MATHEMATICAL MODELING AND COMPUTING, Vol. 10, No. 3, pp. 956–964 (2023)

Semilinear periodic equation with arbitrary nonlinear growth and data measure: mathematical analysis and numerical simulation

El Ghabi M., Alaa H., Alaa N. E.

Laboratory LAMAI, Faculty of Science and Technology,
Cadi Ayyad University,
40000 Marrakesh, Morroco

(Received 12 June 2022; Revised 16 August 2023; Accepted 26 August 2023)

In this work, we are interested in the existence, uniqueness, and numerical simulation of weak periodic solutions for some semilinear elliptic equations with data measures and with arbitrary growth of nonlinearities. Since the data are not very regular and the growths are arbitrary, a new approach is needed to analyze these types of equations. Finally, a suitable numerical discretization scheme is presented. Several numerical examples are given which show the robustness of our algorithm.

Keywords: *periodic solution; semilinear equation; optimisation method; numerical simulation.*

2010 MSC: 35B10, 35D30, 35K59, 35K55, 34C25 **DOI:** 10.23939/mmc2023.03.956

1. Introduction

Periodic equations play an important role in the development of mathematical analysis of differential and partial differential equations. These problems appear in the modeling of many real-world phenomena, including fluid mechanics, pseudo-plastic flows, chemical reactions (resistivity of materials), nerve impulses (Fitzhugh–Nagumo problem), population dynamics (Lotka–Volterra system), combustion, morphogenesis, genetics, etc. Hundreds of articles on periodic problems have been published in various journals and conference proceedings, although there are still more questions than answers. We refer the reader to [1–10] for a good introduction to periodic problems. These references contain review articles on ordinary periodic differential equations, which focus on the mathematical modeling of nonlinear equations and expose different solving methods. Among them are degree theory, variational methods, compactness methods, monotone methods, lower and upper solutions techniques, etc.

The purpose of this paper is to conduct a mathematical analysis and a numerical simulation of weak solutions for the semilinear equation with periodic boundary conditions.

Consider the following model equation

$$\begin{cases} u(t) - u''(t) + j(t, u(t)) = f \ \text{ in } \ (0, T), \\ u(0) = u(T), \quad u'(0) = u'(T), \end{cases} \tag{1}$$

where $T > 0$ is a period, $j\colon [0, T] \times \mathbb{R} \to [0, +\infty[$ is measurable continuous function with respect to u, T-periodic with respect to t, i.e., $j(0, r) = j(T, r)$ $\forall r$ (it allows to expand j into a continuous periodic function on $\mathbb{R}$, by $j(t, r + kT) = j(t, r)$ $\forall r \in (0, T)$) and f is a given positive bounded Radon measure on $]0, T[$, T-periodic in the sense of the following definition.

Definition 1. *We denote by $\mathcal{M}_B^+(0, T)$ the set of positive bounded Radon measures on $]0, T[$.*

$f \in \mathcal{M}_B^+(0, T)$ is said to be T-periodic if there exists $f_\varepsilon \in C([0, T])^+$ such that $f_\varepsilon(0) = f_\varepsilon(T)$ and

$$\forall \phi \in C([0, T]),\ \langle f, \phi \rangle = \lim_{\varepsilon \to 0} \int_0^T f_\varepsilon(t) \phi(t)\, dt.$$

An example of a Radon measure to be 1-periodic is $f = \delta_{\frac{1}{2}}$ since the Lorentzian sequence:

$$f_\varepsilon(t) = \frac{1}{\pi\varepsilon}\frac{1}{1+\frac{(t-1/2)^2}{\varepsilon^2}} \tag{2}$$

is 1-periodic continuous (in the sense that f_ε is defined on $[0,1]$ by (1), and its extension beyond $]0,1[$ is given by $f_\varepsilon(t+k) = f_\varepsilon(t)$ with $k \in \mathbb{Z}$ and $t \in (0,1)$. One can prove that f_ε is convergent in the sense of measure to $\delta_{\frac{1}{2}}$.

The case: $j \equiv 0$ corresponds to the linear periodic problem. It has been widely studied in the literature due to the regularity of f. When f is T-periodic and $f \in C([0,T])$ and $r \longrightarrow j(t,r)$ is globally Lipschitz, Coster et al. [10] proves the existence of a periodic solution $u \in W^{1,2}([0,T])$. Takemura et al. [11] considered the case where f is 1-periodic and $f \in L^2(0,T)$ and they prove existence and uniqueness of a periodic solution $u \in H^2(0,1)$, in addition u is expressed as

$$u(t) = \int_0^T G(t-s)f(s)\,ds \quad (0<t<T),$$

where G is the Green function given by

$$G(t-s) = \begin{cases} \frac{1}{T}\frac{e^{\frac{(t-s)}{T}-1}}{(e^{-1}-1)^2}\left(1+\left(e^{-1}-1\right)\frac{(t-s)}{T}\right) & (0<s\leqslant t<T), \\ \frac{1}{T}\frac{e^{\frac{(t-s)}{T}}}{(e^{-1}-1)^2}\left(e^{-1}+\left(e^{-1}-1\right)\frac{(t-s)}{T}\right) & (0<t<s<T). \end{cases}$$

In the case where j actually depends on t and u, i.e. $j = j(t,u(t))$, the problem is said to be semilinear. It has been analyzed by Ciarlet et al. [12], by using an optimization method and under the following assumptions: $f \in C([0,T])$, $r \to j(t,r)$ is differentiable nondecreasing and $\forall t \in [0,T]$, $\left|\frac{\partial j(t,r)}{\partial r}\right|$ is bounded on the bounded set of $\mathbb{R}$.

In the present work, we are particularly interested in cases where f is irregular and the growth of j with respect to u is arbitrary. Obviously, classical methods fail to prove the existence and new techniques must be used. We describe some of them here.

The other analysis that we deal with in this paper, is the simulation of the periodic solution of (1). Several methods for numerical analysis and simulation of periodic equations have been proposed in the literature. One of the numerical methods is the collocation method, see [13,14]. Samoilenko [15] proposed another quasi-linear numerical method. Here we will present the complete discretization of equation (1) by finite differences. Then we reduce the search for a periodic solution to the solution of a nonlinear system whose dimension is the number of nodes of the considered mesh. We then develop an algorithm based on the Newton–Raphson method to numerically simulate a large system and obtain an approximation of our periodic solution.

The rest of this paper is organized as follows. In Section 2, we present the exact problem statement and main results. In Section 3, we give the existence proof for the semilinear problem, if $f \in L^2(0,T)$. In Section 4, we construct an approximate problem for (1) with regular data whose existence will be a consequence of the previous section. After performing a priori estimations, we pass to the limit in the approximated problem and prove the main existence result. The last section is devoted to numerical simulation of our general problem. After proposing a numerical scheme based on finite differences, we present several numerical examples to demonstrate the efficiency and robustness of our proposed algorithm.

2. Statement of the main theoretical result

Throughout this paper we assume:
A_1) $f \in \mathcal{M}_B^+(0,T)$ T-periodic (in the sens of Definition 1);
A_2) $j\colon [0,T]\times\mathbb{R} \to [0,+\infty[$ a mesurable T-periodic function;
A_3) $\forall t$, $r \to j(t,s)$ is continuous and nondecreasing and $j(t,0)=0$;
A_4) $\forall r \in \mathbb{R}$, $j(t,r) \in L^1(0,T)$.

Consider for $1 \leqslant p \leqslant \infty$,

$$W^{1,p}_{per}(0,T) = \left\{u \in W^{1,p}(0,T), \text{ such that } u(0) = u(T)\right\}$$

equipped with the norm induced by $W^{1,p}(0,T)$

$$\|u\|_{1,p} = \|u\|_p + \|u'\|_p.$$

In the case $p = 2$, this space is noted by $H^1_{per}(0,T)$.

Now we introduce the notion of weak periodic solution of the problem (1) used here.

Definition 2. *A function u is said to be a weak T-periodic solution of the problem (1), if*

$$\begin{cases} u \in W^{1,1}_{per}(0,T) \\ \int_0^T u(t)\phi(t)\,dt + \int_0^T u'(t)\phi'(t)\,dt + \int_0^T j(t,u(t))\phi(t)\,dt = \langle f,\phi\rangle \text{ for all } \phi \in W^{1,\infty}_{per}(0,T). \end{cases} \tag{3}$$

Remark 1.

i) for all $1 \leqslant p \leqslant \infty$, $W^{1,p}_{per}(0,T) \subset C([0,T])$ with compact injection.

ii) $\langle , \rangle$ denotes the duality bracket between $\mathcal{M}_B(0,T)$ and $L^\infty(0,T)$.

iii) if $u \in W^{1,\infty}_{per}(0,T)$, since j satisfy (A_4), then $j(t,u(t)) \in L^1(0,T)$, therefore all terms in (3) make sense.

Till the end of this paper, we denote by C every generic and positive constant. We have the following main result.

Theorem 1. *Assume that $(A_2) - (A_4)$ holds. Then for all $f \in \mathcal{M}^+_B(0,T)$ T-periodic, there exists a weak nonnegative T-periodic solution u of (1).*

3. An auxiliary existence result

Consider $f \in L^2(0,T)$. One can obtain the following result.

Theorem 2. *Let $f \in L^2(0,T)$ be T-periodic and j satisfy (A_2)-(A_4). Then there exists a unique nonnegative weak T-periodic solution of the problem*

$$\begin{cases} u \in H^1_{per}(0,T) \\ \int_0^T u(t)\phi(t)\,dt + \int_0^T u'(t)\phi'(t)dt + \int_0^T j(t,u(t))\phi(t)\,dt = \int_0^T f(t)\phi(t)\,dt \text{ for all } \phi \in H^1_{per}(0,T). \end{cases} \tag{4}$$

In addition, if $f \geqslant 0$ then $u(t) \geqslant 0$ $\forall t \in [0,T]$.

Proof. Let us define the functional

$$J\colon \begin{array}{l} H^1_{per}(0,T) \to \mathbb{R} \\ v \to \dfrac{1}{2}\displaystyle\int_0^T |v(t)|^2 dt + \dfrac{1}{2}\int_0^T |v'(t)|^2 dt + \int_0^T J_p(t,v(t))\,dt - \int_0^T f(t)v(t)\,dt, \end{array}$$

where $J_p(t,r) = \int_0^r j(t,s)\,ds$. Since J_p and $\|u\|^2_{1,2}$ are convex then J is convex. Now we will prove that J is lower semi-continous. Consider for $C \in \mathbb{R}$, the set

$$A = [J \leqslant C] = \{v \in H^1_{per}(0,T) \text{ such that } J(v) \leqslant C\}.$$

We are going to prove that A is a closed set in $H^1_{per}(0,T)$. Let us consider a sequence $v_n \in A$ and $v_n \to v$ in $H^1_{per}(0,T)$, we have

$$\frac{1}{2}\int_0^T |v_n(t)|^2 dt + \frac{1}{2}\int_0^T |v_n'(t)|^2 dt + \int_0^T J_p(t,v_n(t))\,dt - \int_0^T f(t)v_n(t)\,dt \leqslant C. \tag{5}$$

Since $H^1_{per}(0,T) \subset C([0,T])$ with a compact injection, we can extract a subsequence v_{nk} such that

$$v_{nk} \to v \text{ in } C([0,1]),$$

since $v \in H^1_{per}(0,T)$ we get also $\int_0^T f v_{n_k} \to \int_0^T f v$ and by using Fatou's Lemma, we get

$$\int_0^T J_p(v(t))\,dt \leqslant \liminf_{k\to+\infty} \int_0^T J_p\big(t, v_n(t)_k\big)\,dt.$$

Passing to the limit in (5), we obtain $J(v) \leqslant \liminf\limits_{n\to+\infty} J(v_{nk}) \leqslant C$. Therefore $v \in A$.

Now we prove that J is infinite at infinity. We have
$$J(v) \geqslant \frac{1}{2}\|u\|_{1,2}^2 - \|v\|_{1,2}\|f\|_{L^2}$$
then
$$\liminf_{n\to+\infty} \frac{J(v)}{\|u\|_{1,2}} = +\infty.$$
Consequently, J attains a unique global minimum
$$\inf_{v\in H^1_{per}(0,T)} J(v) = \min_{v\in H^1_{per}(0,T)} J(v) = J(u).$$
Let us finally show that u is a solution of (4). By choosing $v = u + s\phi$ for any s in the neighborhood of 0 and any $\phi \in H^1_{per}(0,T)$, we get:
$$\frac{1}{2}\int_0^T |u(t)+s\phi(t)|^2 dt + \frac{1}{2}\int_0^T |u'(t)+s\phi'(t)|^2 dt + \int_0^T J_p(t,u(t)+s\phi(t))\,dt$$
$$-\frac{1}{2}\int_0^T |u(t)|^2 dt - \frac{1}{2}\int_0^T |u'(t)|^2 dt - \int_0^1 J_p(t,u(t))\,dt \geqslant s\int_0^T f(t)\phi(t)\,dt.$$
We divide the inequality by $s>0$, then $s<0$, the limit when s approaches 0 gives us:
$$\lim_{s\to 0}\frac{J(u+s\phi)-J(u)}{s} = 0,$$
then
$$\frac{d}{ds}_{|s=0} J(u+s\phi) = 0.$$
Which, in turn, yields
$$\int_0^T u(t)\phi(t)\,dt + \int_0^T u'(t)\phi'(t)\,dt + \int_0^T j(t,u(t))\phi(t)\,dt = \int_0^T f(t)\phi(t)\,dt \quad \forall \phi \in H^1_{per}(0,T).$$
Finally, suppose $f \geqslant 0$ a.e. in $(0,T)$, since j is nonnegative, we consider the equation (1) with
$$\hat{j}(t,r) = \begin{cases} j(t,r) \text{ if } r \geqslant 0, \\ 0 \text{ if } r < 0 \end{cases}$$
instead of j. It is clear that if $r \geqslant 0$, $\hat{j} = j$.

We introduce the function sign^- defined on $\mathbb{R}$ by
$$\operatorname{sign}^-(r) = \begin{cases} -1 \text{ if } r < 0, \\ \ \ 0 \text{ if } r \geqslant 0, \end{cases}$$
as sign^- is an increasing function, we consider the convex function ρ_ε, which is a twice differentiable function such that
$$\rho'_\varepsilon(r) \to \operatorname{sign}^- r \text{ when } \varepsilon \to 0.$$
Let us take $\rho'_\varepsilon(u)$ as a test function, then, we get
$$\int_0^T u(t)\rho'_\varepsilon(u(t))\,dt + \int_0^T u'^2(t)\rho''_\varepsilon(u(t))\,dt + \int_0^T \hat{j}(t,u(t))\rho'_\varepsilon(u(t)) = \int_0^T \rho'_\varepsilon(u(t))f(t)\,dt$$
using the convexity of ρ_ε, we deduce that
$$\int_0^T u'^2(t)\rho''_\varepsilon(u(t))\,dt \geqslant 0$$
for the other terms, we have
$$\lim_{\varepsilon\to 0}\int_0^T \hat{j}(t,u(t))\rho'_\varepsilon(u(t))\,dt = \lim_{\varepsilon\to 0}\int_{[u\geqslant 0]} \hat{j}(t,u(t))\rho'_\varepsilon(u(t))\,dt + \int_{[u<0]} \hat{j}(t,u(t))\rho'_\varepsilon(u(t))\,dt$$
$$= \int_{[u<0]} \hat{j}(t,u)\,dt = 0.$$
It follows that
$$\lim_{\varepsilon\to 0}\int_0^T u(t)\rho'_\varepsilon(u(t))\,dt \leqslant \lim_{\varepsilon\to 0}\int_0^T \rho'_\varepsilon(u(t))f(t)\,dt,$$

which implies that

$$\int_0^T u^-(t)\,dt \leqslant -\int_0^T f(t)\,dt \leqslant 0,$$

which allows us to conclude that $u(t) \geqslant 0$ a.e. t in $[0,T]$. ■

4. Proof of the main result

Since $f \in \mathcal{M}_B^+(0,T)$ T-periodic then there exist $f_n \in C([0,T])$ $f_n(0) = f_n(1)$, $f_n \geqslant 0$ such that $\|f_n\|_{L^1} \leqslant \|f\|_{\mathcal{M}_B}$ and which converge to f in $\mathcal{M}_B^+(0,T)$. According to the Theorem 2, there exists u_n, nonnegative weak T-periodic solution of

$$\begin{cases} u_n \in H^1_{per}(0,T),\ u_n \geqslant 0, \\ \int_0^T u_n(t)\phi(t)\,dt + \int_0^T u_n'(t)\phi'(t)\,dt + \int_0^T j_n(t,u_n(t))\phi(t)\,dt = \int_0^T f_n(t)\phi(t)\,dt \quad \forall \phi \in H^1_{per}(0,T). \end{cases} \tag{6}$$

We have the following estimates.

Lemma 1. *Let u_n be the sequence defined as above, then we have:*
i) $\int_0^T |u_n|\,dt \leqslant \|f\|_{\mathcal{M}_B}$;
ii) $\int_0^T |j_n(t,u_n)|\,dt \leqslant \|f\|_{\mathcal{M}_B}$;
iii) $\int_0^T |u_n''(t)| \leqslant C\|f\|_{\mathcal{M}_B}$.

Proof. Take $\phi \equiv 1$ in (6), and as $j(\cdot,u_n) \geqslant 0$, it comes that

$$\int_0^T u_n(t)\,dt + \int_0^T j_n(t,u_n(t))\,dt = \int_0^1 f_n(t)\,dt \leqslant \|f\|_{\mathcal{M}_B},$$

since u_n and $j(t,u_n) \geqslant 0$, then we obtain i) and ii).

Finally, we deduce from (6) that

$$\begin{cases} u_n'' = u_n + j(t,u_n) - f_n \text{ in } \mathcal{D}'(0,T), \\ u_n(0) = u_n(T), \\ u_n'(0) = u_n'(T). \end{cases} \tag{7}$$

Since u_n, $j(t,u_n)$, $f_n \in L^1(0,1)$, then $u_n'' \in L^1(0,T)$ and one get

$$\int_0^T \left|u_n''(t)\right|\,dt \leqslant C\|f\|_{\mathcal{M}_B},$$

which proves iii).

Furthermore, u_n' is continue and $u_n'(0) = u_n'(T)$, then there exists t_{0n} such that $u_n'(t_{0n}) = 0$, hence $u_n'(t) = \int_{t_{0n}}^t u_n''(s)\,ds$. According to ii) of Lemma 1, we get $\int_0^T |u_n'(t)|\,dt \leqslant C\|f\|_{\mathcal{M}_B}$. Then u_n is bounded in $W^{1,1}_{per}(0,T)$.

Since $W^{1,1}_{per}(0,T) \subset C([0,T])$ with compact injection, then there exists $u \in W^{1,1}_{per}(0,T)$ and a subsequence noted by u_n such that $u_n \to u$ in $C[0,1]$. Therefore, due to (A_3), $j(t,u_n) \to j(t,u)$ in $L^1(0,T)$.

This allows us to go to the limit in the equation (6) and obtain that u is a weak periodic solution of the equation (1). ■

5. Numerical simulation

In this section, we propose a numerical simulation of the equation (1) using finite differences. The first subsection is devoted to discretizing our periodic problem using the finite difference method, and then we present a solution algorithm based on the Newton–Raphson method. In the second subsection, we show numerical results obtained depending on the case where the source f is a regular function or a Radon measure.

5.1. Discretization and numerical algorithm

For that we discrete the interval $[0,T]$ in $N+1$ points $x_k=(k-1)*h$, for $k=1,\dots,N+1$, where $h=\frac{T}{N}$. Let us set $u_k=u(x_k)$ and add two fictious points $x_0=-h$, $x_{N+2}=T+h$. Let us denote $u_0=u(x_0)$ and $u_{N+2}=u(x_{N+2})$. Therefore, since u is periodic we have $u_{N+1}=u_1$, $u_0=u_N$ and $u_{N+2}=u_2$, we then have N unknowns u_k, $k=1,2,\dots,N$.

Our problem (1) can then be discretized in space as follows:

$$\begin{cases} u_1-\frac{1}{h^2}(u_N-2u_1+u_2)+j(0,u_1)=f(0), \\ u_i-\frac{1}{h^2}(u_{i+1}-2u_i+u_{i-1})+j(x_i,u_i)=f(x_i) \ \text{ for } \ 2\leqslant i\leqslant N-1, \\ u_N-\frac{1}{h^2}(u_1+u_{N-1}-2u_N)+j(x_N,u_N)=f(x_N). \end{cases}$$

This can be written in matrix form:

$$G(U)=(\mathcal{I}_N-\frac{1}{h^2}A)*U+J(U)-F=0, \tag{8}$$

where $U=(u_i)_{1\leqslant i\leqslant N}$ is the unknows vector, $F=(f(x_i))_{1\leqslant i\leqslant N}$, $\mathcal{I}_N$ is the identity matrix of order N, the matrix A is given by

$$\mathbf{A}=\begin{bmatrix} -2 & 1 & 0 & \cdots & 0 & 1 \\ 1 & -2 & 1 & 0 & \cdots & 0 \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ 0 & \cdots & 0 & 1 & -2 & 1 \\ 1 & 0 & \cdots & 0 & 1 & -2 \end{bmatrix}$$

and the nonlinear term vector $J(U)$ is given by

$$J(U)=\begin{bmatrix} j(0,u_1) \\ \cdots \\ j(x_i,u_i) \\ \cdots \\ j(x_N,u_N) \end{bmatrix}$$

We will use the Newton–Raphson method to solve equation (8) starting from the initial U_0 which is the solution of the linear system:

$$(\mathcal{I}_N-\frac{1}{h^2}A)*U_0=F. \tag{9}$$

Our algorithm is therefore the following.

Algorithm 1

Input: choose $k_{\max}$ the maximum number of iterations, the tolerance ε_0, we get $N+1$ points $x_i=(i-1)*h$, $h=\frac{T}{N}$
set $k=0$ and set $U=U_0$
repeat
  set $k=k+1$
  if $(k=k_{\max})$ then exit convergence
  endif
  set $Y=G(U)$
  solve $(\mathcal{I}_N-\frac{1}{h^2}A+DJ(U))*D=-Y$
  set $U=U+D$
until $\|D\|<\varepsilon_0$
Output: U

5.2. Numerical examples

We present some numerical examples depending on the cases if the source f is regular or not. The first example is the following

$$\begin{cases} u(t) - u''(t) + u(t)^4 = 1 + t\sin(\pi t) \text{ in } (0,1), \\ u(0) = u(1), \quad u'(0) = u'(1). \end{cases} \tag{10}$$

The simulation we give here corresponds to $T = 1$, $\varepsilon_0 = 1\,e^{-9}$, $N = 400$, $k_{\max} = 8$.

Figure 1 shows the shape of the periodic solution and Figure 2 shows the decrease of the norm between two successive iterations as a function of the iteration number.

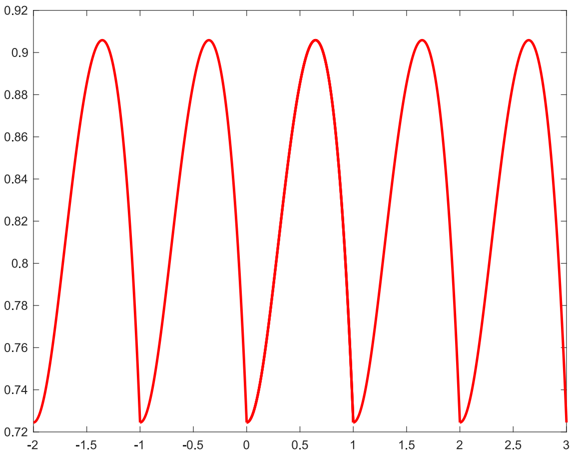

Fig. 1. Shape of the periodic solution of (10).

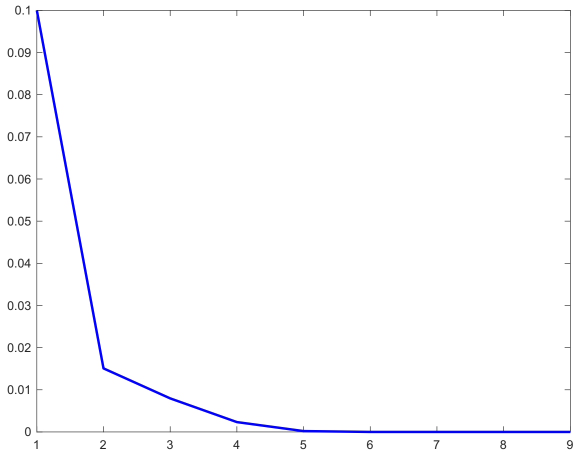

Fig. 2. Error of Newton.

The second example is the following

$$\begin{cases} u(t) - u''(t) + u^4(t) = \delta_{\frac{1}{2}} \text{ in } (0,1), \\ u(0) = u(1), \quad u'(0) = u'(1). \end{cases} \tag{11}$$

We have approximated the Dirac mass $\delta_{\frac{1}{2}}$ by the sequence of Lorentzian 1-periodic function (f_ε) we defined before (2). The simulation we give here corresponds to $T = 1$, $\varepsilon = 1\,e^{-12}$, $\varepsilon_0 = 1\,e^{-9}$, $N = 500$, $k_{\max} = 8$.

Figure 3 shows the shape of the periodic solution and Figure 4 shows the decrease of the norm between two successive iteration as a function of the iteration number.

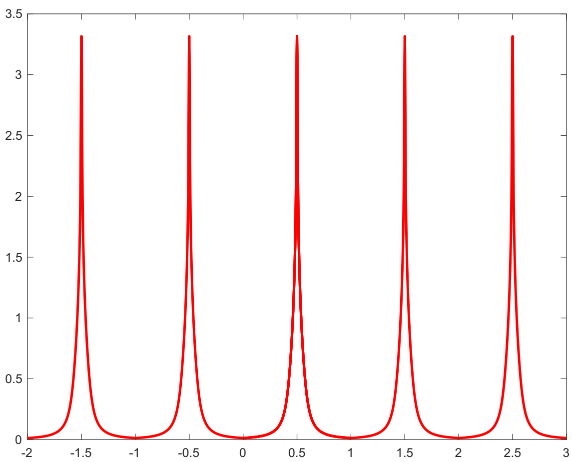

Fig. 3. Shape of the periodic solution of (11).

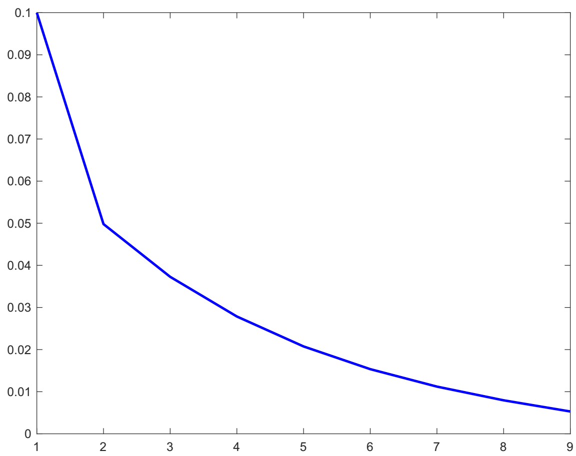

Fig. 4. Error of Newton.

6. Conclusions

In this work, we have been interested in the mathematical analysis and numerical simulation of a class of periodic nonlinear equations with non-regular data. If the data is regular, we prove the existence and uniqueness of the periodic solution through optimization methods. With the data only nonnegative measure, we construct a sequence of periodic solutions based on the regular case, and after obtaining a priori estimates, we show that we can extract a subsequence that converges to the solution to the problem we consider. We then propose a numerical algorithm to simulate these periodic solutions, giving some examples of when the data is regular or irregular. Numerical simulations demonstrate that our algorithm is efficient and robust. Besides, in the future we will focus on analyzing other numerical methods for simulating periodic equations, such as FEM (Finite Element Method), ANN (Artificial Neural Networks), LBM (Lattice Boltzmann Method) etc. and analyzed the performance differences between these methods, comparing their accuracy, time consumption, etc.

Acknowledgments

The authors wish to thank you, our esteemed Referees, for your effort and time spent evaluating our article.

[1] Alaa N. Quasilinear elliptic equations with arbitrary growth nonlinearity and data measures. Extracta Mathematicae. **11** (3), 405–411 (1996).

[2] Alaa N., Iguernane M. Weak periodic solutions of some quasilinear parabolic equations with data measure. Journal of Inequalities in Pure and Applied Mathematics. **3** (3), 46 (2002).

[3] Charkaoui A., Kouadri G., Selt O., Alaa N. Existence results of weak periodic solution for some quasilinear parabolic problem with L1 data. Annals of the University of Craiova, Mathematics and Computer Science Series. **46** (1), 66–77 (2019).

[4] Charkaoui A., Kouadri G., Alaa N. Some Results on The Existence of Weak Periodic Solutions For Quasilinear Parabolic Systems With L1 Data. Boletim da Sociedade Paranaense de Matemática. **40**, 1–15 (2022).

[5] Elaassri A., Uahabi L. K., Charkaoui A., Alaa N. E., Mesbahi S. Existence of weak periodic solution for quasilinear parabolic problem with nonlinear boundary conditions. Annals of the University of Craiova, Mathematics and Computer Science Series. **46** (1), 1–13 (2019).

[6] Alaa N. E., Charkaoui A., Elaassr A. Periodic parabolic problem with discontinuous coefficients. Mathematical Analysis and Numerical Simulation. **41** (6), 1251–1271 (2022).

[7] Alaa H., El Ghabi M., Charkaoui A. Semilinear Periodic Parabolic Problem with Discontinuous Coefficients: Mathematical Anlysis and Numerical Simulation. Filomat. **37** (7), 2151–2164 (2023).

[8] Canada A., Drabek P., Fonda A. Handbook of Differential Equations: Ordinary Differential Equations. North Holland (2008).

[9] Farkas M. Periodic Motions. Springer, New York (1994).

[10] De Coster C., Habets P. Chapter III – Relation with Degree Theory. Two-Point Boundary Value Problems: Lower and Upper Solutions. Mathematics in Science and Engineering. **205**, 135–188 (2006).

[11] Takemura K., Kametaka Y., Watanabe K., Nagai A., Yamagishi H. Sobolev type inequalities of time-periodic boundary value problems for Heaviside and Thomson cables. Boundary Value Problems. **2012**, 95 (2012).

[12] Ciarlet P. G., Scitultz H., Vag R. S. Numerical Methods of High-Order Accuracy for Nonlinear Boundary Value Problems, IV. Periodic Boundary Conditions. Numerische Mathematik. **12**, 266–279 (1968).

[13] Aubin J.-P., Ekeland I. Applied Nonlinear Analysis. Wiley, Hoboken (1984).

[14] Amjed D., Zaraiqati F., Al-Zoubhi H., Abu Hamma M. Numerical methods for finding periodic solutions of ordinary differential equations with strong nonlinearity. Journal of Mathematical and Computational Science. **11** (6), 6910–6922 (2021).

[15] Samoilenko A. M. Certain questions of the theory of periodic and quasi-periodic systems. D.Sc. Dissertation, Kiev (1967).

Напівлінійне періодичне рівняння з довільною нелінійністю зростання та мірою даних: математичний аналіз та чисельне моделювання

Ель Габі М., Алаа Х., Алаа Н. Е.

Лабораторія LAMAI, факультет науки і техніки, Університет Каді Айяд, 40000 Марракеш, Марокко

У цій роботі цікавимося існуванням, єдиністю та чисельним моделюванням слабких періодичних розв'язків для деяких напівлінійних еліптичних рівнянь із мірами даних та з довільними нелінійностями зростання. Оскільки дані не дуже регулярні, а зростання є довільним, необхідний новий підхід для аналізу цих типів рівнянь. Нарешті, наведено відповідну чисельну схему дискретизації. Наведено декілька числових прикладів, які демонструють надійність запропонованого алгоритму.

Ключові слова: *періодичний розв'язок; напівлінійне рівняння; метод оптимізації; чисельне моделювання.*