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ALGEBRAIC-DIFFERENTIAL EQUATIONS OF A NONLINEAR PASS-THROUGH QUADRIPOLE

Vasil Tchaban, Taras Ryzhyi

Lviv Polytechnic National University, Ukraine

vasyl.y.chaban@lpnu.ua, taras-oleh.i.ryzhyi@lpnu.ua

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Abstract. A method of forming algebraic-differential equations of a nonlinear pass-through active quadripole, which connect its independent pole currents and independent polar voltages, is proposed. The difficulty of the analysis lies in the fact that some of both internal and external unknowns may be under the symbol of differentiation. The common differential equations of the system of internal and external currents and voltages act as starting information for this formation. The method is demonstrated on two cases of the formation of corresponding algebraic-differential equations of systems as formed by nonlinear two-port elements. The analysis is significantly simplified in the case of internal D-degeneracies of the system or purely resistive circuits.

Keywords: nonlinear algebraic-differential equations, active quadripole, independent pole currents and polar voltages.

1. Introduction

A nonlinear pass-through quadripole can have any complex and unpredictable structure, extremely complex physical processes can take place in it, so establishing its independent equations sometimes requires extraordinary talent and skill of the researcher . It is not always possible to describe a physical process with sufficient accuracy, so you often have to limit yourself to approximate methods or use experimental dependencies taken on a real device. Sometimes, to describe an element in-depth knowledge of the physical process from related fields (physics, mechanics, heat engineering, chemistry, etc.) is necessary. It is almost impossible to describe all unforeseen situations. But further on, we will consider a fairly universal approach based on the description of the element as a calculation circuit built from two-port elements.

Any pass-through quadripole contains two channels (input-output). It is characterized relative to the outer part of the circuit by a system of two equations expressing the relationships between its independent voltages and currents determined by the nature of the element itself [1]. These equations do not depend on the nature of the outer circle. They must be satisfied both in the idle state of the element (when it is not included in the circuit at all) and in the short-circuit state (when all its poles are shortened to each other).

No restrictions are imposed on the form of the equations of functional dependencies between independent currents and voltages of such a quadripole. But the most common of them are the dependence of voltages on currents (Z-equation), the dependence of currents on voltages (Y-equation), the channel-by-channel dependence of input on output, or output on input (A-equation). According to the markings of the quadripole equivalent circuit shown in Fig. 1, the mentioned equations can be written in matrix form as:

$$\mathbf{u} = \mathbf{Z}\mathbf{i} + \mathbf{E}\,; \tag{1}$$

$$\mathbf{i} = \mathbf{Y}\mathbf{u} + \mathbf{J}\,; \tag{2}$$

$$\mathbf{x} = \mathbf{A}\mathbf{y} + \mathbf{B}\,, \tag{3}$$

where **Z, Y, A** – matrices of coefficients; **E, J, B** – columns of active elements;

$$\mathbf{u} = \begin{vmatrix} u_1 \\ u_2 \end{vmatrix};\quad \mathbf{i} = \begin{vmatrix} i_1 \\ i_2 \end{vmatrix};\quad \mathbf{x} = \begin{vmatrix} u_1 \\ i_1 \end{vmatrix};\quad \mathbf{y} = \begin{vmatrix} u_2 \\ i_2 \end{vmatrix}. \tag{4}$$

The transition from one form of writing quadripole equations to another is carried out according to the general rules of theoretical electrical engineering, and more preferable rules of matrix calculation.

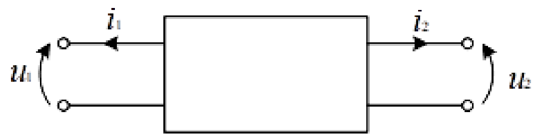

Fig. 1. Scheme of independent pole currents and polar voltages of a pass-through nonlinear active quadripole.

In our computerized time, it is perhaps not necessary to talk about the equations of a quadripole as themselves, but when forming the equations of complex electrical circuits composed of such elements, they acquire fundamental importance. This is because the matrices of coefficients (1)–(3) in this case take on the role of submatrices of matrices of equations of a branched external circuit in accordance with one or another chosen method of analysis: node voltages, loop currents, variable states, etc. [2-5]!

This important point may be somewhat lost in the field of implicit methods of integrating the equations of circuit state, but in the field of explicit methods it is relevant. According to this, we focus on the columns of active elements (1)−(3), as those that are formed not only by physical sources of voltages and currents, but also pseudophysical ones as a result of time discretization of the differential equations of the physical reactive elements of the circuit, for example, according to the simplest difference scheme

$$\frac{du_C}{dt} = \frac{e_C(t+\Delta t) - u_C(t)}{\Delta t};$$
$$\frac{di_L}{dt} = \frac{j_L(t+\Delta t) - i_L(t)}{\Delta t}, \qquad (5)$$

where e_C, j_L are the reactive sources of capacitor voltages and inductor currents as components together with corresponding active sources of columns of active elements (1)-(3); Δt is the time step of discretization.

Since there is a time difference between the reactive sources and the corresponding physical voltages and currents Δt, then in the process of forming the quadripole equations at a given time step, the reactive sources will be considered given next to the real physical sources present in the structure of the device.

We emphasize that this joint work is a direct continuation of the theoretical development [6] published earlier on the pages of this journal.

2. Formation of the equations of a nonlinear pass-through active quadripole.

1. On the basis of the calculation diagram of the quadripole electric circuit, we build a equivalent generator circuit, which is obtained by replacing the capacitors with their reactive voltage sources, and the inductance coils with similar current sources.

2. We write down the initial state equations of the equivalent generator circuit as structural equations

$$\sum_k u_k, e_k = 0; \quad \sum_k i_k, j_k = 0; \qquad (6)$$

and equations of internal bipolar resistors

$$u_{Rk} = R_k i_k, \qquad (7)$$

where $R_k = R_k\left(i_{Rk}\right)$ are the static resistances of resistors.

3. To determine the columns of active elements **E, J, B** according to (5), we form autonomous differential equations of the reactive elements of the quadripole

$$\frac{di_{Lk}}{dt} = \frac{u_{Lk}}{L_k}; \quad \frac{du_{Ck}}{dt} = \frac{i_{Ck}}{C_k}, \qquad (8)$$

where $L_k = L_k(i_{Lk}), C_k = C_k(u_{Ck})$ are the differential inductances of coils and capacitances of capacitors.

4. From systems (6), (7) we exclude all unknown currents and voltages, with the exception of elements of columns of independent polar voltages **u** and pole currents **i**, and at the same time we give the resulting equations one or another form (1)−(3).

Currents **i** and voltages **u** refer to the external characteristics of the element, they represent the physical state of the element in the external circuit. The remaining currents and voltages are internal and, if possible, are not considered (cases of linear subelements, or selected entire linear parts). If we are dealing with nonlinear subelements, then the mentioned equations, although in an autonomous form, are subject to a compatible solution with the basic equations of the device (1)−(3).

In the case of a resistive nonlinear circuit, as already mentioned, such a problem of excluding internal electromagnetic quantities does not exist. It is also simplified in D-degenerate circuits, where the derivatives of all unknowns in the initial equations of the multipole are present. In this case, it is enough to make a simple substitution in (6), (7). $x_i \to dx_i / dt, i = 1, 2$ and further analysis becomes obvious [6]. Based on this, the first problem to be solved was the formation of nonlinear differential equations of multipole elements with totally - *D*- degenerate internal electromagnetic circuits [4], which are electric machines and transformers of electric machine and electric power systems [5].

Below we will consider two typical examples of the formation of such equations of a nonlinear pass-through active quadripole.

Example 1. Fig. 2 shows the calculation diagram of the electric circuit of a nonlinear pass-through quadripole, and at the same time its equivalent generator circuit.

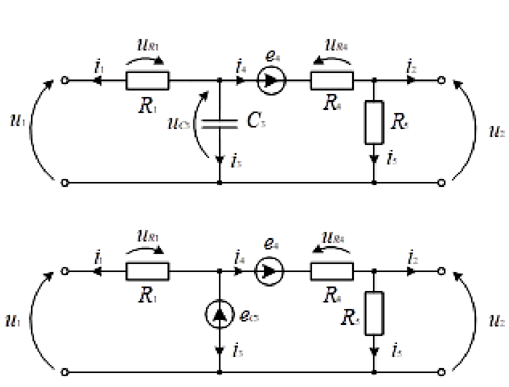

Fig. 2. Calculation diagram of the electric circuit of a nonlinear pass-through quadripole (above), and at the same time its equivalent generator circuit (below). The case of the presence of a nonlinear capacitor.

Ininitial equations of state (6), (7) of the equivalent generator circuit of the quadripole are written as

$$\begin{aligned} & i_1 + i_3 + i_4 = 0; \quad i_2 - i_4 + i_5 = 0; \\ & e_{C3} - u_{R1} - u_1 = 0; \; u_2 - e_{C3} + u_{R4} - e_4 = 0;; \\ & u_{R1} = R_1 i_1; \quad u_{R4} = R_4 i_4; \quad u_{R5} = R_5 i_5. \end{aligned} \tag{9}$$

On the basis of (9), excluding internal unknown currents and voltages, we form matrix *Z*-equations (1) in expanded form

$$\begin{array}{|c|}\hline u_1 \\ \hline u_2 \\ \hline \end{array} = \begin{array}{|c|c|}\hline -R_1 & \\ \hline & -\dfrac{R_4 R_5}{R_4 + R_5} \\ \hline \end{array} \begin{array}{|c|}\hline i_1 \\ \hline i_2 \\ \hline \end{array} + \begin{array}{|c|}\hline e_{C3} \\ \hline \dfrac{R_5}{R_4 + R_5}(e_{C3} + e_4) \\ \hline \end{array}. \tag{10}$$

In the same way, if desired, the corresponding Y-equations (2) can be constructed, which will have an expanded form

$$\begin{array}{|c|}\hline i_1 \\ \hline i_2 \\ \hline \end{array} = \begin{array}{|c|c|}\hline -\dfrac{1}{R_1} & \\ \hline & -\left(\dfrac{1}{R_4} + \dfrac{1}{R_5}\right) \\ \hline \end{array} \begin{array}{|c|}\hline u_1 \\ \hline u_2 \\ \hline \end{array} + \begin{array}{|c|}\hline \dfrac{e_{C3}}{R_1} \\ \hline \dfrac{e_{C3} + e_4}{R_4} \\ \hline \end{array}. \tag{11}$$

Equation (11) could be obtained formally from (10). For this, it was enough to multiply the left and right parts of equation (10) on the left by the inverse matrix $\mathbf{Z}^{-1} = \mathbf{Y}$.

To determine the columns of active elements in the right part of (10) and (11) according to (8), (9), we form the autonomous differential equation of the nonlinear capacitor of the quadripole

$$\frac{de_{C3}}{dt} = \frac{i_{C3}}{C_3} = \frac{-i_1 - i_2 - \dfrac{u_2}{R_5}}{C_3}, \tag{12}$$

which at each time step of the integration is subject to a solution compatible with (10) or (11).

If necessary, based on (9), or (10) or (11), A-equation (3) can be obtained. But we will do this in the next example, where other surprises await us.

Example 2. Fig. 3 shows another calculation scheme of the electric circuit of a nonlinear pass-through quadripole, as well as its equivalent generator circuit.

Initial equations of state (6), (7) of the equivalent generator circuit of this quadripole are written as

$$\begin{aligned} & i_1 + i_2 + j_{L3} = 0; \quad u_1 + u_{R1} - u_{L3} = 0; \\ & e_2 - u_2 + u_{L3} = 0; \; u_{R1} = R_1 i_1. \end{aligned} \tag{13}$$

It can be verified that on the basis of (13) due to the combination of ideal sources in the second circuit of the equivalent generator circuit of the device, it is not possible to construct Z-equation (1) of the device. Therefore, we will consider only Y-equation (2) and A-equation (3).

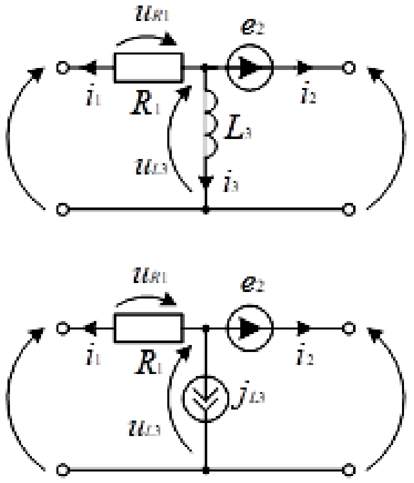

Fig. 3. Calculation diagram of the electric circuit of a nonlinear pass-through quadripole (above), and at the same time its equivalent generator circuit (below). The case of the presence of a nonlinear inductance coil.

On the basis of (13), excluding internal unknown currents and voltages, we form matrix *Y*-equations (2) in the form

$$\begin{array}{|c|}\hline i_1 \\ \hline i_2 \\ \hline \end{array} = \begin{array}{|c|c|}\hline -\dfrac{1}{R_1} & \dfrac{1}{R_1} \\ \hline \dfrac{1}{R_1} & -\dfrac{1}{R_1} \\ \hline \end{array} \begin{array}{|c|}\hline u_1 \\ \hline u_2 \\ \hline \end{array} + \begin{array}{|c|}\hline -\dfrac{e_2}{R_1} \\ \hline \dfrac{e_2}{R_1} - j_{L3} \\ \hline \end{array}. \tag{14}$$

Note that the Y-parameter matrix in (14) is special because its determinant is $\Delta = 0$. So, the matrix $\mathbf{Z} = \mathbf{Y}^{-1}$ does not exist, which confirms the above that Z-equation (1) of the given quadripole cannot be formed. Here it is necessary to refer to the rules of analysis of degenerate electric circuits [4], or to endow ideal energy sources with dissipative properties - to replace them with real ones.

At the same time, on the basis of (13), one can easily construct A-equation (3)

$$\begin{array}{|c|}\hline u_1 \\ \hline i_1 \\ \hline \end{array} = \begin{array}{|c|c|}\hline 1 & R_1 \\ \hline & -1 \\ \hline \end{array} \begin{array}{|c|}\hline u_2 \\ \hline i_2 \\ \hline \end{array} + \begin{array}{|c|}\hline R_1 j_{L3} - e_2 \\ \hline -j_{L3} \\ \hline \end{array}. \tag{15}$$

To determine the columns of active elements in the right-hand part of (15) according to (13), we form an autonomous differential equation of the nonlinear inductance coil

$$\frac{dj_{L3}}{dt} = \frac{u_{L3}}{L_3} = \frac{u_2 - e_2}{L_3}, \qquad (16)$$

which at each time step of the integration is subject to a solution compatible with (15).

Since the determinant of the matrix of coefficients (15) $\Delta = -1$ and $\mathbf{A}^{-1} = \mathbf{A}$, then it is not difficult to build the corresponding inverse equations

$$\begin{bmatrix} u_2 \\ i_2 \end{bmatrix} = \begin{bmatrix} 1 & R_1 \\ & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ i_1 \end{bmatrix} + \begin{bmatrix} e_2 \\ -j_{L3} \end{bmatrix}, \qquad (17)$$

In the case of a resistive nonlinear circuit, as already mentioned, the problem of excluding internal electromagnetic quantities does not exist.

3. Conclusion

A method of forming nonlinear algebraic-differential equations of a quadripole is proposed, which connect its independent pole currents and independent polar voltages, based on the differential equations of a non-degenerate system of its external and internal voltages and currents.

4. References

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АЛҐЕБРО-ДИФЕРЕНЦІАЛЬНІ РІВНЯННЯ НЕЛІНІЙНОГО ПРОХІДНОГО ЧОТИРИПОЛЮСНИКА

Чабан Василь, Рижий Тарас

Запропоновано метод формування алгебро-диференціальних рівнянь нелінійного прохідного активного чотириполюсника, які пов’язують між собою його незалежні полюсні струми і незалежні полярні напруги. Трудність аналізу полягає в тому, що частина як внутрішніх, так і зовнішніх невідомих можуть перебувати під символом диференціювання. Стартовою інформацією для даного формування виступають спільні диференціальні рівняння системи внутрішніх і зовнішніх струмів і напруг. Метод продемонстровано на двох випадках формування відповідних алгебро-диференціальних рівнянь систем як таких, що утворені нелінійними двополюсними елементами. Аналіз суттєво спрощується у випадку наявности внутрішніх D-вироджень системи або чисто резистивних кіл.

Vasil Tchaban: D.Sc, full professor at Lviv Polytechnic National University (Ukraine), as well as Lviv Agrarian National University, Editor-in-Chief of Technical News journal. His research interests are in the areas of mathematical modelling of electromechanical processses in electric and gravity fields theories, surrealistic short story writer. He is the authorof 550 scientific publications and 800 surrealistic short stories including 52 books (13 monographs, 17 didactic, 12 humanistic and 10 of the arts), 1500 aphorisms. The total number of publications is over 1500.

Taras Ryzyi - senior lecturer, graduated from Electric Power Engineering Faculty of Lviv Polytechnic Institute, Ukraine. Since 1983 he has been working at the Department of Theoretical Electrical Engineering at Lviv Polytechnic National University.

His research interests are connected with modeling of power systems and problems of didactics in electrical engineering.

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