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# ALGORITHM FOR CONSTRUCTING THE SUBSOIL DENSITY DISTRIBUTION FUNCTION CONSIDERING ITS VALUE ON THE SURFACE

The conventional approach to constructing a three-dimensional distribution of the Earth's masses involves using Stokes constants incrementally up to a certain order. However, this study proposes an algorithm that simultaneously considers all of these constants, which could potentially provide a more efficient method. The basis for this is a system of equations obtained by differentiating the Lagrange function, which takes into account the minimum deviation of the three-dimensional mass distribution of the planet's subsoil from one-dimensional referential one. An additional condition, apart from taking into account the Stokes constants, for an unambiguous solution to the problem is to specify the value of the function on the surface of the ellipsoidal planet. It is possible to simplify the calculation process by connecting the indices of summation values in a series of expansions to their one-dimensional analogues in the system of linear equations. The study presents a control example illustrating the application of the given algorithm. In its implementation, a simplified variant of setting the density on the surface of the ocean is taken. The preliminary results of calculations confirm the expediency of this approach and the need to expand such a technique with other conditions for unambiguously solving the inverse problem of potential theory. Objectives. To create and implement the algorithm that takes into account the density of the planet's subsoil on its surface. Method. The mass distribution function of the planet's subsoil is represented by a decomposition into biorthogonal series, the coefficients of decomposition which are determined from a system of linear equations. The system of equations is obtained from the condition of minimizing the deviation function of the desired mass distribution from the initially determined two-dimensional density distribution (PREM reference model). Results. On the basis of the described algorithm, a three-dimensional model of the density distribution of subsoil masses in the middle of the Earth is obtained, which takes into account Stokes constants up to the eighth order inclusively and corresponds to the surface distribution of masses of the oceanic model of the Earth. Its concise interpretation is also presented.

*Key words:* distribution function, Earth, Stokes constants, Lagrange function.

## *Introduction*

Determining the distribution function of the planet's mass according to its external gravitational field is an ambiguous task, but for the unity of the solution, the imposition of additional conditions is required. Detailed studies in this direction have been constantly carried out in theoretical and practical aspects. The theoretical works include the series of works by Strakhov [Strahov, 1977], Moritza [Moritz, 1973; Moritz, 1994]. Practical results were obtained by a number of scientists, in particular, Martinenc and Pec [Martinenc & Pec, 1986], Meshcheryakov [Meshcheryakov, 1991] and others. At the same time, it should be noted that the main point in these studies is the selection of additional conditions imposed on the mass distribution functions of the planet's subsoil during the generation of the planet's external gravitational field. Reaching the minimum energy of the gravitational field of a celestial body (conditions of hydrostatic equilibrium of the planet [Morits, 1994; Fys et al., 2019; Meshcheryakov, 1991], closeness to a known and predetermined function [Morits, 1994; Fys et al., 2021; Meshcheryakov, 1991], specification of the density function on the surface of the figure of the celestial body, etc. [Fys, Gubar, 1999] can serve as such conditions. Another option for solving the given problem can be formulated as the proximity of the desired distribution function to a fixed (reference) geophysical radial model of the density function of the planet's subsoil, and the inhomogeneity of the Earth's external gravitational field is connected with its three-dimensionality [Fys et al., 2018; Shcherbakov, 1978; Fys et al., 2016]. For the clarity of the given task, it is necessary to add additional conditions. Traditionally, the minimal deviation from the presented one-dimensional mass distribution model is taken as such. However, other conditions can be set. One such condition may be an additional presentation of the mass distribution function at individual points in the middle of the Earth, for example, in the center of the Earth [Fys et al., 2019], or on its surface. The mathematical formulation of such a problem is given in [Meshcheryakov, 1989]. The proposed method of its solution is based on the theory of harmonic, biharmonic, and three-harmonic functions, which are used in the theory of elasticity and contains a series and are quite cumbersome. In addition, the practical implementation of this technique is associated with the receiving of the data obtained from observations (values on the surface of the Earth) which can be too imprecise. Therefore, it should be noted that the solution is not exact and the extent of its approximation may not always be determinable. For this reason, we plan to provide a simplified version of the solution in the future, which will be based on an approximation method that considers the density of the Earth's surface.

### *Formulation of the problem*

Determine the mass distribution function of the planet's interior, which generates its external gravitational field and takes a given value on the surface.

### *Presenting main material*

For the simplicity of the presentation of the material, we take the shape of the planet in the form of a triaxial ellipsoid [Tserklevych et al., 2016], and the mass distribution function in its middle in the form of a decomposition by biorthogonal polynomials [Meshcheryakov, 1991]:

$$\delta(x_1, x_2, x_3) = \sum_{m+n+k=0}^{N} b_{mnk} W_{mnk}(x_1, x_2, x_3) \quad (1)$$

where

$$b_{mnk} = \frac{\int_\tau \delta\omega_{mnk}\, d\tau}{\int_\tau W_{mnk}\omega_{mnk}\, d\tau},$$

$$W_{mnk} = \frac{1}{m!\,n!\,k!\,2^N} \frac{\partial^N}{\partial x_1^m \partial x_2^n \partial x_3^k} \left( \frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + \frac{x_3^2}{a_3^2} - 1 \right)^N, \quad (2)$$

$b_{mnk}$ – decomposition coefficients, $W_{mnk}$ – three-dimensional generalization of Legendre polynomials [Meshcheryakov, 1991].

On the surface of the ellipsoid, we consider the mass distribution function to be known, i.e.:

$$\delta\big|_\Omega (x_1, x_2, x_3) = \varphi\,(x_1, x_2, x_3). \quad (3)$$

The task of determining the density function is reduced to finding the expansion coefficients $b_{mnk}$ in (1), which generates the external gravitational field of the Earth [Fys et al., 2019], adequately represented by a set of Stokes constants [Pavlis et al., 2008], which through the inner spherical functions [Fys et al., 2021]:

$$U_{nk} + iV_{nk} =$$

$$= \frac{1}{Ma_e^n} \frac{RR(n-k)!}{2^k} \sum_{m=0}^{\left[\frac{n-k}{2}\right]} \frac{x_3^{n-k-2m}(-1)^m (x_1^2 + x_2^2)^m}{(n-k-2m)!(k+m)!m!} (x_1 + ix_2)^k = \quad (4)$$

$$= \sum_{p+q+s=n} (\alpha_{pqs}^n + i\beta_{pqs}^n) x_1^p x_2^q x_3^s,$$

are presented as follows:

$$C_{nk} + iS_{nk} = \int_\tau (u_{nk} + iv_{nk})\,\delta\,(x_1, x_2, x_3)\, d\tau\,, \quad n, k = 0, 1, 2, \ldots, N. \quad (5)$$

Substituting the equation (4) into relation (5) and taking into account (1), we get

$$C_{n,k} + iS_{n,k} = \sum_{t=0}^{n} \left( C_{n,k}^t + iS_{n,k}^t \right), \quad (6)$$

where

$$C_{n,k}^t + iS_{n,k}^t = \frac{RR(n-k)!}{2^k Ma_e^n} \sum_{p+q=s=t} \int_\tau \frac{b_{pqs}}{2^t\, p!\,q!\,s!} \frac{\partial^t \left(\rho^2 - 1\right)^t}{a_1^m a_2^n a_3^k \partial x_1^p \partial x_2^q \partial x_3^s} \left( \sum_{m=0}^{\left[\frac{n-k}{2}\right]} \frac{(-1)^m x_3^{n-k-2m} (x_1^2 + x_2^2)^m}{(n-k-2m)!\,m!\,(k+m)!} (x_1 + ix_2)^k \right) d\tau\,, \quad (7)$$

and $RR = \begin{cases} 1, & n = 0 \\ 2, & n \neq 0 \end{cases}.$

Representation of the density distribution function in the form (1) covers the entire set of possible distributions. However, in the discontinuity regions, the convergence of series (1) is extremely slow (in mathematics, it is called the "Gibbs effect"). To accelerate the convergence of series (1) in the future, it is advisable to present the density distribution function, including as a separate term the already well-studied one-dimensional spherical model of mass distribution (for the Earth, it is the reference model PREM [Dziewonski et al., 1977]), i.e.:

$$\delta(x_1, x_2, x_3) = \delta^0(\rho) + \sum_{m+n+k=0}^{N} b_{mnk} W_{mnk}(x_1, x_2, x_3). \quad (8)$$

Then the system will take the form [Fys et al., 2019]:

$$C_{n,k} + iS_{n,k} = C_{n,k}^{pr} + iS_{n,k}^{pr} + \sum_{t=0}^{n} \left( C_{n,k}^t + iS_{n,k}^t \right), \quad (9)$$

then

$$C_{n,k}^{pr} + iS_{n,k}^{pr} = \frac{3RR(n-k)!k!\delta_C}{2^k (n+1)!!} \int_0^1 \delta^0(\rho)\rho^{n+2} d\rho\, \frac{1}{(n+1)!} \sum_{m=0}^{\left[\frac{n-k}{2}\right]} \frac{(-1)^k (n-k-2m-1)!!k!}{2^{2m}(n-k-2m)!(k+m)!} \sum_{l=0}^{n-k-2m} \alpha^{2m+k-l} \beta^l \sum_{i+j=l} \frac{(-1)^{\frac{j}{2}} (2m+k-l-1)!!(l-1)!!}{(m-i)!i!(k-j)!j!} \gamma^{j-2k}, \quad (10)$$

where $\alpha = \frac{a_1}{a_e}$, $\beta = \frac{a_2}{a_e}$, $\gamma = \frac{a_3}{a_e}$, $\delta_C = \frac{M}{V_e}$ – average density of the Earth, $\delta_n = \int_0^1 \delta^0(\rho)\rho^{n+2}d\rho$, $C_{n,k}$, $S_{n,k}$ – even non-zero Stokes constants calculated by the PREM model of the triaxial ellipsoid.

For the ellipsoid of rotation ( $\alpha_1 = a_2$ ), respectively

$$C^p_{n,0} = \frac{3\delta_n}{\delta_C}\frac{n!}{(n+3)!!}\sum_{m=0}^{\frac{m}{2}}\frac{(-1)^m\gamma^{n-2m}(2m)!!}{(n-2m)!!(m)!2^{2m}}\alpha^{2m} = \frac{3\delta_n}{\delta_C}\frac{\left(\frac{n}{2}\right)!(n-1)!!2^{\frac{n}{2}}}{(n+3)!!}\sum_{m=0}^{\frac{m}{2}}\frac{(-1)^m\gamma^{n-2m}\left(\alpha^2\right)^m 2^m}{\left(\frac{n}{2}-m\right)!(m)!2^{3m}2^{\frac{n}{2}-m}} = \frac{3\delta_n}{\delta_C}\frac{\left(\gamma^2-1\right)^{\frac{n}{2}}}{(n+1)(n+3)}, \quad (11)$$

where $C^p_{n,0}$ are the corresponding nonzero even Stokes constants for the biaxial ellipsoid.

The total number of system equations is (10) $N^2$, and the number of sought coefficients $b_{mnk}$ is $\frac{(N+1)(N+2)(N+3)}{6}$. Therefore, the system has many solutions. For clarity, it is necessary to add additional conditions that select dependencies on the given task. The condition of minimal deviation of the three-dimensional mass distribution function from its one-dimensional counterpart, which was proposed by Professor G.O. Meshcheryakov (truth in terms of the problem of moments [Meshcheryakov, 1991]) can be considered as the mostly used one. Other conditions for obtaining a single solution can be given. For example, if there is a question of studying the hydrostatic state of the planet, related to the condition of minimum gravitational energy [Fys et al., 2019], then this is also a defining condition of unambiguity.

In this study, we propose to use the value of the density of the substance on the surface, which in principle can be considered known. However, it is practically impossible to analytically describe such a distribution due to the specifics of the setting on the surface. Therefore, it is appropriate to consider the deviation of the given and approximated function in the sense of mean square deviation. Since we take the shape of the planet to be an ellipsoid, on which the value of the elements of the series is determined:

$$W_{mnk} = \frac{N!x_1^m x_2^n x_3^k}{m!n!k!}, \quad (12)$$

then this condition will take the form:

$$\min \oint_\Omega \left(\delta - \sum_{m+n+k=0}^{\infty}\frac{N!x_1^m x_2^n x_3^k}{m!n!k!}b_{mnk}\right)^2 d\Omega. \quad (13)$$

Expression (13) is a surface integral over a closed surface, which is connected to the volume integral through the Ostrogradsky formula. Therefore, the value of the surface density function affects its structure throughout the body, and as a result, it is appropriate to study the nature of this influence.

Considering this remark, we mathematically formulate our problem as a conditional extremum problem, as it is done in the article [Fys et al., 2019], accordingly, the Lagrange function for our problem has the following form:

$$F = \oint_\Omega\left(\delta(x_1,x_2,x_3) - \delta_0 - \sum_{m+n+k=0}^{\infty}\frac{N!x_1^m x_2^n x_3^k}{m!n!k!}b_{mnk}\right)^2 d\Omega + \int_\tau\left(\sum_{m+n+k=0}^{\infty} b_{mnk}W_{mnk}\right)^2 d\tau + \sum_{i=1}^{2}\sum_{n,k}\left(\lambda^i_{n,r}c^i_{n,k} - \sum_{t=0}^{n}\sum_{p+q+s=t}\alpha^{nk}_{pqs}b_{pqs}\right), \quad (14)$$

where $\lambda^i_{n,r}$ $(i=1,2)$ is the Lagrange multiplier, $c^1_{nr} = c_{nk}$, $c^2_{nr} = S_{nk}$, $\alpha^{1,nk}_{pqs} = \alpha^{nk}_{pqs}$, $\alpha^{2,nk}_{pqs} = \beta^{nk}_{pqs}$.

The minimum of expression (14) by variables $\lambda^i_{n,r}$, $b_{mnk}$ determines the solution of the problem. To find it, differentiation of function (14) by variables $b_{mnk}$, $\lambda^i_{n,r}$ gives a system of equations relative to these unknowns. A detailed analysis shows that its structure has a block form, which allows presenting the system in the form of subsystems:

$$\begin{cases} T_iB_i - R_i\lambda_i = D_i \\ \alpha_iB_i = C_i \end{cases} (i = I - VIII), \quad (15)$$

In matrix form, the course of solving system (14) and its solution can be presented as follows:

$$\begin{cases} T_iB_i = R_i\lambda_i + D_i, B_i = T_i^{-1}\left(R_i\lambda_i + D_i\right), \alpha_iT_i^{-1}\left(R_i\lambda_i + D_i\right) = C_i \\ \lambda_i = \left(\alpha_iT_i^{-1}R_i\right)^{-1}\left(C_i - \alpha_iT_i^{-1}D_i\right) \\ B_i = T_i^{-1}\left(R_i\left(\alpha_iT_i^{-1}R_i\right)^{-1}\left(C_i - \alpha_iT_i^{-1}D_i\right) + D_i\right) \end{cases} \quad (16)$$

where the columns $c_i$ are defined as follows

$$c_{I,VI} = \begin{pmatrix} c_{T,T} \\ \cdots \\ c_{2N+T,T} \\ c_{2N+T,T+2} \\ \cdots \\ c_{2N+T,T+2N} \end{pmatrix}, c_{III} = \begin{pmatrix} c_{2,1} \\ \cdots \\ c_{2N,1} \\ c_{2N,3} \\ \cdots \\ c_{2N,2N-1} \end{pmatrix},$$

$$c_V=\begin{pmatrix} c_{1,0}\\ \dots\\ c_{2N+1,0}\\ c_{2N+1,2}\\ \dots\\ c_{2N+1,2N}\end{pmatrix},\quad c_{II,VIII}=\begin{pmatrix} s_{T+2,T+2}\\ \dots\\ s_{2N+T,T}\\ s_{2N+T,T+2}\\ \dots\\ s_{2N+T,T+2N}\end{pmatrix},$$

$$c_{IV}=\begin{pmatrix} s_{2,1}\\ \dots\\ s_{2N,1}\\ s_{2N,3}\\ \dots\\ s_{2N,2N-1}\end{pmatrix},\quad c_{VI}=\begin{pmatrix} s_{32}\\ \dots\\ s_{2N+1,2}\\ s_{2N+1,2}\\ \dots\\ s_{2N+1,2N}\end{pmatrix},\qquad (17)$$

$$T=N-2\left[\frac{N}{2}\right].$$

Accordingly, the unknown coefficients $b_{mnk}$ in subsystems (15) have the form:

$$b_I=\begin{pmatrix} b_{002}\\ \dots\\ b_{002N}\\ b_{20(2N-2)}\\ \dots\\ b_{02N0}\end{pmatrix},\quad b_{II}=\begin{pmatrix} b_{110}\\ \dots\\ b_{11(2N-2)}\\ b_{31(2N-4)}\\ \dots\\ b_{1(2N-1)0}\end{pmatrix},$$

$$b_{III}=\begin{pmatrix} b_{101}\\ \dots\\ b_{10(2N-1)}\\ b_{30(2N-3)}\\ \dots\\ b_{1(2N-2)1}\end{pmatrix},\quad b_{IV}=\begin{pmatrix} b_{011}\\ \dots\\ b_{01(2N-1)}\\ b_{21(2N-3)}\\ \dots\\ b_{0(2N-1)1}\end{pmatrix},\qquad (18)$$

$$b_V=\begin{pmatrix} b_{001}\\ \dots\\ b_{00(2N+1)}\\ b_{20(2N-1)}\\ \dots\\ b_{0(2N)1}\end{pmatrix},\quad b_{VI}=\begin{pmatrix} b_{111}\\ \dots\\ b_{11(2N-1)}\\ b_{31(2N-3)}\\ \dots\\ b_{1(2N-1)1}\end{pmatrix},$$

$$b_{VII}=\begin{pmatrix} b_{100}\\ \dots\\ b_{102N}\\ b_{30(2N-2)}\\ \dots\\ b_{12N0}\end{pmatrix},\quad b_{VIII}=\begin{pmatrix} b_{010}\\ \dots\\ b_{012N}\\ b_{21(2N-3)}\\ \dots\\ b_{0(2N+1)0}\end{pmatrix}.$$

$\tau$ – serial number of placement in columns $b$ and $D$ , $m$, $n$, $k$ and $\tau$ connected by dependencies (21).

The value of the expression

$$d_\tau=\int_\Omega \left(\varphi\left(x_1,x_2,x_3\right)\right)\frac{N!x_1^m x_2^n x_3^k}{m!n!k!}d\Omega$$

is calculated according to a known function $\varphi$ , which in most cases is specified discretely. With the regard to this, the integral will be replaced by a finite sum. Since in the future in our research we consider the figure to be a geoid, on which the density – value of the water surface is constant and equal to unity, then the calculation of integrals is reduced to the integration of the expression:

$$d_\tau=\int_\Omega \frac{N!x_1^m x_2^n x_3^k}{m!n!k!}d\Omega. \qquad (19)$$

In the future, we take the shape of the Earth as an ellipsoid of rotation with half axes $a=b.$

Accordingly, the integral (19) after reduction to double will take the form:

$$d_\tau=\int_\Omega \frac{N!x_1^m x_2^n x_3^k}{m!n!k!}d\Omega=S\frac{N!}{k!m!!n!!\left(m+n\right)!!}\iint_{D_z}\rho^{m+n+1}\left(1-\rho^2\right)^{\frac{k}{2}-1}d\rho=$$

$$=S\frac{N!}{k!!m!!n!!\left(m+n\right)!!}\left(\frac{1}{m+n+2}+\sum_{l=1}^{\infty}\frac{\left(-1\right)^l e^{2l}}{l!\left(m+n+2l\right)!!}\right).$$

Matrix coefficients $T_i$ are defined as follows $t_{i,j}=t_{i,j}^1+t_{i,j}^2$.

If $N=N_1$, $m+m_1$, $n+n_1$ and $k+k_1$ even, then

$$t_{i,j}^1=\frac{N!N_1!}{m!n!k!m_1!n_1!k_1!}\int_\Omega x_1^{m+m_1}x_2^{n+n_1}x_3^{k+k_1}d\Omega, \qquad (20)$$

or

$$t_{i,j}^1=\begin{cases}\dfrac{N!N_1!(m+m_1-1)!!(n+n_1-1)!!(k+k_1-1)!!}{m!n!k!m_1!n_1!k_1!(N+N_1+1)!!}, & m+m_1,\ n+n_1,\ k+k_1 - \text{even}\\ 0, & \text{otherwise}\end{cases}$$

Similarly,

$$t_{i,j}^1=\begin{cases}\displaystyle\int_\tau W_{mnk}W_{m_1n_1k_1}=\dfrac{(N!)^2(m+m_1-1)!!(n+n_1-1)!!(k+k_1-1)!!}{(N+N_1+3)!!}, & m+m_1,\ n+n_1,\ k+k_1 - \text{even}\\ 0, & \text{otherwise}\end{cases}$$

The indices $i, j$ correspond to their own $m$, $n$, $k$ and $m_1$, $n_1$, $k_1$ selected from (18) by dependence (21).

Elements $r_{t,s}$ of matrices $R_i$, – coefficients $\alpha_{mnk}^{nk}$ or $\beta_{mnk}^{nk}$, $r$ is the row number in equality (17), $s$ is the corresponding index $m$, $n$, $k$ in (18) and connected to them by formula (21) $\lambda_i$ is the column of unknowns $\lambda_t(\lambda_{nk}, \gamma_{nk})$.

Each of the subsystems of equations (14) gives part of the coefficients $b_{mnk}$ that appear in them with one index, similarly as in [Zayats, 2006]. Therefore, there is a need to establish a connection between the number of unknowns in the systems and the index of coefficients $b_{mnk}$: $m$, $n$, $k$, $(N^* = m + n + k)$ in a row

The scheme for establishing this connection looks like this. Auxiliary order $N^*$ of the general index $r$ choose from the condition:

$$N_1 = \frac{(N^*+1)(N^*+2)(N^*+3)}{6},$$

$$N_2 = \frac{(N^*+2)(N^*+3)(N^*+4)}{6},$$

$$N_1 < r \le N_2.$$

Next, we define another natural parameter $l$ as follows:

$$r_1 = r - N_1, \; l = \left[\frac{1+\sqrt{8r_1 - 7}}{2}\right],$$

$$l_1 = \frac{l(l-1)}{2}, \; r_2 = r_1 - l_1.$$

Then the indexes $m$, $n$, $k$ for each of the subsystems are defined as follows:

I. Supposing, we have in the equation (paired Stokes constants $C_{2n2k}$) an unknown with the number $r$.

$$N = 2(N^*+1), k = N - 2(l-1), m = 2(l - r2).$$

II. Paired Stokes constants $S_{2n2\kappa}$

$$N = 2(N^*+4), k = N - 2l, m = 2(l - r2) + 1.$$

III. Paired Stokes constants $C_{2n2k+1}$.

$$N = 2(N^*+2), k = N - 2l + 1, m = 2(l - r2) + 1.$$

IV. Paired Stokes constants $S_{2n2k+1}$

$$N = 2(N^*+2), k = N - 2l + 1, m = 2(l - r2).$$

V. Odd Stokes constants $C_{2n+12k}$.

$$N = 2N^* + 3, r1 = r - N1,$$
$$k = N - 2l + 1, m = 2(l - r2) + 1.$$

VI. Odd Stokes constants $S_{2n+12k}$

$$N = 2N^* + 3, r1 = r - N1,$$
$$k = N - 2l + 1, m = 2(l - r2). \quad (21)$$

VII. Odd Stokes constants $C_{2n+12k+1}$

$$N = 2N^* + 3, k = N - 2l + 1, m = 2(l - r2).$$

VIII. Odd Stokes constants $S_{2n+12k+1}$

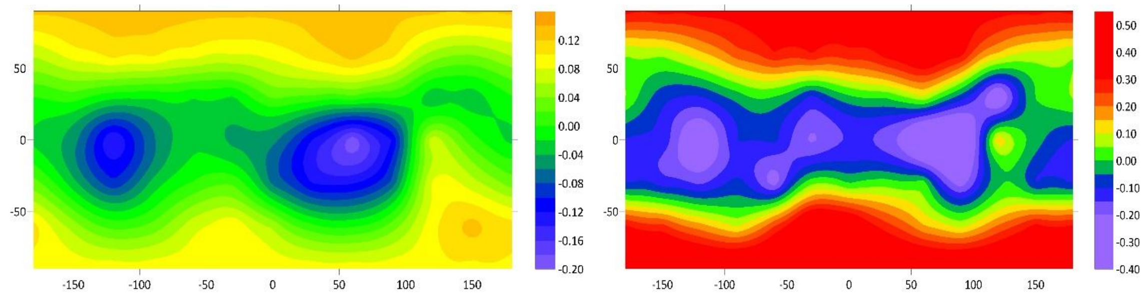
$$N = 2N^* + 3, k = N - 2l + 1, m = 2(l - r2) + 1.$$

The connection I–VIII allows the transition to the use of the found coefficients for various options for their determination, and therefore is an important link in the implementation of the calculation of the density in the middle of the planet.

***Practical implementation of the algorithm on the example of building a model distribution of subsoil masses and preliminary interpretation of the obtained results.***

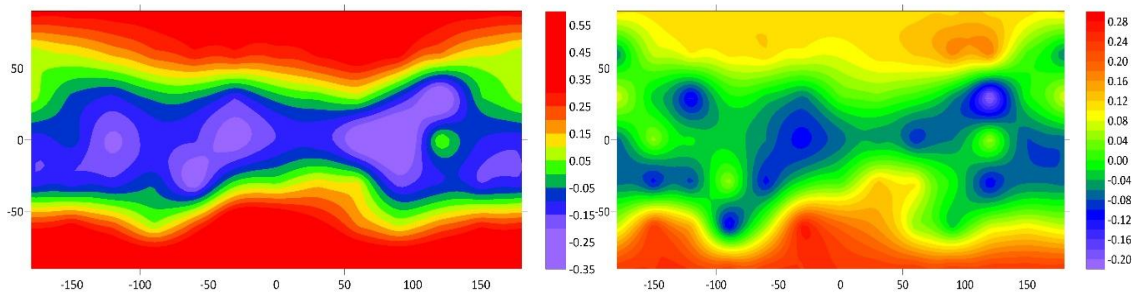
Let u's apply the algorithm described above for a specific case. As a basic model, we take the generally accepted reference one-dimensional distribution of the Earth's mass density subsurface - PREM, and its three-dimensional external gravitational field of the Earth is determined by one of the modern EGM 2008 models [Pavlis et al., 2008]. Note that the choice of the model practically does not affect the final result, since it is used to construct powers up to ten, which are practically the same for all variants of the description of the potential. It is also should be noted that the low order of approximation is associated with difficulties in implementing the method for calculations. First of all, this affects the use of large data sets and the possible accumulation of calculation errors, and therefore requires a more detailed analysis during calculations. Since the task is the implementation of the method and its possible application in interpretation, such aspects are not considered in this work and can be raised during specific constructions of three-dimensional density models, which significantly complements research using other approaches [Tserklevych, 2005; Woodhouse & Dziewonski, 1986; Panning & Romanowicz, 2006; Dziewonski, 1984].

We will construct maps of deviations of the three-dimensional mass distribution from the reference and generally accepted radial one. Below we present their cartographic image at different depths. The analysis of these maps clearly shows the differentiation of the three-dimensional deviation from the one-dimensional deviation depending on the depth of the masses. Two areas with a negative mass balance are clearly visible at the core-mantle boundary. It can be related to the Stokes constant of the second order $C_{2,0}$. It is at these depths that geophysicists attribute its formation. With the transition to the mantle region, three-dimensional detailing is observed [Tarakanov & Cherevko, 1978; Furman, 2018]. Although the centers of negative masses are concentrated at these longitudes ($\lambda = 50°$, $\varphi = 0°$) ($\lambda = 200°$, $\varphi = 0°$), new concentrations of negative heterogeneity begin to appear next to them: $\lambda = 30°$, j = 0°. Such differentiation even more intensifies when approaching the surface of the planet. The influence of negative heterogeneity with coordinates l = 30°, j = 0° at a depth of 500 km (Fig. 2a), which breaks up into several local extrema with a minus sign, is significantly weakened. A minimum of around a point ( l = 200°, j = 0° ) erodes at a depth of 500 km, forming several smaller ones. The map of inhomogeneities at a depth of 50 km deserves special attention (Fig. 3). In areas near the North Pole, there is almost zero increase in density (Arctic Ocean), while in the South there is a tendency to increase the value of density (mainland-Antarctica [Marchenko et al., 2012]). For other latitudinal areas, there is an inherent mass deficit, which is characterized by many local minima. Characteristically, on all maps in the polar regions, the three-dimensional deviation from the radial one is kept with a plus sign. It is difficult to find any explanation for this fact. Obviously, more detailed studies are needed, which include considering the distribution of masses in the earth's crust [Laske et al., 2013], and possibly in the center of mass [Bullen., 1962]. Also, for a more complete picture, it is necessary to increase the order of approximation of the three-dimensional mass distribution, which involves taking into account the Stokes constant of higher orders.




a b

**Fig. 1.** Deviation of the three-dimensional mass distribution function from the one-dimensional one at a depth of 2890 km – a), 1000 km – b).




a b

**Fig. 2.** Deviation of the three-dimensional mass distribution function from the one-dimensional one at a depth of 500 km – a), 100 km – b).

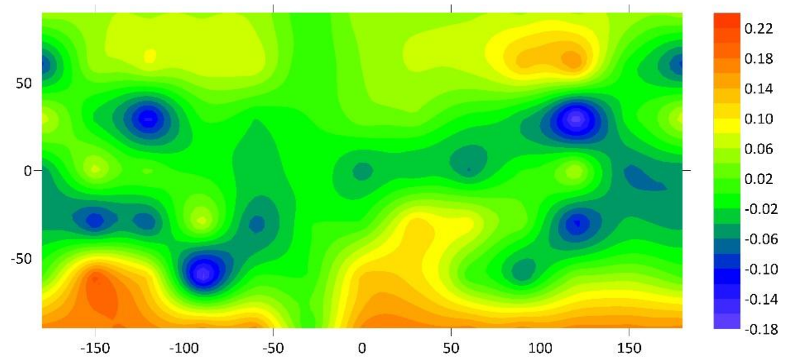



**Fig. 3.** Deviation of the three-dimensional mass distribution function from the one-dimensional one at a depth of 50 km.

### *Conclusions*

The proposed method of approximate construction of the three-dimensional mass distribution of the planet's subsoil, considering the Stokes constants, significantly complements the existing methods of studying the internal structure of the Earth.

Taking into account the Stokes orders in their entirety alters the picture of mass distribution in the middle of the Earth.

Setting the values of the density of the planet significantly complements the picture of the placement of inhomogeneities, which emphasizes the growing role of setting values on the surface when approaching it.

A more detailed study of this role, as well as an increase in the order of approximation, is planned to be performed in further studies.

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## АЛГОРИТМ ПОБУДОВИ ФУНКЦІЇ РОЗПОДІЛУ ГУСТИНИ ЗЕМНИХ НАДР З УРАХУВАННЯМ ЇЇ ЗНАЧЕННЯ НА ПОВЕРХНІ

На відміну від широко вживаного ітераційного методу побудови тривимірного розподілу мас Землі, що використовує поетапно стоксові постійні до встановленого порядку, в роботі запропонований алгоритм одночасного їх урахування. Функція розподілу мас надр планети подається сумою многочленів трьох змінних, коефіцієнти розкладу якої визначаються з системи рівнянь. Ця система одержується диференціюванням функції Лагранжа, яка будується з урахуванням мінімального відхилення тривимірного розподілу мас надр планети від референтного одновимірного. Додатковою умовою, крім урахування стоксових постійних, для однозначного розв’язання задачі є задання значення функції на поверхні еліпсоїдальної планети. Кліткова структура матриці системи дає можливість апроксимації високих порядків та можливість збільшити його у вісім разів, що є наслідком групування стоксових постійних, а отриманий зв'язок між індексами величин сумування в ряд розкладу та їх одновимірними аналогами в системі лінійних рівнянь дає можливість просто реалізовувати процес обчислень. Подається контрольний приклад, що ілюструє ефективність застосування наведеного алгоритму. При його реалізації береться спрощений варіант задання густини на поверхні океану, що приймається за одиницю. В подальшому планується використати одну з моделей густини земної кори та провести чисельне інтегрування поверхневих інтегралів для більш повного відображення реальності Результати обчислень узгоджуються з дослідженнями, проведеними за допомогою інших методів, наприклад, методів сейсмічної томографії, що підтверджує доцільність такого підходу та необхідність розширення даної методики та, можливо, долученням інших умов для однозначного розв’язування оберненої задачі теорії потенціалу. Мета. Створити та реалізувати алгоритм, який враховує значення густини надр планети на її поверхні. Методика. Функція розподілу мас надр планети подається за допомогою розкладу в біортогональні ряди, коефіцієнти розкладу якого визначаються з системи лінійних рівнянь. Система рівнянь отримується з умови мінімізації функції відхилення шуканого розподілу мас від початково визначеного двовимірного розподілу густини (референцна модель PREM). Результати. На основі описаного алгоритму отримана тримірна модель густин розподілу мас надр в середині Землі, що враховує стоксові постійні до восьмого порядку включно та відповідає поверхневому розподілу мас океанічної моделі Землі, а також подано її стислу інтерпретацію.

*Ключові слова:* функція розподілу, Земля, стоксові постійні, функція Лагранжа.

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