12}^2,\quad L_{13}=A_{12}/R\partial_1,\\ L_{14}=B_{11}\partial_{11}^2+B_{66}/R^2\partial_{22}^2,\quad L_{15}=(B_{12}+B_{66})/R\,\partial_{12}^2,\\ L_{16}=(A_{13}+B_{12}/R)\partial_1,\quad L_{22}=A_{66}\partial_{11}^2+A_{22}/R^2\partial_{22}^2-k'A_{55}/R^2,\\ L_{23}=(A_{22}+k'A_{55})/R^2\partial_2,\quad L_{24}=(B_{12}+B_{66})/R\,\partial_{12}^2,\\ L_{25}=B_{66}\partial_{11}^2+B_{22}/R^2\partial_{22}^2+k'A_{55}/R,\quad L_{26}=\big(A_{23}/R+(B_{22}+k'B_{55})/R^2\big)\partial_2,\\ L_{33}=-k'A_{44}\partial_{11}^2-k'A_{55}/R^2\partial_{22}^2+A_{22}/R^2,\quad L_{34}=(B_{12}/R-k'A_{44})\partial_1,\\ L_{35}=(B_{22}/R-k'A_{55})/R\,\partial_2,\quad L_{36}=-k'B_{44}\partial_{11}^2+\left(B_{22}-k'B_{55}\partial_{22}^2\right)/R^2+A_{23}/R,\\ L_{44}=D_{11}\partial_{11}^2+D_{66}/R^2\partial_{22}^2-k'A_{44},\quad L_{45}=(D_{12}+D_{66})/R\,\partial_{12}^2,\\ L_{46}=(D_{12}/R+B_{13}-k'B_{44})\partial_1,\quad L_{55}=D_{66}\partial_{11}^2+D_{22}/R^2\partial_{22}^2-k'A_{55},\\ L_{56}=\left((B_{23}-k'B_{55})/R+D_{22}/R^2\right)\partial_2,\\ L_{66}=A_{33}+2B_{23}/R+D_{22}/R^2-k'D_{44}\partial_{11}^2-k'D_{55}/R^2\partial_{22}^2,\\ b_1=A_{11}^t\partial_1T_1+B_{11}^t/h\,\partial_1T_2-q_1,\quad b_2=A_{22}^t/R\,\partial_2T_1+B_{22}^t/(Rh)\partial_2T_2-q_2,\\ b_3=A_{22}^t/R\,T_1+B_{22}^t/(Rh)T_2+q_3,\quad b_4=B_{11}^t\partial_1T_1+D_{11}^t/h\,\partial_1T_2-m_1,\\ b_5=B_{22}/R\,\partial_2T_1+D_{22}^t/(Rh)\partial_2T_2-m_2,\\ b_6=\left(A_{33}^t+B_{22}^t/R\right)T_1+\left(D_{22}^t/R+B_{33}^t\right)/hT_2+m_3.\end{gathered}$$

Here

$$\{A_{ii},B_{ij},D_{ij}\}=\int_{-h}^{h}c_{ij}\{1,z,z^2\}\,dz,\quad \{A_{ii}^t,B_{ii}^t,D_{ii}^t\}=\int_{-h}^{h}\beta_{ii}^t\{1,z,z^2\}\,dz,$$

k' is the shear factor [14].

For the solution uniqueness of the system (8), it is necessary to impose the appropriate boundary conditions. For the shell of a finite length, it is necessary at its ends $x=0$ and $x=l$ to put one value from each of the following pair: $\{N_{11},u_1\}$, $\{N_{12},u_2\}$, $\{N_{13},u_3\}$, $\{M_{11},\gamma_1\}$, $\{M_{12},\gamma_2\}$, $\{M_{13},\gamma_3\}$.

The system of equations (8) with the boundary conditions constitutes the boundary value problem of quasi-static thermoelasticity for inhomogeneous anisotropic cylindrical shells in terms of displacements. By means of the known displacements, we determine the deformations of the mid-surface from the relation (4), and the forces and moments from the equations of state

$$\begin{pmatrix} N_{11} \\ N_{22} \\ N_{33} \\ M_{11} \\ M_{22} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} & A_{13} & B_{11} & B_{12} \\ A_{12} & A_{22} & A_{23} & B_{12} & B_{22} \\ A_{13} & A_{23} & A_{33} & B_{13} & B_{23} \\ B_{11} & B_{12} & B_{13} & D_{11} & D_{12} \\ B_{12} & B_{22} & B_{23} & D_{12} & D_{22} \end{pmatrix} \begin{pmatrix} \partial_1 u_1 \\ (\partial_2 u_2 + u_3)/R \\ \gamma_3 \\ \partial_1 \gamma_1 \\ (\partial_2 \gamma_2 + \gamma_3)/R \end{pmatrix} - \begin{pmatrix} A^t_{11} \\ A^t_{22} \\ A^t_{33} \\ B^t_{11} \\ B^t_{22} \end{pmatrix} T_1 - \begin{pmatrix} B^t_{11} \\ B^t_{22} \\ B^t_{33} \\ D^t_{11} \\ D^t_{22} \end{pmatrix} \frac{T_2}{h},$$

$$\begin{pmatrix} N_{12} \\ M_{12} \end{pmatrix} = \begin{pmatrix} A_{66} & B_{66} \\ B_{66} & D_{66} \end{pmatrix} \begin{pmatrix} \partial_1 u_2 + \partial_2 u_1/R \\ \partial_1 \gamma_2 + \partial_2 \gamma_1/R \end{pmatrix}, \quad \begin{pmatrix} N_{13} \\ M_{13} \end{pmatrix} = k' \begin{pmatrix} A_{44} & B_{44} \\ B_{44} & D_{44} \end{pmatrix} \begin{pmatrix} \gamma_1 + \partial_1 u_3 \\ \partial_1 \gamma_3 \end{pmatrix},$$

$$\begin{pmatrix} N_{23} \\ M_{23} \end{pmatrix} = k' \begin{pmatrix} A_{55} & B_{55} \\ B_{55} & D_{55} \end{pmatrix} \begin{pmatrix} \gamma_2 + (\partial_2 u_3 - u_2)/R \\ \partial_2 \gamma_3 / R \end{pmatrix}. \tag{9}$$

The temperature deformations and stresses in the shell are found using the formulae (3) and (5).

3.2. System of heat conduction equations

The integral temperature characteristics T_1 and T_2 included in the absolute terms of the system (8) and in the state equation (9) can be determined from the corresponding equations of heat conduction under the boundary conditions imposed on the surfaces $z = \pm h$ and at the ends of the shell. For convective heat exchange on the surfaces $z = \pm h$, the system of heat conduction equations provided the linear dependence of temperature on the transverse coordinate (2) can be written in the form:

$$\Delta_{(1)} T_1 - \varepsilon^t_1 T_1 + \Delta_{(2)} T_2 + \left(\frac{\lambda^{(1)}_{33}}{hR} - \varepsilon^t_2 \right) T_2 - C^{(1)} \partial_\tau T_1 - C^{(2)} \partial_\tau T_2 = -f_1,$$

$$\Delta_{(2)} T_1 - \varepsilon^t_2 T_1 + \Delta_{(3)} T_2 + \left(\frac{\lambda^{(2)}_{33}}{hR} - \frac{\lambda^{(1)}_{33}}{h^2} - \varepsilon^t_1 \right) T_2 - C^{(2)} \partial_\tau T_1 - C^{(3)} \partial_\tau T_2 = -f_2. \tag{10}$$

Here

$$\Delta_{(k)} = \Lambda^{(k)}_{11} \partial^2_{11} + \frac{\Lambda^{(k)}_{22}}{R^2} \partial^2_{22}; \quad \{\Lambda^{(k)}_{ij}, C^{(k)}\} = \int_{-h}^{h} \{\lambda_{ij}, c_\varepsilon\} \left(\frac{z}{h}\right)^{k-1} dz, \quad (k = 1,2,3)\,; \quad \partial_\tau = \frac{\partial}{\partial \tau};$$

$$f_j = t^z_1 \varepsilon^t_j + t^z_2 \varepsilon^t_{3-j} + W^t_j = Q_j(x,\theta) F_j(\tau); \quad \varepsilon^t_j = \alpha^+_z - (-1)^j \alpha^-_z; \quad t^z_j = \frac{1}{2}\left(t^+_c - (-1)^j t^-_c\right);$$

$$W^t_j = \int_{-h}^{h} w_t \left(\frac{z}{h}\right)^{j-1} dz, \quad (j = 1,2);$$

$\lambda_{ij}(z)$ are the coefficients of thermal conductivity; $c_\varepsilon(z)$ is the specific volumetric heat capacity; τ is a time variable; $\alpha^\pm_z$ are the coefficients of heat dissipation from the surfaces $z = \pm h$; $t^\pm_c$ is the ambient temperature on these surfaces; w_t is the power of heat sources.

For the solution uniqueness of the system (10) at the edges $x = 0$ and $x = l$ we need to specify a combination of values $a_0 T_1 + a_1 \frac{\partial T_1}{\partial x}$, $a_2 T_2 + a_3 \frac{\partial T_2}{\partial x}$, where $a_i = \text{const}$; and at the initial moment of time we need to specify the value of temperature characteristics T_1 and T_2.

4. The method of solving the basic systems of equations

Let us consider a cylindrical shell being antisymmetric relative to the middle surface, composed of an even number of orthotropic layers with the same thickness and properties, the material axes of which are oriented at the angle of 0° or 90° to the axis of the shell. Let the edges $x = 0$ and $x = l$ of the shell be hinged and assumed to have a zero temperature. Then we have the following boundary conditions for the defining functions:

$$u_3 = u_2 = 0, \quad \gamma_3 = \gamma_2 = 0, \quad N_{11} = M_{11} = 0, \tag{11}$$

$$T_1 = T_2 = 0. \tag{12}$$

At the initial moment of time, the temperature characteristics T_1, T_2 are given as coordinate functions:

$$T_1(x,\theta,0) = T^0_1(x,\theta), T_2(x,\theta,0) = T^0_2(x,\theta). \tag{13}$$

4.1. Finding the temperature field

The equations (10), after applying the double finite Fourier transformation with respect to the coordinates (x, θ), according to the boundary conditions (12), take the form:

$$\frac{dT_{1mn}}{d\tau_1} + g_1 T_{1mn} + g_2 T_{2mn} = f_{1mn},$$
$$\frac{dT_{2mn}}{d\tau_1} + g_3 T_{1mn} + g_4 T_{2mn} = f_{2mn}. \tag{14}$$

Here $g_1 = L_{11}^{(1)}\mu_n^2 + L_{22}^{(2)}\delta^2 m^2 + \mathrm{Bi}_1$, $g_2 = L_{11}^{(2)}\mu_n^2 + L_{22}^{(2)}\delta^2 m^2 - \delta + \mathrm{Bi}_2$, $g_3 = \tilde{C}\big(L_{11}^{(2)}\mu_n^2 + L_{22}^{(2)}\delta^2 m^2 + \mathrm{Bi}_2\big)$,

$$g_4 = \tilde{C}\big(L_{11}^{(3)}\mu_n^2 + L_{22}^{(3)}\delta^2 m^2 + \mathrm{Bi}_1 + 1\big), \quad \mu_n = \frac{\pi n h}{l}, \quad \delta = \frac{h}{R}, \quad \tau_1 = \frac{\Lambda_{33}^{(1)}}{h^2 C^{(1)}}\tau, \quad \tilde{C} = \frac{C^{(1)}}{C^{(3)}},$$

$$L_{ii}^{(j)} = \frac{\Lambda_{ii}^{(j)}}{\Lambda_{33}^{(1)}}, \quad \mathrm{Bi}_i = \frac{\varepsilon_i^t h^2}{\Lambda_{33}^{(1)}}, \quad f_{1mn} = Q_{1mn}(x,\theta)F_1(\tau) = \mathrm{Bi}_1 t_{1mn}^z + \mathrm{Bi}_2 t_{2mn}^z + W_{1mn}^t \frac{h^2}{\Lambda_{33}^{(1)}},$$

$$f_{2mn} = Q_{2mn}(x,\theta)F_2(\tau) = \left(\mathrm{Bi}_2 t_{1mn}^z + \mathrm{Bi}_1 t_{2mn}^z + W_{2mn}^t \frac{h^2}{\Lambda_{33}^{(1)}}\right)\tilde{C}.$$

The solution of the system (14) under the initial conditions (13) is obtained by the method of the integral Laplace transform in the form:

$$T_{1mn} = \sum_{\substack{j=1\\k\neq j}}^{2} \frac{\left\{(p_j - g_4)Q_{1nm}Z_1^{(j)}(\tau) + g_2 Q_{2nm}Z_2^{(j)}(\tau) + \left[(p_j - g_4)T_{1nm}^0 + g_2 T_{2nm}^0\right]e^{-p_j\tau_1}\right\}}{p_j - p_k},$$

$$T_{2mn} = \sum_{\substack{j=1\\k\neq j}}^{2} \frac{\left\{(p_j - g_1)Q_{2nm}Z_2^{(j)}(\tau) + g_3 Q_{1nm}Z_1^{(j)}(\tau) + \left[(p_j - g_1)T_{2nm}^0 + g_3 T_{1nm}^0\right]e^{-p_j\tau_1}\right\}}{p_j - p_k}. \tag{15}$$

Here

$$p_j = \frac{g_1 + g_4}{2} + (-1)^j\sqrt{\frac{(g_1 - g_4)^2}{4} + g_2 g_3},$$

$$\left\{Q_{jnm}, T_{jnm}^0\right\} = \frac{\varsigma}{\pi l}\int_0^l \int_{-\pi}^{\pi} \left\{Q_j, T_j^0\right\}(x,\theta)\sin\frac{\pi n}{l}x\cos m\theta\,dx\,d\theta, \quad \varsigma = \begin{cases} 1, & m = 0,\\ 2, & m \neq 0,\end{cases} \tag{16}$$

$$Z_k^{(j)} = \int_0^{\tau_1} F_k(u)\,e^{-p_j(\tau_1 - u)}du, \quad (k, j = 1, 2). \tag{17}$$

The temperature characteristics T_1, T_2 through the Fourier coefficients T_{1mn}, T_{2mn} are described by formulae

$$\{T_1, T_2\} = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\{T_{1mn}, T_{2mn}\}\sin\frac{\pi n}{l}x\,\cos m\theta. \tag{18}$$

4.2. Finding the generalized displacements

The solution of the system of equilibrium equations (8), which satisfies the boundary conditions (11), under the known temperature field (18) is found by the method of finite double Fourier transforms with respect to the coordinates x, q. As a result, we obtain a system of algebraic equations for determining the Fourier coefficients y_{kmn} of the sought functions. Let us write this system in a matrix form:

$$AY = ST_{1mn} + GT_{2mn}. \tag{19}$$

Here the matrices $A = (a_{rk})_{6\times 6}$, $Y = (y_{kmn})_{6\times 1}$, $S = (s_k)_{6\times 1}$, $G = (g_k)_{6\times 1}$, while $y_{imn} = U_{imn}$ are the Fourier coefficients for displacements u_i, and $y_{3+i,mn} = \Gamma_{imn}$ are the Fourier coefficients for γ_i $(i = 1, 2, 3)$. The coefficients a_{rk}, s_k and g_k of the specified matrices we calculate from the expressions of the differential operators of the system (8).

From the system (19), we obtain the solution:

$$y_{kmn} = \frac{1}{|A|}\sum_{r=1}^{6}(s_r T_{1mn} + g_r T_{2mn})\, B_{rk}, \quad (k = 1, 2, \dots, 6),$$

where $|A|$ is the determinant of the matrix A, and B_{rk} is the algebraic adjunct to the element a_{rk} of this matrix.

The generalized displacements in terms of Fourier coefficients are given by the formulae:

$$\{u_1, \gamma_1\} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\{U_{1mn}, \Gamma_{1mn}\}\cos\frac{\pi n}{l}x\cos m\theta,$$

$$\{u_2, \gamma_2\} = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\{U_{2mn}, \Gamma_{2mn}\}\sin\frac{\pi n}{l}x\sin m\theta,$$

$$\{u_3, \gamma_3\} = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\{U_{3mn}, \Gamma_{3mn}\}\sin\frac{\pi n}{l}x\cos m\theta. \tag{20}$$

Based on the known generalized displacements (20) and the temperature field (18), all other characteristics of the stress-strain state of the shell are determined by the formulae (3), (4), (5) and (9) given above.

5. Numerical analysis of the thermoelastic state of a two-layer cylindrical shell of a regular antisymmetric structure

We assume that the shell is heated by the temperature field given at the initial moment of time: $T_1^{(0)}(x,\theta) = \phi(x,\theta)$, $T_2^{(0)} = 0$, or the shell is heated by the environment with temperature $t_c^+(x,\theta,\tau) = \phi(x,\theta)S_+(\tau)$, $t_c^-(x,\theta,\tau) = 0$ given respectively on the surfaces $z = \pm h$ of the shell by convective heat exchange. There are no heat sources.

As the function of temperature distribution depending on the coordinates (x,θ), we choose the function $\varphi(x,\theta)$:

$$\varphi(x,\theta) = T^*\left(1 - \frac{(x-x_0)^2}{d^2}\right)\left(1 - \frac{\theta^2}{\eta^2}\right)\left[S_-(x - x_0 + d) - S_+(x - x_0 - d)\right]\left[S_-(\theta + \eta) - S_+(\theta - \eta)\right].$$

Here $T^* = \text{const}$, $2d$ and 2η are the width and angle of the heating area, respectively; $(x_0, 0)$ are the coordinates of the middle of this area, $S_+(x) = \begin{cases} 1, & x > 0, \\ 0, & x \leqslant 0, \end{cases}$ $S_-(x) = \begin{cases} 1, & x \geqslant 0, \\ 0, & x < 0 \end{cases}$ are asymmetric unit functions.

Fourier coefficients T_{inm}^0, Q_{inm} included in the solution (15) are calculated according to the formula (16). We obtained the expressions:

$$\left\{T_{1n0}^0, Q_{in0}\right\} = \left\{1, \frac{\mathrm{Bi}}{2}\right\}\frac{16}{3}\frac{\eta T^*}{\pi^3 n^2 (d/l)^2}\left(\frac{1}{\pi n}\sin\frac{\pi n d}{l} - \frac{d}{l}\cos\frac{\pi n d}{l}\right)\sin\frac{\pi n x_0}{l}, \quad T_{2n0}^0 = 0,$$

$\{T_{1nm}^0, Q_{inm}\} = \left\{1, \frac{\mathrm{Bi}}{2}\right\}\frac{32T^*}{\pi^3 n^2 m^2 \eta^2 (d/l)^2}\left(\frac{1}{\pi n}\sin\frac{\pi n d}{l} - \frac{d}{l}\cos\frac{\pi n d}{l}\right)\left(\frac{1}{m}\sin m\eta - \eta\cos m\eta\right)\sin\frac{\pi n x_0}{l}$, $T_{2nm}^0 = 0$, $(m \neq 0)$. Accordingly, the function of time $Z_k^{(j)}(\tau)$, which is determined by the formula (17), will have the form:

$$Z_k^{(j)}(\tau) = \frac{1}{p_j}\big(1 - \exp(-p_j\tau_1)\big)S_+(\tau).$$

The layers of the shell are made of orthogonal reinforced composite with the following physical and mechanical properties [1,2]: $E_L = 150\,\mathrm{GPa}$, $E_T = 110\,\mathrm{GPa}$, $G_{LT} = 35\,\mathrm{GPa}$, $G_{TT} = 41\,\mathrm{GPa}$, $\nu_{LT} = \nu_{TT} = 0.33$, $\alpha_L = 7.6\cdot 10^{-6}\,1/\mathrm{K}$, $\alpha_T = 14.0\cdot 10^{-6}\,1/\mathrm{K}$, $\lambda_L = 105\,\mathrm{W/m\cdot K}$, $\lambda_T = 75\,\mathrm{W/m\cdot K}$, where the indices L and T indicate the parallel and perpendicular direction to the fibers of reinforcement.

The values of the other parameters are as follows: $h/R = 0.05$, $l/R = 5$, $\eta = \pi/4$, $d/l = (R/l)\sin\eta$, $x_0 = l/2$, $k' = 5/6$, $\mathrm{Bi} = 1$.

During the numerical experiment, there are calculated: the dimensionless temperature field $T_1' = \frac{T_1}{T^*}$, the deflection $w' = \frac{w}{R\alpha_L T^*}$, and the stresses $\sigma_i' = \frac{\sigma_{ii}}{E_L\alpha_L T^*}$, $\sigma_{23}' = \frac{\sigma_{23}}{E_L\alpha_L T^*}$ $(i = 1, 2)$ for the four values of dimensionless time $\tau' = \frac{\lambda_T \tau}{c_\varepsilon h^2}$: 0.01, 0.1, 0.4, and 1.

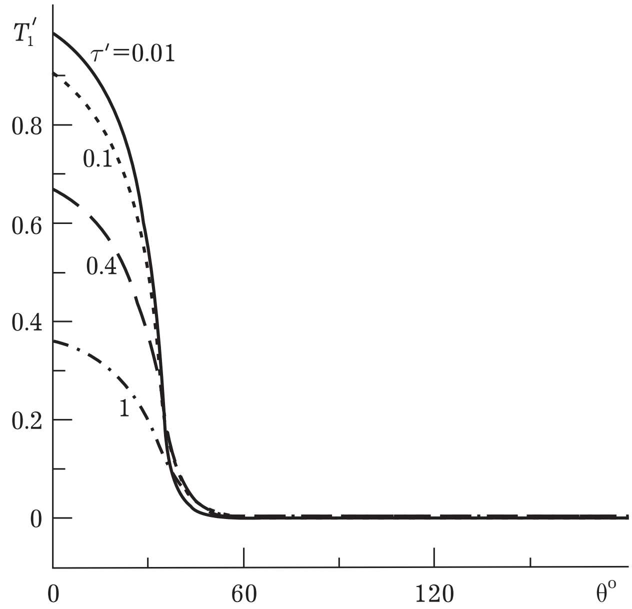

Fig. 1. Change in average temperature T_1' along guiding line $x' = 0.5$.

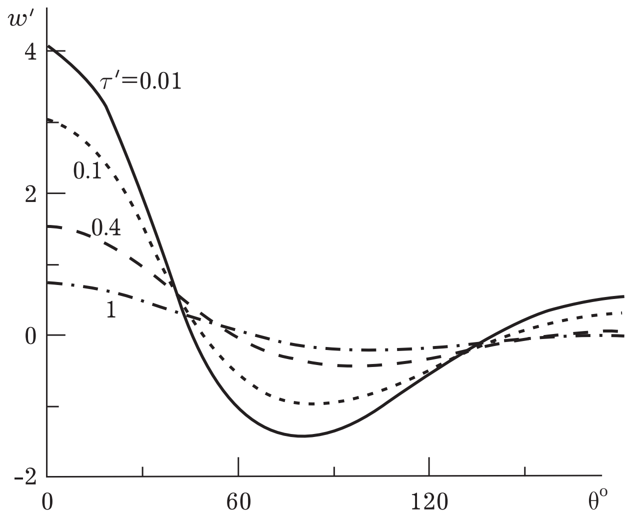

Fig. 2. Change of radial deflection w' along guiding line $x' = 0.5$.

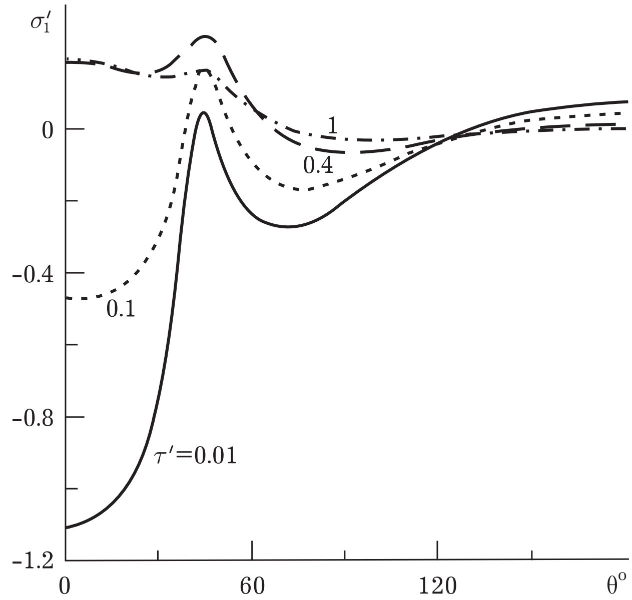

Fig. 3. Change in axial stress σ_1' along guiding line $x' = 0.5$.

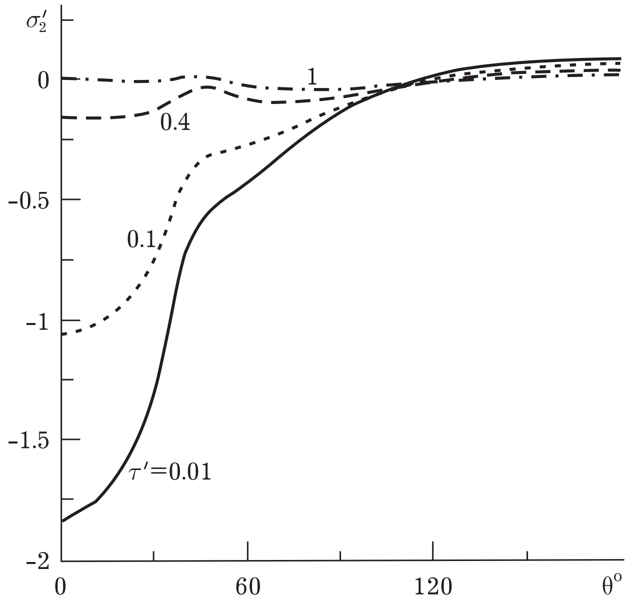

Fig. 4. Change in circular stress σ_2' along guiding line $x' = 0.5$.

In Figures 1–4, the changes in the average temperature T_1', in the radial deflection w', as well as in the axial σ_1' and circular σ_2' stresses along the guiding line $x' = 0.5$ from the middle of the heated area to the middle of the unheated area $(0 \leqslant \theta \leqslant \pi)$ are illustrated.

The maximum values of temperature and radial deflections are observed in the middle of the heated area. The values of deflections along the guiding line alternate between positive and negative values, and along the generator they monotonically decrease to zero. The normal stresses σ_1', σ_2' are calculated on the outer surface $z' = z/h = 1$, where at the initial moment of time in the heated area the stresses are compressive, and their maximum values are observed at the point $(0.5; 0)$. Over time, stresses and displacements in the heated area and outside it are equalized.

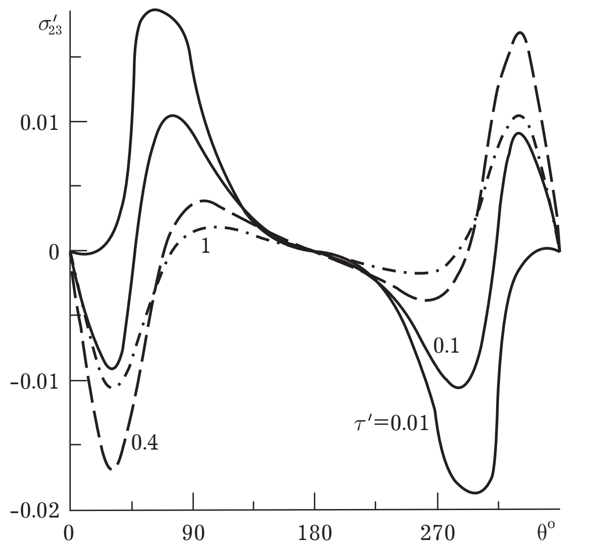

Fig. 5. Change in shear stress σ_{23}' along guiding line $x' = 0.5$.

Figure 5 illustrates the change in the shear stress along the guiding line. The shear stresses are calculated on the middle surface of the shell. It was found that these stresses reach their maximum values at the interface of heated and unheated areas. When passing through the middle of these areas, these stresses change their sign.

6. Conclusion

Based on the equations of the six-modal linear shear theory of the first order, the algorithm for determining the stress-strain state of a layered orthotropic circular closed cylindrical shell, which is heated by the temperature field specified at the initial moment, is proposed. Using the integral Fourier transforms of spatial variables and Laplace transform with respect to the time variable, a closed solution of the non-stationary problem of thermal conductivity and the quasi-static problem of unbound thermoelasticity for a finite hinged cylindrical shell supported at the ends is written. A numerical analysis is performed for a two-layer shell of a regular antisymmetric structure. The regularities of temperature, deflection, and stress dependence on the circular coordinate at different moments have been established.

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Визначення і аналіз термопружного стану шаруватих ортотропних циліндричних оболонок

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Записано вихідні співвідношення квазістатичної задачі термопружності для скінченної шаруватої ортотропної циліндричної оболонки антисиметричної структури. За конвективного теплообміну на поверхнях даної оболонки і лінійної залежності температури від поперечної координати приведено вихідну систему рівнянь на інтегральні характеристики температури. Запропоновано метод розв'язування сформульованих задач термопружності і теплопровідності, який використовує подвійне скінченне інтегральне перетворення Фур'є за відповідними координатами перетворення і Лапласа за часом. Приведено результати числового аналізу температури, прогинів і напружень для розглядуваної двошарової шарнірно обпертої по краях оболонки за локального нагріву початково заданим температурним полем.

Ключові слова: *ортотропна; шарувата; циліндрична оболонка; температура; термонапружений стан; теплообмін.*