

Regularity for entropy solutions of degenerate parabolic equations with L^m data

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In this paper, we study the regularity results for entropy solutions of a class of parabolic nonlinear parabolic equations with degenerate coercivity, when the right-hand side is in L^m with $m > 1$.

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1. Introduction and preliminary results

1.1. Introduction

This paper will deal with the following problem

$$\begin{cases} \frac{\partial u}{\partial t} + Au = f, & \text{in } Q, \\ u = 0, & \text{on } \Gamma = \partial\Omega \times (0, T), \\ u(x, 0) = 0, & \text{in } \Omega, \end{cases} \quad (1)$$

where

$$Au = -\operatorname{div}(a(x, t, u)\widehat{a}(x, t, u, \nabla u)),$$

$f \in L^m(Q)$, $m \geq 1$, Ω is an open bounded subset of \mathbb{R}^N ($N \geq 2$), Q is the cylinder $\Omega \times (0, T)$ ($T > 0$), Γ the lateral surface $\partial\Omega \times (0, T)$.

Let $a: Q \times \mathbb{R} \rightarrow \mathbb{R}$ be a Carathéodory function satisfying for almost every $(x, t) \in Q$ and every $s \in \mathbb{R}$

$$\frac{\alpha}{(1 + |s|)^\theta} \leq a(x, t, s) \leq \beta, \quad (2)$$

and

$$0 \leq \theta < p - 1 + \frac{p}{N}, \quad (3)$$

where p is a real number such that $2 < p < N$, and α, β are two positive constants.

We assume that $\widehat{a}: \Omega \times]0, T[\times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}^N$ is a Carathéodory function, satisfying for a.e. $(x, t, s) \in Q \times \mathbb{R}$, $\forall \xi, \xi' \in \mathbb{R}^N$:

$$\widehat{a}(x, t, s, \xi) \cdot \xi \geq |\xi|^p, \quad (4)$$

$$|\widehat{a}(x, t, s, \xi)| \leq b(x, t) + |s|^{p-1} + |\xi|^{p-1}, \quad (5)$$

$$(\widehat{a}(x, t, s, \xi) - \widehat{a}(x, t, s, \xi')) \cdot (\xi - \xi') > 0, \quad (6)$$

b is a nonnegative function in $L^{p'}(Q)$, where $p' = \frac{p}{p-1}$.

When the degenerate term does not appear in (1) (i.e., $a(x, t, u) \equiv 1$) and $u(x, 0) = u_0 \in L^1(\Omega)$, the existence and regularity of entropy solution of (1) are proved in [1]. The uniqueness results has been developed in [2]. If $\widehat{a}(x, t, u, \nabla u) = |\nabla u|^{p-2} \nabla u$, in [3], existence and regularity results for the

problem (1) were proved. The existence and uniqueness of a renormalised solution of problem (1) proved in [4]. In the case $\theta \neq 0$, $p = 2$, $0 \leq \theta < 1 + \frac{2}{N}$ and $f \in L^1(Q)$, the existence and regularity of entropy solutions studied in [5]. In [6] the authors prove the following result

Theorem 1. *Under the hypotheses (2)–(6), if $f \in L^m(Q)$ with $m > \frac{N}{p} + 1$, then there exists a bounded weak solution $u \in L^p(0, T; W_0^{1,p}(\Omega)) \cap L^\infty(Q)$ to problem (1).*

1.2. Preliminary results

Let $k > 0$ and $T_k: \mathbb{R} \rightarrow \mathbb{R}$ the truncating function equal to $T_k(s) := \text{sgn}(s) \min\{|s|, k\}$, and its primitive $S_k: \mathbb{R} \rightarrow \mathbb{R}^+$

$$S_k(x) = \int_0^x T_k(s) ds. \quad (7)$$

It results

$$\frac{1}{2}|T_k(s)|^2 \leq S_k(s) \leq k|s|, \quad \forall k > 0, \quad \forall s \in \mathbb{R}. \quad (8)$$

We use the following definition of the entropy solutions.

Definition 1. *A measurable function $u \in L^\infty(0, T; L^1(\Omega))$ will be called an entropy solution to problem (1) if $T_k(u) \in L^p(0, T; W_0^{1,p}(\Omega))$, for every $k > 0$, and if*

$$\int_\Omega S_k(u(t) - \phi(t)) dx \in C([0, T]), \quad (9)$$

$$\begin{aligned} \int_\Omega S_k(u(T) - \phi(T)) dx - \int_\Omega S_k(-\phi(0)) dx + \int_0^T \langle \phi_t, T_k(u - \phi) \rangle dt \\ + \int_Q a(x, t, u) \widehat{a}(x, t, u, \nabla u) \nabla T_k(u - \phi) dx dt \leq \int_Q f T_k(u - \phi) dx dt, \end{aligned} \quad (10)$$

for every $k > 0$ and $\phi \in L^p(0, T; W_0^{1,p}(\Omega)) \cap L^\infty(Q)$ such that

$$\phi_t \in L^p(0, T; W^{-1,p'}(\Omega)) + L^1(Q).$$

Lemma 1. *For every $k > 0$, if $T_k(u) \in L^p(0, T; W_0^{1,p}(\Omega))$, then there exists a unique measurable function $v: Q \mapsto \mathbb{R}^N$ such that $\nabla T_k(u) = v \chi_{\{|u| < k\}}$ a.e. in Q , where $\chi_{\{|u| < k\}}$ denotes the characteristic function over the set $\{|u| < k\}$. Defining the derivative ∇u of u as the unique function v which satisfies the above equality. Furthermore, $u \in L^p(0, T; W_0^{1,p}(\Omega))$ if and only if $v \in L^p(Q)$, and then $v \equiv \nabla u$ in the usual weak sense.*

Proof. The proof of Lemma 1 is the same as that of Lemma 2.2 in [7], we omit the details. ■

Definition 2 (Refs. [8, 9]). *Let q be a positive number. The Marcinkiewicz space $\mathcal{M}^q(Q)$ is the set of all measurable functions $u: Q \rightarrow \mathbb{R}$ such that*

$$\text{meas}(\{(x, t) \in Q: |u(x, t)| > k\}) \leq \frac{C}{k^q}, \quad \text{for every } k > 0, \quad (11)$$

for some constant $C > 0$. The norm of u in $\mathcal{M}^q(Q)$ is defined by

$$\|u\|_{\mathcal{M}^q(Q)}^q = \inf\{C > 0 \text{ such that (11) holds}\}.$$

The alternate name of weak L^q space is due to the fact that, if Q has finite measure, then

$$\begin{cases} \mathcal{M}^q(Q) \subset \mathcal{M}^\gamma(Q), \\ L^q(Q) \subset \mathcal{M}^q(Q) \subset \mathcal{M}^\gamma(Q), \end{cases}$$

for every $\gamma < q$.

We also recall a consequence of the Gagliardo–Nirenberg embedding theorem.

Lemma 2 (Ref. [13]). Let $v \in L^h(0, T; W_0^{1,h}(\Omega)) \cap L^\infty(0, T; L^\varrho(\Omega))$, $h, \varrho \geq 1$. Then v belongs to $L^q(Q)$, where $q = h \frac{N+\varrho}{N}$, and there exists a positive constant M_1 depending only on N, h, ϱ such that

$$\int_Q |v(x, t)|^q dx dt \leq M_1 \left(\operatorname{ess\,sup}_{0 < t < T} \int_\Omega |v(x, t)|^\varrho dx \right)^{\frac{h}{N}} \int_Q |Dv(x, t)|^h dx dt. \quad (12)$$

Before the proof, we need a technical lemma.

Lemma 3. Let u be a measurable function in $\mathcal{M}^\mu(Q)$ for some $\mu > 0$, and assume that there exist two nonnegative constants $\nu > \gamma$ such that

$$\int_Q |\nabla T_k(u)|^p dx dt \leq M_2(1+k)^\gamma k^{\nu-\gamma}, \quad \forall k > 0,$$

where M_2 is a positive constant independent of k . Then $|\nabla u|$ belongs to $\mathcal{M}^s(Q)$, with $s = \frac{p\mu}{\mu+\nu}$.

Proof. We follow the lines of the proof of [7], Lemma 4.1. and 4.2. Let λ be a fixed positive real number. We have, for every $k > 0$,

$$\begin{aligned} \operatorname{meas}(\{|\nabla u|^p > \lambda\}) &= \operatorname{meas}(\{|\nabla u|^p > \lambda, |u| \leq k\}) + \operatorname{meas}(\{|\nabla u|^p > \lambda, |u| > k\}) \\ &\leq \operatorname{meas}(\{|\nabla u|^p > \lambda, |u| \leq k\}) + \operatorname{meas}(\{|u| > k\}). \end{aligned} \quad (13)$$

Moreover,

$$\begin{aligned} \operatorname{meas}(\{|\nabla u|^p > \lambda, |u| \leq k\}) &= \frac{1}{\lambda} \int_{\{|\nabla u|^p > \lambda, |u| \leq k\}} \lambda dx dt \leq \frac{1}{\lambda} \int_{\{|u| \leq k\}} |\nabla u|^p dx dt \\ &\leq \frac{1}{\lambda} \int_Q |\nabla T_k(u)|^p dx dt \leq M \frac{(1+k)^\gamma k^{\nu-\gamma}}{\lambda}. \end{aligned}$$

If $k > 1$, then the above inequality turns into

$$\operatorname{meas}(\{|\nabla u|^p > \lambda, |u| \leq k\}) \leq M \frac{(2k)^\gamma k^{\nu-\gamma}}{\lambda} \leq 2^\gamma M \frac{k^\nu}{\lambda}.$$

By Definition 2 of the Marcinkiewicz space and $u \in \mathcal{M}^\mu(Q)$, then there exists a positive constant M_1 independent of k such that

$$\operatorname{meas}(\{|u| > k\}) \leq M_1 \frac{1}{k^\mu}. \quad (14)$$

Using (13)–(14), we obtain

$$\operatorname{meas}(\{|\nabla u|^p > \lambda\}) \leq 2^\gamma M \frac{k^\nu}{\lambda} + M_1 \frac{1}{k^\mu} \leq M_2 \left(\frac{k^\nu}{\lambda} + \frac{1}{k^\mu} \right), \quad (15)$$

where $M_2 = \max\{2^\gamma M, M_1\}$, and (15) holds for every $k > 1$. Minimizing with respect to k , we easily prove that as $k = \left(\frac{\mu}{\nu}\right)^{\frac{1}{\mu+\nu}} \lambda^{\frac{1}{\mu+\nu}}$, the minimum value of the right side term in (15) is achieved, and setting $\lambda = h^p$ for every $h > 0$ we get

$$\operatorname{meas}(\{|\nabla u| > h\}) \leq M_2 \min_k \left(\frac{k^\nu}{h^p} + \frac{1}{k^\mu} \right) \leq M_2 \left[\left(\frac{\mu}{\nu} \right)^{\frac{\nu}{\mu+\nu}} + \left(\frac{\mu}{\nu} \right)^{-\frac{\mu}{\mu+\nu}} \right] \frac{1}{h^{\frac{p\mu}{\mu+\nu}}} \leq \frac{M_3}{h^{\frac{p\mu}{\mu+\nu}}} \leq \frac{M_3}{h^s}, \quad (16)$$

where M_3 is a positive constant independent of h . However, the above conclusion is obtained under the assumption $k > 1$, that is $h > \left(\frac{\nu}{\mu}\right)^{\frac{1}{p}}$. If $h \leq \left(\frac{\nu}{\mu}\right)^{\frac{1}{p}}$, since Q is bounded, the above inequality obviously holds. By (16) and Definition 2 yield $|\nabla u| \in \mathcal{M}^s(Q)$. ■

2. Statement of main results and approximate solutions

2.1. Statement of main results

Theorem 2. Under the hypotheses (2)–(6), if $f \in L^m(Q)$ with

$$1 \leq m \leq \max \left\{ \frac{N + \theta + 2}{(p-1)N + p + 1 - \theta(N-1)}, 1 \right\}, \quad (17)$$

then there exists an entropy solution u to problem (1) in the sense of Definition 1 with

$$u \in \mathcal{M}^\delta(Q), \quad \text{and} \quad |\nabla u| \in \mathcal{M}^q(Q), \quad (18)$$

where $\mathcal{M}^\delta(Q)$, $\mathcal{M}^q(Q)$ are Marcinkiewicz spaces defined in Definition 2, and

$$\delta = \frac{m(p + N(p-1-\theta))}{N + p - pm}, \quad \text{and} \quad q = \frac{m[N(p-1-\theta) + p]}{N + 1 - (\theta + 1)(m-1)}. \quad (19)$$

Remark 1. If $0 \leq \theta < p-1 + \frac{p}{N} - \frac{N+1}{N}$, then (17) becomes $m = 1$, thus $\delta = \frac{p+N(p-1-\theta)}{N} > 1$, $q = \frac{N(p-1-\theta)+p}{N+1} > 1$. By the embedding theorems between Marcinkiewicz and Lebesgue spaces, we can deduce that u belongs to $L^s(0, T; W_0^{1,s}(\Omega))$ for every $1 \leq s < q = \frac{N(p-1-\theta)+p}{N+1}$.

Remark 2. If $p-1 + \frac{p}{N} - \frac{N+1}{N} \leq \theta < p-1 + \frac{p}{N}$, then (17) becomes $1 \leq m \leq \frac{N+\theta+2}{(p-1)N+p+1-\theta(N-1)}$ and $q \leq 1$. It is not possible to deduce that $|\nabla u|$ belongs to some Sobolev space even if $1 < m \leq \frac{N+\theta+2}{(p-1)N+p+1-\theta(N-1)}$.

2.2. Approximate solutions

In the remainder of this section, we denote by c various positive constants depending only on the data of the problem, but not on n and k .

Let (f_n) be a sequence of bounded functions defined in Q , where $f_n \in \mathcal{D}(Q)$ and satisfy

$$\|f_n\|_{L^m(Q)} \leq \|f\|_{L^m(Q)} \leq c, \quad \forall n, \quad (20)$$

$$f_n \rightarrow f, \quad \text{strongly in } L^m(Q). \quad (21)$$

We approximate the problem (1) by the following problems

$$\begin{cases} \frac{\partial u_n}{\partial t} - \operatorname{div}(a(x, t, T_n(u_n))\widehat{a}(x, t, u_n, \nabla u_n)) = f_n, & \text{in } Q, \\ u_n = 0, & \text{on } \Gamma, \\ u_n(x, 0) = 0, & \text{in } \Omega. \end{cases} \quad (22)$$

For $n \in \mathbb{N}$, we define the operator A_n by $A_n = -\operatorname{div}(a(\cdot, \cdot, T_n(u))\widehat{a}(\cdot, \cdot, u, \nabla u))$.

From (2) and (4), we have

$$\int_Q a(x, t, T_n(u))\widehat{a}(x, t, u, \nabla u) \cdot \nabla u \, dx \, dt \geq g(n) \int_Q |\nabla u|^p \, dx \, dt, \quad \text{with} \quad g(n) = \frac{1}{(1 + |n|)^\theta},$$

so that the operator A_n from $L^p(0, T; W_0^{1,p}(\Omega))$ into its dual $L^{p'}(0, T; W^{-1,p'}(\Omega))$ is coercive and satisfies the classical Leary–Lions conditions. Then from the well-known result of [10], there exists at least a solution u_n in $C([0, T]; L^2(\Omega))$ to problem (22) such that $u'_n \in L^{p'}(0, T; W^{-1,p'}(\Omega))$ and satisfies

$$\int_Q u'_n \phi \, dx \, dt + \int_Q a(x, t, T_n(u_n))\widehat{a}(x, t, u_n, \nabla u_n) \nabla \phi \, dx \, dt = \int_Q f_n \phi \, dx \, dt,$$

for any $\phi \in L^p(0, T; W_0^{1,p}(\Omega))$ and $u_n(x, 0) = 0$.

3. A priori estimates

Throughout this section we assume that hypotheses (20)–(21) hold. Let u_n be a solution of problem (22).

In this section, we prove some a priori estimates for the approximate solutions u_n and its partial derivatives.

Lemma 4. *Let $f \in L^m(Q)$, with m satisfies (17), and (2)–(6) hold. Then there exists a positive constant c such that*

$$\text{meas}(\{|u_n| > k\}) \leq \frac{c}{k^\delta}, \quad (23)$$

$$\text{meas}(\{|\nabla u_n| > k\}) \leq \frac{c}{k^q}, \quad (24)$$

$$\|u_n\|_{L^\infty(0,T;L^1(\Omega))} \leq c, \quad \text{and} \quad (25)$$

$$\|T_k(u_n)\|_{L^p(0,T;W_0^{1,p}(\Omega))} \leq c(1+k)^{\frac{1+\theta}{p}}, \quad (26)$$

where δ and q as in (19).

Proof. The proof is divided into three cases.

Case 1. Suppose that $m > \frac{p(N+2)}{(p-1)N+2p}$. Choosing $T_k(u_n(x, t))\chi_{(0,\tau)}(t)$ a test function for problem (22), using (7), (2), (4) and Hölder's inequality, we get

$$\int_{\Omega} S_k(u_n(x, \tau)) dx + \alpha \int_0^\tau \int_{\Omega} \frac{|\nabla u_n|^p}{(1+|u_n|)^\theta} dx dt \leq \|f_n\|_{L^m(Q)} \left(\int_0^\tau \int_{\Omega} |T_k(u_n)|^{m'} dx dt \right)^{\frac{1}{m'}}. \quad (27)$$

By (8) and (27), we have

$$\text{ess sup}_{0 \leq t \leq T} \int_{\Omega} |T_k(u_n(x, t))|^2 dx + 2\alpha \int_0^\tau \int_{\Omega} \frac{|\nabla u_n|^p}{(1+|u_n|)^\theta} dx dt \leq 2\|f\|_{L^m(Q)} \left(\int_0^\tau \int_{\Omega} |T_k(u_n)|^{m'} dx dt \right)^{\frac{1}{m'}}. \quad (28)$$

Moreover

$$\begin{aligned} \int_Q |\nabla T_k(u_n)|^p dx dt &= \int_Q \frac{|\nabla T_k(u_n)|^p}{(1+|T_k(u_n)|)^\theta} (1+|T_k(u_n)|)^\theta dx dt \\ &\leq \frac{\|f\|_{L^m(Q)}}{\alpha} (1+k)^\theta \left(\int_Q |T_k(u_n)|^{m'} dx dt \right)^{\frac{1}{m'}}. \end{aligned} \quad (29)$$

If $m > \frac{p(N+2)}{(p-1)N+2p}$, we have $m' < \frac{p(N+2)}{N}$, thus we can choose $\rho < p$ such that $\frac{\rho(N+2)}{N} = m'$. Then

$$\rho = \frac{Nm}{(N+2)(m-1)}. \quad (30)$$

For the above ρ , (28) and Hölder's inequality imply that

$$\begin{aligned} \int_Q |\nabla T_k(u_n)|^\rho dx dt &= \int_Q \frac{|\nabla T_k(u_n)|^\rho}{(1+|T_k(u_n)|)^\frac{\theta\rho}{p}} (1+|T_k(u_n)|)^\frac{\theta\rho}{p} dx dt \\ &\leq \left(\int_Q \frac{|\nabla T_k(u_n)|^p}{(1+|T_k(u_n)|)^\theta} dx dt \right)^\frac{\rho}{p} \left(\int_Q (1+|T_k(u_n)|)^\frac{\theta\rho}{p-\rho} dx dt \right)^\frac{p-\rho}{p} \\ &\leq c \left(\int_Q |T_k(u_n)|^{m'} dx dt \right)^\frac{\rho}{pm'} \left(\int_Q (1+|T_k(u_n)|)^\frac{\theta\rho}{p-\rho} dx dt \right)^\frac{p-\rho}{p}. \end{aligned} \quad (31)$$

By Lemma 2, applied to $v(x, t) = T_k(u_n(x, t))$, $\varrho = 2$, and $h = \rho$, using (28), (31), we obtain

$$\begin{aligned}
\int_Q |T_k(u_n)|^{\frac{(N+2)\rho}{N}} dx dt &\leq \left(\operatorname{ess\,sup}_{0 \leq t \leq T} \int_{\Omega} |T_k(u_n)|^2 dx \right)^{\frac{p}{N}} \int_Q |DT_k(u_n)|^\rho dx dt \\
&\leq c \left(\int_Q |T_k(u_n)|^{m'} dx dt \right)^{\frac{\rho}{Nm'}} \left(\int_Q |T_k(u_n)|^{m'} dx dt \right)^{\frac{\rho}{pm'}} \\
&\quad \times \left(\int_Q (1 + |T_k(u_n)|)^{\frac{\theta\rho}{p-\rho}} dx dt \right)^{\frac{p-\rho}{p}} \\
&\leq c \left(\int_Q |T_k(u_n)|^{m'} dx dt \right)^{\frac{\rho(N+p)}{pNm'}} \left(\int_Q (1 + |T_k(u_n)|)^{\frac{\theta\rho}{p-\rho}} dx dt \right)^{\frac{p-\rho}{p}}. \quad (32)
\end{aligned}$$

Now $m > \frac{p(N+2)}{(p-1)N+2p}$ and (17) imply

$$m \leq \frac{N + \theta + 2}{(p-1)N + p + 1 - \theta(N-1)}. \quad (33)$$

However, by virtue of $\theta < p-1 + \frac{p}{N}$, then

$$\frac{N + \theta + 2}{(p-1)N + p + 1 - \theta(N-1)} < \frac{p(N+2) - N\theta}{(p-1)N + 2p - N\theta}. \quad (34)$$

Thus from (30), (33) and (34), we can deduce that $\frac{\theta\rho}{p-\rho} > m'$, if $k \geq 1$, (32) yields

$$\begin{aligned}
\int_Q |T_k(u_n)|^{\frac{(N+2)\rho}{N}} dx dt &= \int_Q |T_k(u_n)|^{m'} dx dt \\
&\leq c \left(\int_Q |T_k(u_n)|^{m'} dx dt \right)^{\frac{\rho(N+p)}{pNm'}} \left(\int_Q (1 + |T_k(u_n)|)^{\frac{\theta\rho}{p-\rho} - m'} (1 + |T_k(u_n)|^{m'}) dx dt \right)^{\frac{p-\rho}{p}} \\
&\leq c \left(\int_Q |T_k(u_n)|^{m'} dx dt \right)^{\frac{\rho(N+p)}{pNm'}} (2k)^{\left(\frac{\theta\rho}{p-\rho} - m'\right)\frac{p-\rho}{p}} \left(\int_Q (1 + |T_k(u_n)|)^{m'} dx dt \right)^{\frac{p-\rho}{p}} \\
&\leq c \left(\int_Q |T_k(u_n)|^{m'} dx dt \right)^{\frac{\rho(N+p)}{pNm'}} (2k)^{\left(\frac{\theta\rho}{p-\rho} - m'\right)\frac{p-\rho}{p}} \left(2^{m'} |Q| + 2^{m'} \int_Q |T_k(u_n)|^{m'} dx dt \right)^{\frac{p-\rho}{p}} \\
&\leq ck^{\frac{\theta\rho}{p} - \frac{(p-\rho)m'}{p}} \left(\int_Q |T_k(u_n)|^{m'} dx dt \right)^{\frac{\rho(N+p)}{pNm'}} \left(1 + \int_Q |T_k(u_n)|^{m'} dx dt \right)^{\frac{p-\rho}{p}}. \quad (35)
\end{aligned}$$

If $\int_Q |T_k(u_n)|^{m'} dx dt \geq 1$, it follows from (35) that

$$\int_Q |T_k(u_n)|^{m'} dx dt \leq c2^{\frac{p-\rho}{p}} k^{\frac{\theta\rho}{p} - \frac{(p-\rho)m'}{p}} \left(\int_Q |T_k(u_n)|^{m'} dx dt \right)^{\frac{\rho(N+p)}{pNm'} + \frac{p-\rho}{p}}.$$

Hence

$$\left(\int_Q |T_k(u_n)|^{m'} dx dt \right)^{1 - \frac{\rho(N+p)}{pNm'} - \frac{p-\rho}{p}} \leq c2^{\frac{p-\rho}{p}} k^{\frac{\theta\rho}{p} - \frac{(p-\rho)m'}{p}}.$$

Thus we get

$$\int_Q |T_k(u_n)|^{m'} dx dt \leq ck^{\left[\frac{\theta\rho}{p} - \frac{(p-\rho)m'}{p}\right] \frac{1}{1 - \frac{\rho(N+p)}{pNm'} - \frac{p-\rho}{p}}}. \quad (36)$$

From (30) we obtain

$$\left(\frac{\theta\rho}{p} - \frac{(p-\rho)m'}{p} \right) \frac{1}{1 - \frac{\rho(N+p)}{pNm'} - \frac{p-\rho}{p}} = -m \frac{N((p-1)m-p) + 2p(m-1) - \theta N(m-1)}{(m-1)(N-pm+p)}. \quad (37)$$

It follows from (36)–(37) that

$$\int_Q |T_k(u_n)|^{m'} dx dt \leq ck^{-m \frac{N((p-1)m-p)+2p(m-1)-\theta N(m-1)}{(m-1)(N-pm+p)}}. \quad (38)$$

New $\theta < p - 1 + \frac{p}{N}$, (33) and (34) imply

$$\begin{cases} N - pm + p > 0, & \text{and} \\ N((p-1)m-p) + 2p(m-1) - \theta N(m-1) < 0. \end{cases} \quad (39)$$

Combining (37) and (39), we obtain

$$-m \frac{N((p-1)m-p) + 2p(m-1) - \theta N(m-1)}{(m-1)(N-pm+p)} > 0.$$

If $\int_Q |T_k(u_n)|^{m'} dx dt \leq 1$, by virtue of $k \geq 1$, then

$$\int_Q |T_k(u_n)|^{m'} dx dt \leq 1 \leq k^{-m \frac{N((p-1)m-p)+2p(m-1)-\theta N(m-1)}{(m-1)(N-pm+p)}}. \quad (40)$$

By (38) and (40) we get for any $k \geq 1$,

$$\int_Q |T_k(u_n)|^{m'} dx dt \leq ck^{-m \frac{N((p-1)m-p)+2p(m-1)-\theta N(m-1)}{(m-1)(N-pm+p)}}. \quad (41)$$

The condition $m > 1$ ensures that

$$m' > -m \frac{N((p-1)m-p) + 2p(m-1) - \theta N(m-1)}{(m-1)(N-pm+p)}. \quad (42)$$

If $k \leq 1$, using (42), we have

$$\int_Q |T_k(u_n)|^{m'} dx dt \leq |Q|k^{m'} \leq |Q|k^{-m \frac{N((p-1)m-p)+2p(m-1)-\theta N(m-1)}{(m-1)(N-pm+p)}}.$$

It follows from (41) and (43) that for any $k > 0$,

$$\int_Q |T_k(u_n)|^{m'} dx dt \leq ck^{-m \frac{N((p-1)m-p)+2p(m-1)-\theta N(m-1)}{(m-1)(N-pm+p)}}. \quad (43)$$

Therefore we have

$$k^{m'} \text{meas}\{(x, t) \in Q : |u_n(x, t)| > k\} \leq ck^{-m \frac{N((p-1)m-p)+2p(m-1)-\theta N(m-1)}{(m-1)(N-pm+p)}}.$$

Namely,

$$\text{meas}\{(x, t) \in Q : |u_n(x, t)| > k\} \leq ck^{-m \frac{N((p-1)m-p)+2p(m-1)-\theta N(m-1)}{(m-1)(N-pm+p)} - m'} \leq ck^{-\frac{m(p+N(p-1-\theta))}{N+p-pm}} \leq ck^{-\delta}.$$

Thus (23) is proved.

Now, (29) and (43) yield

$$\begin{aligned} \int_Q |DT_k(u_n)|^p dx dt &\leq c(1+k)^\theta \left(\int_Q |T_k(u_n)|^{m'} dx dt \right)^{\frac{1}{m'}} \\ &\leq c(1+k)^\theta k^{-\frac{N((p-1)m-p)+2p(m-1)-\theta N(m-1)}{N-pm+p}}. \end{aligned}$$

Thus, by the Lemma 3, applied to $v(x, t) = u(x, t)$, $\mu = \delta$, $\gamma = \theta$, $s = q$ and $\nu = \frac{-(N(p-1)m-Np+2p(m-1))+\theta(Nm-pm+p)}{N-pm+p}$, we can obtain (24).

Case 2. Suppose that $1 < m \leq \frac{p(N+2)}{(p-1)N+2p}$. Note that $m' \geq \frac{p(N+2)}{N}$. Then we have

$$\begin{aligned} \left(\int_Q |T_k(u_n)|^{m'} dx dt \right)^{\frac{1}{m'}} &\leq \left(\int_Q |T_k(u_n)|^{\frac{p(N+2)}{N}} |T_k(u_n)|^{m' - \frac{p(N+2)}{N}} dx dt \right)^{\frac{1}{m'}} \\ &\leq k^{1 - \frac{p(N+2)}{m'N}} \left(\int_Q |T_k(u_n)|^{\frac{p(N+2)}{N}} dx dt \right)^{\frac{1}{m'}}. \end{aligned} \quad (44)$$

From (28)–(29) and (44), we have

$$\operatorname{ess\,sup}_{0 \leq t \leq T} \int_{\Omega} |T_k(u_n(x, t))|^2 dx \leq ck^{1 - \frac{p(N+2)}{m'N}} \left(\int_Q |T_k(u_n)|^{\frac{p(N+2)}{N}} dx dt \right)^{\frac{1}{m'}}, \quad (45)$$

and

$$\int_Q |DT_k(u_n)|^p dx dt \leq c(1+k)^{\theta} k^{1 - \frac{p(N+2)}{m'N}} \left(\int_Q |T_k(u_n)|^{\frac{p(N+2)}{N}} dx dt \right)^{\frac{1}{m'}}. \quad (46)$$

Thus, by the Gagliardo–Nirenberg inequality (12) (Lemma 2), applied to $v(x, t) = T_k(u_n(x, t))$, $\varrho = 2$, and $h = p$, using (45)–(46), we have

$$\begin{aligned} \int_Q |T_k(u_n)|^{\frac{p(N+2)}{N}} dx dt &\leq \left(\operatorname{ess\,sup}_{0 \leq t \leq T} \int_{\Omega} |T_k(u_n(x, t))|^2 dx \right)^{\frac{p}{N}} \int_Q |DT_k(u_n)|^p dx dt \\ &\leq c(1+k)^{\theta} k^{(1 - \frac{p(N+2)}{Nm'}) (\frac{p}{N} + 1)} \left(\int_Q |T_k(u_n)|^{\frac{p(N+2)}{N}} dx dt \right)^{\frac{p+N}{Nm'}}. \end{aligned}$$

By virtue of $m \leq \frac{p(N+2)}{(p-1)N+2p}$, then $1 - \frac{p+N}{Nm'} > 0$. Thus we get

$$\left(\int_Q |T_k(u_n)|^{\frac{p(N+2)}{N}} dx dt \right)^{1 - \frac{p+N}{Nm'}} \leq c(1+k)^{\theta} k^{(1 - \frac{p(N+2)}{Nm'}) (\frac{p}{N} + 1)}.$$

Hence

$$\begin{aligned} \int_Q |T_k(u_n)|^{\frac{p(N+2)}{N}} dx dt &\leq c \left[(1+k)^{\theta} k^{(1 - \frac{p(N+2)}{Nm'}) (\frac{p}{N} + 1)} \right]^{\frac{1}{1 - \frac{p+N}{Nm'}}} \\ &\leq c(1+k)^{\frac{Nm\theta}{N-pm+p}} k^{\frac{(N+p)(Nm-p(N+2)(m-1))}{N(N-pm+p)}}. \end{aligned} \quad (47)$$

If $k \geq 1$, it follows from (47) that

$$\int_Q |T_k(u_n)|^{\frac{p(N+2)}{N}} dx dt \leq ck^{\frac{(N+p)(Nm-p(N+2)(m-1)) + \theta N^2 m}{N(N-pm+p)}}. \quad (48)$$

If $k \leq 1$. Now $\theta < p - 1 + \frac{p}{N}$, imply

$$\frac{p(N+2)}{N} > \frac{(N+p)(Nm-p(N+2)(m-1)) + \theta N^2 m}{N(N-pm+p)},$$

which implies

$$\int_Q |T_k(u_n)|^{\frac{p(N+2)}{N}} dx dt \leq |Q| k^{\frac{p(N+2)}{N}} \leq |Q| k^{\frac{(N+p)(Nm-p(N+2)(m-1)) + \theta N^2 m}{N(N-pm+p)}}. \quad (49)$$

It follows from (48)–(49) that for any $k > 0$,

$$\int_Q |T_k(u_n)|^{\frac{p(N+2)}{N}} dx dt \leq ck^{\frac{(N+p)(Nm-p(N+2)(m-1)) + \theta N^2 m}{N(N-pm+p)}}. \quad (50)$$

Therefore from (50) we can obtain (23). Finally, (24) can be deduced from (46), (50) and Lemma 3.

Case 3. Suppose that $m = 1$. We only need to replace $\left(\int_Q |T_k(u_n)|^{m'} dx dt\right)^{\frac{1}{m'}}$ with $|Q|^{\frac{1}{m'}} k$ in (27)–(29). That is

$$\int_{\Omega} S_k(u_n(x, \tau)) dx + \alpha \int_0^{\tau} \int_{\Omega} \frac{|\nabla u_n|^p}{(1 + |u_n|)^{\theta}} dx dt \leq \|f_n\|_{L^m(Q)} |Q|^{\frac{1}{m'}} k,$$

so,

$$\operatorname{ess\,sup}_{0 \leq t \leq T} \int_{\Omega} |T_k(u_n(x, t))|^2 dx + \alpha \int_Q \frac{|\nabla u_n|^p}{(1 + |u_n|)^{\theta}} dx dt \leq ck. \quad (51)$$

Therefore

$$\int_Q |\nabla T_k(u_n)|^p dx dt = \int_Q \frac{|\nabla T_k(u_n)|^p}{(1 + |T_k(u_n)|)^{\theta}} (1 + |T_k(u_n)|)^{\theta} dx dt \leq c(1 + k)^{\theta} k. \quad (52)$$

By (51)–(52) and Lemma 2 (here $v(x, t) = T_k(u_n(x, t))$, $h = p$, $\varrho = 2$), going through the same process as that of (51), we obtain

$$\int_Q |T_k(u_n)|^{\frac{p(N+2)}{N}} dx dt \leq ck^{\frac{N+p+\theta N}{N}}. \quad (53)$$

Thus it's easy to get (23) by (53). Now (52)–(53) and Lemma (3) imply that (24) holds.

Taking $T_1(u_n)\chi_{(0, \tau)}(t)$ as a test function for problem (22), and using (2), (4) and Hölder's inequality, we get

$$\int_{\Omega} S_1(u_n(x, \tau)) dx + \alpha \int_0^{\tau} \int_{\Omega} \frac{|\nabla T_1(u_n)|^p}{(1 + |u_n|)^{\theta}} dx dt \leq \|f_n\|_{L^m(Q)} \left(\int_0^{\tau} \int_{\Omega} |T_1(u_n)|^{m'} dx dt \right)^{\frac{1}{m'}}.$$

Note that by (7)–(8) for any $s \in \mathbb{R}$, $|s| - \frac{1}{2} \leq S_1(s) \leq |s|$. Then we have

$$\operatorname{ess\,sup}_{0 \leq t \leq T} \int_{\Omega} |u_n(x, t)| dx \leq \|f_n\|_{L^m(Q)} |Q|^{\frac{1}{m'}} + \frac{1}{2} |\Omega|. \quad (54)$$

So, (20) and (54) yield (25).

By (53), and Hölder's inequality, we obtain

$$\int_Q |T_k(u_n)|^p dx dt \leq \left(\int_Q |T_k(u_n)|^{\frac{p(N+2)}{N}} dx dt \right)^{\frac{N}{N+2}} |Q|^{\frac{2}{N+2}} \leq ck^{\frac{N+p+\theta N}{N+2}} |Q|^{\frac{2}{N+2}},$$

New by (52), we have

$$\int_Q |DT_k(u_n)|^p dx dt \leq c|Q|^{\frac{1}{m'}} (1 + k)^{\theta} k.$$

The above two inequalities imply (26). ■

4. Proof of the main theorem

Proof. Let

$$h_k(s) = 1 - |T_1(s - T_k(s))|, \quad H_k(s) = \int_0^s h_k(\tau) d\tau, \quad \forall s \in \mathbf{R}, \quad \forall k > 0.$$

Taking $\phi = h_k(u_n)$ in (22), we get in the sense of distributions

$$\begin{aligned} (H_k(u_n))_t &= \operatorname{div}(h_k(u_n)a(x, t, T_n(u_n))\hat{a}(x, t, u_n, \nabla u_n)) \\ &\quad - a(x, t, T_n(u_n))\hat{a}(x, t, u_n, \nabla u_n)\nabla u_n h'_k(u_n) + f_n h_k(u_n). \end{aligned} \quad (55)$$

Note that $\operatorname{supp}(h_k) \subseteq [-k - 1, k + 1]$, $0 \leq h_k \leq 1$, $|h'_k| \leq 1$, if $n > k + 1$,

$$h_k(u_n)a(x, t, T_n(u_n))\widehat{a}(x, t, u_n, \nabla u_n) = h_k(u_n)a(x, t, T_{k+1}(u_n))\widehat{a}(x, t, T_{k+1}(u_n), \nabla T_{k+1}(u_n)),$$

and

$$\begin{aligned} a(x, t, T_n(u_n))\widehat{a}(x, t, u_n, \nabla u_n)\nabla u_n h'_k(u_n) \\ = a(x, t, T_{k+1}(u_n))\widehat{a}(x, t, T_{k+1}(u_n), \nabla T_{k+1}(u_n))\nabla T_{k+1}(u_n)h'_k(u_n). \end{aligned}$$

By Lemma 4, (9) and the above equalities, for fixed $k > 0$, we can deduce that

$$\begin{aligned} h_k(u_n)a(x, t, T_n(u_n))\widehat{a}(x, t, u_n, \nabla u_n) & \text{ is bounded in } L^p(Q), \\ a(x, t, T_n(u_n))\widehat{a}(x, t, u_n, \nabla u_n)\nabla u_n h'_k(u_n) & \text{ is bounded in } L^1(Q). \end{aligned}$$

Hence

$$(H_k(u_n))_t \text{ is bounded in } L^{p'}(0, T; W^{-1,p'}(\Omega)) + L^1(Q). \quad (56)$$

(56) implies $(H_k(u_n))_t$ is bounded in $L^1(0, T; W^{-1,s}(\Omega))$ for all $s > 1$. By virtue of $\nabla H_k(u_n) = h_k(u_n)\nabla u_n = h_k(u_n)\nabla T_{k+1}(u_n)$, (26) implies that $H_k(u_n)$ is bounded in $L^p(0, T; W_0^{1,p}(\Omega))$.

So, we can use Corollary 4 of [11] to see that $H_k(u_n)$ is relatively compact in $L^1(Q)$. By Theorem 1.1 in [12], we have $H_k(u_n) \in C([0, T], L^1(\Omega))$. Thus there exists a subsequence of $\{H_k(u_n)\}$ (still denoted by $\{H_k(u_n)\}$) such that it also converges in measure and almost everywhere in Q .

Let σ , k , and ε be positive numbers. Noting that

$$\text{meas}\{|u_n - u_m| > \sigma\} \leq \text{meas}\{|u_n| > k\} + \text{meas}\{|u_m| > k\} + \text{meas}\{|H_k(u_n) - H_k(u_m)| > \sigma\}. \quad (57)$$

By (23) in Lemma 4, we can choose k large enough to have

$$\text{meas}\{|u_n| > k\} + \text{meas}\{|u_m| > k\} < \frac{\varepsilon}{2}, \quad \forall n, m. \quad (58)$$

Furthermore, for the above fixed k , we can choose a large N_0 such that

$$\text{meas}\{|H_k(u_n) - H_k(u_m)| > \sigma\} < \frac{\varepsilon}{2}, \quad \forall n, m > N_0. \quad (59)$$

(57)–(59) yield

$$\text{meas}\{|u_n - u_m| > \sigma\} < \varepsilon, \quad \forall n, m > N_0. \quad (60)$$

Now, (60) implies that $\{u_n\}$ is a Cauchy sequence in measure in Q . Hence there exists a measurable function u such that

$$u_n \rightarrow u \text{ a.e. in } Q. \quad (61)$$

Thus we get

$$H_k(u_n) \rightarrow H_k(u) \text{ a.e. in } Q. \quad (62)$$

Since $|H_k| \leq k + 1$, (62) and Lebesgue's dominated convergence theorem yield

$$H_k(u_n) \rightarrow H_k(u) \text{ strongly in } L^p(Q). \quad (63)$$

Since $H_k(u_n)$ is bounded in $L^p(0, T; W_0^{1,p}(\Omega))$ and noting that (63) holds, we have

$$H_k(u_n) \rightharpoonup H_k(u) \text{ weakly in } L^p(0, T; W_0^{1,p}(\Omega)).$$

Now, (61) yields

$$T_k(u_n) \rightarrow T_k(u) \text{ a.e. in } Q. \quad (64)$$

Using Lebesgue's dominated convergence theorem once again, we get

$$T_k(u_n) \rightarrow T_k(u) \text{ strongly in } L^p(Q). \quad (65)$$

From (26) and (65), it follows that

$$T_k(u_n) \rightharpoonup T_k(u) \text{ weakly in } L^p(0, T; W_0^{1,p}(\Omega)).$$

Then (25), (61) and Fatou's lemma yield $u \in L^\infty(0, T; L^1(\Omega))$.

Similarly to Theorem 2.1 in [12], we can prove

$$T_k(u_n) \rightarrow T_k(u) \text{ strongly in } L^p(0, T; W_0^{1,p}(\Omega)). \quad (66)$$

Hence

$$\nabla T_k(u_n) \rightarrow \nabla T_k(u) \text{ a.e. in } Q. \quad (67)$$

Choosing $T_1(u_n - T_k(u_n))$ as a test function for problem (22), using (4) we obtain

$$\int_{\Omega} \tilde{T}(u_n(T)) dx + \int_{\{k < |u_n| \leq k+1\}} a(x, t, T_n(u_n)) |\nabla u_n|^p dx dt \leq \int_{\{|u_n| \geq k\}} |f_n| dx dt,$$

where

$$\tilde{T}(u_n(T)) = \int_0^{u_n(T)} T_1(s - T_k(s)) ds.$$

It is easy to see that $\tilde{T}(u_n(T)) \geq 0$ a.e. in Ω . Hence we have

$$\int_{\{k < |u_n| \leq k+1\}} a(x, t, T_n(u_n)) |\nabla u_n|^p dx dt \leq \int_{\{|u_n| \geq k\}} |f_n| dx dt. \quad (68)$$

Letting $n \rightarrow \infty$ in (68) and using Fatou's lemma in the left side and Vitali's theorem on the right side of (68), we get

$$\int_{\{k < |u| \leq k+1\}} a(x, t, u) |\nabla u|^p dx dt \leq \int_{\{|u| \geq k\}} |f| dx dt. \quad (69)$$

Thus from (69) we can deduce that

$$\lim_{k \rightarrow \infty} \int_{\{k < |u| \leq k+1\}} a(x, t, u) |\nabla u|^p dx dt = 0. \quad (70)$$

Then (61), (64), (66) and Vitali's theorem imply that

$$k(u_n) a(x, t, T_n(u_n)) \hat{a}(x, t, u_n, \nabla u_n) \rightarrow h_k(u) a(x, t, T_{k+1}(u)) \hat{a}(x, t, T_{k+1}(u), \nabla T_{k+1}(u))$$

strongly in $L^p(Q)$, and

$$a(x, t, T_n(u_n)) \hat{a}(x, t, u_n, \nabla u_n) \nabla u_n h'_k(u_n) \rightarrow a(x, t, T_{k+1}(u)) \hat{a}(x, t, T_{k+1}(u), \nabla T_{k+1}(u)) \nabla T_{k+1}(u) h'_k(u)$$

strongly in $L^1(Q)$. Let $n \rightarrow \infty$ in (55). We obtain in the sense of distributions that

$$(H_k(u))_t = \operatorname{div}(h_k(u) a(x, t, T_{k+1}(u)) \hat{a}(x, t, T_{k+1}(u), \nabla T_{k+1}(u))) - a(x, t, T_{k+1}(u)) \hat{a}(x, t, T_{k+1}(u), \nabla T_{k+1}(u)) \nabla T_{k+1}(u) h'_k(u) + f h_k(u). \quad (71)$$

Hence $(H_k(u))_t \in L^{p'}(0, T; W^{-1,p'}(\Omega)) + L^1(Q)$. By Theorem 1.1 in [12], we have $H_k(u) \in C([0, T], L^1(\Omega))$. Since $H_k(u_n(0)) = 0$, thus we get $H_k(u(0)) = 0$. For every $\phi \in L^p(0, T; W_0^{1,p}(\Omega)) \cap L^\infty(Q)$ such that $\phi_t \in L^{p'}(0, T; W^{-1,p'}(\Omega)) + L^1(Q)$ and for all $\tau \in (0, T]$, using $T_l(H_k(u) - \phi) \chi_{(0,\tau)}(t)$ as a test function in (71), and integrating by parts we obtain

$$\begin{aligned} & \int_{\Omega} S_l(H_k(u) - \phi)(\tau) dx - \int_{\Omega} S_l(-\phi(0)) dx + \int_0^\tau \langle \phi_t, T_l(H_k(u) - \phi) \rangle dt \\ & + \int_0^\tau \int_{\Omega} h_k(u) a(x, t, T_{k+1}(u)) \hat{a}(x, t, T_{k+1}(u), \nabla T_{k+1}(u)) \nabla T_l(H_k(u) - \phi) dx dt \\ & + \int_0^\tau \int_{\Omega} a(x, t, T_{k+1}(u)) \hat{a}(x, t, T_{k+1}(u), \nabla T_{k+1}(u)) \nabla T_{k+1}(u) h'_k(u) T_l(H_k(u) - \phi) dx dt \\ & = \int_0^\tau \int_{\Omega} f h_k(u) T_l(H_k(u) - \phi) dx dt. \end{aligned}$$

Noting that if $k \rightarrow \infty$, we have

$$h_k(u) \rightarrow 1 \text{ a.e. in } Q, \quad (72)$$

$$H_k(u) \rightarrow u \text{ a.e. in } Q. \quad (73)$$

Since $h'_k(u) = -\text{sign}(u)\chi_{\{k \leq |u| \leq k+1\}}$, $\text{sign}(H_k(u)) = \text{sign}(u)$, and $|H_k(u)| > k$ if $|u| > k$; and $H_k(u) = u$ if $|u| \leq k$. Moreover, if $|H_k(u)| > l + \|\phi\|_{L^\infty(Q)} = L$, we have $\nabla T_l(H_k(u) - \phi) = 0$. Hence if $k > L$, thus we have

$$\begin{aligned} \int_0^\tau \int_\Omega h_k(u) a(x, t, T_{k+1}(u)) \widehat{a}(x, t, T_{k+1}(u), \nabla T_{k+1}(u)) \nabla T_l(H_k(u) - \phi) dx dt \\ = \int_0^\tau \int_\Omega a(x, t, T_L(u)) \widehat{a}(x, t, T_L(u), \nabla T_L(u)) \nabla T_l(T_L(u) - \phi) dx dt. \end{aligned} \quad (74)$$

It follows from (70) that

$$\lim_{k \rightarrow \infty} \int_0^\tau \int_\Omega a(x, t, T_{k+1}(u)) \widehat{a}(x, t, T_{k+1}(u), \nabla T_{k+1}(u)) \nabla T_{k+1}(u) h'_k(u) T_l(H_k(u) - \phi) dx dt = 0. \quad (75)$$

Lebesgue's dominated convergence theorem and (72)–(73) imply that

$$\lim_{k \rightarrow \infty} \int_0^\tau \int_\Omega f h_k(u) T_l(H_k(u) - \phi) dx dt = \int_0^\tau \int_\Omega f T_l(u - \phi) dx dt. \quad (76)$$

We can also prove if $k \rightarrow \infty$,

$$T_l(H_k(u) - \phi) \rightarrow T_l(u - \phi) \text{ strongly in } L^p(0, T; W_0^{1,p}(\Omega)), \quad (77)$$

$$T_l(H_k(u) - \phi) \rightarrow T_l(u - \phi) \text{ weak}^* \text{ in } L^\infty(\Omega). \quad (78)$$

From (77) and (78) we get

$$\lim_{k \rightarrow \infty} \int_0^\tau \langle \phi_t, T_l(H_k(u) - \phi) \rangle dt = \int_0^\tau \langle \phi_t, T_l(u - \phi) \rangle dt. \quad (79)$$

Since for a.e. $\tau \in [0, T]$, a.e. $x \in \Omega$,

$$|H_k(u)| \leq |u|, \quad 0 \leq S_l(H_k(u) - \phi)(\tau) \leq l(|u(\tau)| + |\phi(\tau)|),$$

combining with $u \in L^\infty(0, T; L^1(\Omega))$ and $\phi \in C([0, T]; L^1(\Omega))$, by Lebesgue's dominated convergence theorem and (73), we get

$$\lim_{k \rightarrow \infty} \int_\Omega S_l(H_k(u) - \phi)(\tau) dx = \int_\Omega S_l(u - \phi)(\tau) dx. \quad (80)$$

Now (74)–(76), (79)–(80) yield for a.e. $\tau \in [0, T]$,

$$\begin{aligned} \int_\Omega S_l(u - \phi)(\tau) dx - \int_\Omega S_l(-\phi(0)) dx + \int_0^\tau \langle \phi_t, T_l(u - \phi) \rangle dt \\ + \int_0^\tau \int_\Omega a(x, t, u) \widehat{a}(x, t, u, \nabla u) \nabla T_l(u - \phi) dx dt = \int_0^\tau \int_\Omega f T_l(u - \phi) dx dt. \end{aligned} \quad (81)$$

This shows that the first term on the left side of the above equality is almost everywhere equal to a continuous function on $[0, T]$. Replacing l with k in (81), we obtain (9)–(10) and u is an entropy solution to problem (1). By (23), we have

$$\int_Q \chi_{\{|u_n| > k\}} dx dt = \text{meas}\{|u_n| > k\} \leq \frac{c}{k^\delta}. \quad (82)$$

Thus (61), (82) and Fatou's lemma yield

$$\text{meas}\{|u| > k\} = \int_Q \chi_{\{|u| > k\}} dx dt \leq \frac{c}{k^\delta}.$$

Rewriting (73) as follows

$$k \operatorname{meas}\{|u| > k\}^{\frac{1}{\delta}} = c^{\frac{1}{\delta}}. \quad (83)$$

Thus by Definition 2, we obtain $u \in \mathcal{M}^\delta(Q)$.

The complete the proof of (18), we need to prove

$$\nabla u_n \rightarrow \nabla u \text{ a.e. in } Q. \quad (84)$$

If fact, for all $\sigma > 0$ and $\varepsilon > 0$, we have

$$\operatorname{meas}\{|\nabla u_n - \nabla u| > \sigma\} \leq \operatorname{meas}\{|u_n| > k\} + \operatorname{meas}\{|u| > k\} + \operatorname{meas}\{|T_k(u_n) - T_k(u)| > \sigma\}.$$

By (23) and (18), we can choose k large enough to prove

$$\operatorname{meas}\{|u_n| > k\} + \operatorname{meas}\{|u| > k\} < \frac{\varepsilon}{2}, \quad \forall n. \quad (85)$$

For the above k , (67) implies that there exists a large N_0 such that

$$\operatorname{meas}\{|T_k(u_n) - T_k(u)| > \sigma\} < \frac{\varepsilon}{2}, \quad \forall n > N_0. \quad (86)$$

Now, (85) and (86) yield

$$\operatorname{meas}\{|\nabla u_n - \nabla u| > \sigma\} < \varepsilon, \quad \forall n > N_0.$$

Hence from (83), we can deduce that (84) holds. Similarly to (82)–(83), by (24) and (84), we obtain $|\nabla u| \in \mathcal{M}(Q)$. Thus the proof of Theorem 2 is completed. \blacksquare

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Регулярність ентропійних розв'язків вироджених параболічних рівнянь із даними L^m

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У цій статті досліджуються регулярні результати для ентропійних розв'язків класу параболічних нелінійних рівнянь із виродженою коерцитивністю, коли права частина знаходиться в L^m з $m > 1$.

Ключові слова: регулярність; ентропійні розв'язки; вироджена коерцитивність; дані L^m .