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Degenerate elliptic problem with singular gradient lower order term and variable exponents

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In this paper, we prove the existence and regularity of weak solutions for a class of nonlinear elliptic equations with degenerate coercivity and singular lower-order terms with natural growth with respect to the gradient and $L^{m(\cdot)}$ $(m(x) \geqslant 1)$ data. The functional setting involves Lebesgue–Sobolev spaces with variable exponents.

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1. Introduction

1.1. Assumptions and the main results

We consider the elliptic problem

$$\begin{cases} -\operatorname{div}\left(a(x,u)\widehat{a}(x,u,\nabla u)\right) + B\frac{|\nabla u|^{p(x)}}{|u|^{\theta}} = f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where Ω is an open bounded set of $\mathbb{R}^N$ $(N \geqslant 3)$, $B > 0$, f is a positive function belonging to $L^{m(\cdot)}(\Omega)$ $(m(\cdot) \geqslant 1)$, $0 < \theta < 1$ and $p\colon \overline{\Omega} \to [2, +\infty)$, are continuous functions, such that

$$1 < p^- := \min_{x \in \overline{\Omega}} p(x) \leqslant p^+ := \max_{x \in \overline{\Omega}} p(x) < \infty. \tag{2}$$

Suppose that $a\colon \Omega \times \mathbb{R} \to \mathbb{R}$, is Carathéodory function such that for a.e. $x \in \Omega$, for every $s \in \mathbb{R}$,

$$\frac{\alpha}{(1+|s|)^{\gamma(x)}} \leqslant a(x,s) \leqslant \beta, \tag{3}$$

where α, β are strictly positive real numbers and $\gamma\colon \overline{\Omega} \to [0, +\infty[$ is continuous function.

Suppose that $\widehat{a}\colon \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$, is Carathéodory function and satisfying, for a.e. $x \in \Omega$, $\forall s \in \mathbb{R}$ and $\forall \xi, \xi' \in \mathbb{R}^N$, the following

$$\widehat{a}(x,s,\xi) \cdot \xi \geqslant |\xi|^{p(x)}, \tag{4}$$

$$|\widehat{a}(x,s,\xi)| \leqslant c_1 |s|^{\frac{\theta(p(x)-1)}{p(x)}} + c_2 |\xi|^{p(x)-1}, \tag{5}$$

$$\left(\widehat{a}(x,s,\xi) - \widehat{a}(x,s,\xi')\right) \cdot (\xi - \xi') \geqslant c_3 |\xi - \xi'|^{p(x)}, \tag{6}$$

where c_1, c_2, c_3 are strictly positive real numbers.

1.2. Previous results and some remarks

We will state some characteristics of problem (1) as well as the main difficulties we encounter.

Firstly, from hypothesis (5), the operator $Au = -\operatorname{div}\left(a(x,u)\widehat{a}(x,u,\nabla u)\right)$ is well defined between $W_0^{1,p(\cdot)}(\Omega)$ and its dual space $\left(W_0^{1,p(\cdot)}(\Omega)\right)'$. But, from (3) it fails to be coercive if u is large. Due to the lack of coercivity, the classical theory for elliptic operators cannot be applied even if the data f are sufficiently regular.

The second difficulty appears when we give a variable exponential growth condition (5) for $\widehat{a}$. A "bad" result is that in the principal term, $\widehat{a}(x,u,\nabla u)$ may not be in $L^1(\Omega)$, that is to say, maybe the first term in (1) does not make sense even as a distribution; thus, some techniques used in the constant exponent case cannot be carried out for the variable exponent case.

Finally, the lower order term has a growth with respect to the gradient and is singular in the variable u. As we will see, existence and summability of solutions to problem (1) depend on these features. We overcome these difficulties by replacing operator A by another one defined by means of truncations, and approximating the singular term $B\frac{|\nabla u|^{p(x)}}{|u|^{\theta}}$ by nonsingular one in such a way that the corresponding approximated problems have finite energy solutions.

Equations with variable exponents appear in various mathematical models. In some cases, they provide realistic models for the study of natural phenomena in electro-rheological fluids and important applications are related to image processing. We cite some papers that have dealt with the equation (1) or similar problems, we refer the reader to [1–4] and the references therein.

When the singular lower-order term does not appear in (1) (i.e., $B \equiv 0$), the existence and regularity of solutions to problem (1) are proved in [5] under the hypothesis $a\colon \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying the following condition:

$$a(x,t) \geqslant b(x,|t|), \quad \text{with } b(x,t) = \frac{c_0}{(1+t)^{\theta(p(x)-1)}}, \quad c_0 > 0, \quad \theta \geqslant 0.$$

The problem was also considered in [6], when $p(x) = p$, $B \equiv 0$, $\gamma(x)$ was a constant with $0 \leqslant \gamma < p-1$, $f \in L^m(\Omega)$. The authors in [6] mainly considered the regularity of u varying with m. In particular if $p(x) = 2$ and $\gamma(x) = \gamma$, in [7, 8], the authors have shown the existence of solutions to (1). The presence of lower order terms can have a regularizing effect on the solutions. In [9] and [10] three kinds of lower order terms are considered for elliptic problems with degenerate coercivity, with no restriction on p. The corresponding results in the case $p(x) = 2$ and $\gamma(x) = \gamma$ are developed in [11]. Moreover, in the constant case where $1 \leqslant \theta \leqslant 2$, $\gamma > 0$, $p > 2$ and $\widehat{a}(x,u,\nabla u) = |\nabla u|^{p-2}\nabla u$, the existence and some regularity results are proved in [12]. Problem (1) has been studied in the case $p(x) = 2$, $\gamma(x) = 0$, and $f \in L^q(\Omega)$, for $\theta(x) \in C^1(\overline{\Omega})$. The authors of [13] proved the existence of solution $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ with $q \geqslant \frac{N}{2}$ and $\theta(x) < 2$ on $\partial\Omega$.

2. Mathematical preliminaries

In this section we recall some definitions and basic properties of the generalized Lebesgue–Sobolev spaces $L^{p(\cdot)}(\Omega)$, $W^{1,p(\cdot)}(\Omega)$ and $W_0^{1,p(\cdot)}(\Omega)$, where Ω is an open subset of $\mathbb{R}^N$, $(N \geqslant 2)$. For further details on the Lebesgue–Sobolev spaces with variable exponents, we refer to [14, 15] and references therein.

Let $p\colon \overline{\Omega} \to [1,+\infty[$ be a continuous, real-valued function (the variable exponent), we set

$$p^- = \min_{x\in\overline{\Omega}} p(x), \quad \text{and} \quad p^+ = \max_{x\in\overline{\Omega}} p(x). \tag{7}$$

By $L^{p(\cdot)}(\Omega)$ is denoted the space of measerable function $f(x)$ on Ω such that

$$\rho_p(\cdot)(f) = \int_\Omega |f(x)|^{p(x)} dx < \infty.$$

The space $L^{p(\cdot)}(\Omega)$ equipped with the norm (called Luxemburg norm),

$$\|u\|_{L^{p(\cdot)}(\Omega)} = \inf\left\{\mu > 0; \int_\Omega \left|\frac{u(x)}{\mu}\right|^{p(x)} dx \leqslant 1\right\}$$

becomes Banach space. Moreover, if $p^- > 1$, then $L^{p(\cdot)}(\Omega)$ is reflexive and the dual space of $L^{p(\cdot)}(\Omega)$

can be identified with $L^{p'(\cdot)}(\Omega)$, where $\frac{1}{p(x)}+\frac{1}{p'(x)}=1$. For any $u\in L^{p(\cdot)}(\Omega)$ and $v\in L^{p'(\cdot)}(\Omega)$ the Hölder type inequality

$$\int_\Omega |uv|\,dx \leqslant \left(\frac{1}{p^-}+\frac{1}{p'^-}\right)\|u\|_{L^{p(\cdot)}(\Omega)}\|v\|_{L^{p'(\cdot)}(\Omega)}, \tag{8}$$

holds true. We define also the variable Sobolev space

$$W^{1,p(\cdot)}(\Omega)=\left\{u\in L^{p(\cdot)}(\Omega);\ |\nabla u|\in L^{p(\cdot)}(\Omega)\right\}.$$

On $W^{1,p(\cdot)}(\Omega)$ one may consider one of the following equivalent norms

$$\|u\|_{W^{1,p(\cdot)}(\Omega)}=\|u\|_{L^{p(\cdot)}(\Omega)}+\|\nabla u\|_{L^{p(\cdot)}(\Omega)},$$

or

$$\|u\|_{W^{1,p(\cdot)}(\Omega)}=\inf\left\{\mu>0;\int_\Omega\left(\left|\frac{\nabla u(x)}{\mu}\right|^{p(x)}+\left|\frac{u(x)}{\mu}\right|^{p(x)}\right)\leqslant 1\right\}.$$

Next, we define also $W_0^{1,p(\cdot)}(\Omega)$ the Sobolev space with zero boundary values by

$$W_0^{1,p(\cdot)}(\Omega)=\left\{u\in W^{1,p(\cdot)}(\Omega)\colon u=0 \ \text{ on } \ \partial\Omega\right\}$$

endowed with the norm $\|u\|_{W^{1,p(\cdot)}(\Omega)}$. The space $W_0^{1,p(\cdot)}(\Omega)$ is a separable and reflexive provided that $1<p-\leqslant p^+<+\infty$.

An important role in manipulating the generalized Lebesgue and Sobolev spaces is played by the modular $\rho_{p(\cdot)}(u)$ of the space $L^{p(\cdot)}(\Omega)$). We have the following result.

Lemma 1 (Ref. [15]). *If $u_n,u\in L^{p(\cdot)}(\Omega)$ and $p^+<+\infty$, then the following properties hold true:*

$$\min\left(\|u\|_{p(\cdot)}^{p^-},\|u\|_{p(\cdot)}^{p^+}\right)\leqslant \varrho_{p(\cdot)}(u)\leqslant \max\left(\|u\|_{p(\cdot)}^{p^-},\|u\|_{p(\cdot)}^{p^+}\right).$$

Remark 1. The variable exponent $p\colon\overline{\Omega}\to[1,+\infty)$ is said to satisfy the log-continuity condition, if there exists a positive constant C such that

$$\forall x,y\in\overline{\Omega},\quad |x-y|\leqslant 1/2;\quad |p(x)-p(y)|<\frac{C}{|\log|x-y||}. \tag{9}$$

Log-continuity condition (9) is used to obtain several regularity results for Sobolev spaces with variable exponents; in particular, $C^\infty(\Omega)$ is dense in $W^{1,p(\cdot)}(\Omega)$ and $W_0^{1,p(\cdot)}(\Omega)=W^{1,p(\cdot)}(\Omega)\cap W_0^{1,1}(\Omega)$. For $p\in C(\overline{\Omega})$ with $1<p^-\leqslant p^+<N$. If $r\in C(\overline{\Omega})$ and $r(x)<p^*(x)$, for all $x\in\overline{\Omega}$, then the Sobolev embedding holds (see [14])

$$W^{1,p(\cdot)}(\Omega)\hookrightarrow\hookrightarrow L^{r(\cdot)}(\Omega). \tag{10}$$

Moreover, if p satisfies the log-continuity (9) condition and $1<p^-\leqslant p^+<N$, then the Sobolev embedding holds also for $r(\cdot)=p^*(\cdot)=\frac{Np(\cdot)}{N-p(\cdot)}$.

The rest of the paper is organized as follows: in Section 3 we state the main results and we fix the notations used throughout the work. In Section 4 we will use a standard approximation procedure similarly to [11, 16–18]. First, we approximate the problem (1) by a sequence of non-degenerate and non-singular nonlinear elliptic problems. Then, we prove both a priori estimates and convergence results on the sequence of approximating solutions. Next, in Section 5 we prove that the weak limit of the approximate solutions is strictly positive in Ω, applying the strong maximum principle and Harnack inequality. In the end, we pass to the limit in the approximate problem.

3. Statement of results

Let us start giving our definition of solution to problem (1).

Definition 1. *Let $L^{m(\cdot)}(\Omega)$ $(m(\cdot)\geqslant 1)$. A function u is a weak solution of problem* (1)*, if*

$$u \in W_0^{1,1}(\Omega), \quad \widehat{a}(x,u,\nabla u) \in \left(L^1(\Omega)\right)^N, \quad \frac{|\nabla u|^{p(x)}}{u^\theta} \in L^1(\Omega), \quad u > 0 \quad in \quad \Omega,$$

and

$$\int_\Omega a(x,u)\widehat{a}(x,u,\nabla u)\nabla\varphi\,dx + B\int_\Omega \frac{|\nabla u|^{p(x)}}{u^\theta}\varphi\,dx = \int_\Omega f\varphi\,dx, \quad \forall\varphi \in C_0^1(\Omega). \tag{11}$$

Our main results are the following theorems.

Theorem 1. *Let $f \in L^1(\Omega)$ be a positive function and $p\colon \overline{\Omega} \to [1,+\infty)$ is a continuous function, assume that (2)–(6) hold true and*

$$2 \leqslant p(x) < N. \tag{12}$$

Then, the problem (1) has at least one distributional solution $u \in W_0^{1,\eta(\cdot)}(\Omega)$, with

$$\eta(x) < \frac{N(p(x)-\theta)}{N-\theta}, \quad \forall x \in \overline{\Omega}. \tag{13}$$

Theorem 2. *Let $p\colon \overline{\Omega} \to (1,+\infty)$, is a continuous function and $f \in L^m(\Omega)$ be a positive function such that*

$$1 < m < \frac{Np^-}{Np^- - \theta(N-p^-)},$$

assume that (2)–(6) and (12) hold true. Then the problem (1) has at least one distributional solution $u \in W_0^{1,q(\cdot)}(\Omega)$, where q is a continuous function on $\overline{\Omega}$ satisfying

$$q(x) < \frac{mN(p(x)-\theta)}{N-\theta m}, \quad \forall x \in \overline{\Omega}. \tag{14}$$

Theorem 3. *Let $m\colon \overline{\Omega} \to (1,+\infty)$, $p\colon \overline{\Omega} \to [2,+\infty)$ satisfy the log-coninuity condition (9) such that for $x \in \overline{\Omega}$*

$$\frac{Np(x)}{Np(x)-\theta(N-p(x))} \leqslant m(x) < \frac{N}{p(x)}, \tag{15}$$

let $f \in L^{m(\cdot)}(\Omega)$, and assume that (2)–(6) and (12) hold true. Then, the problem (1) has at least one weak solution $u \in W_0^{1,p(\cdot)}(\Omega)$

Remark 2. (1) The assumption (12) implies $1 < \eta(x) < p(x)$.

(2) In Theorem 3, we have inequality (15) is meaningful, because

$$2 \leqslant p(x) < N \Longrightarrow \frac{Np(x)}{Np(x)-\theta(N-p(x))} < \frac{N}{p(x)}.$$

3.1. Notations

For any $q(x) > 1$, $q'(x) = \frac{q(x)}{q(x)-1}$ is the Hölder conjugate exponent of $q(x)$. For fixed $k > 0$ we will use of the truncation T_k defined as $T_k(s) = \max(-k, \min(k,s))$ and $G_k(s) = s - T_k(s)$. We define, for $s \geqslant 0$,

$$H_n(s) = \int_0^s \frac{t(1+T_n(t))^{\gamma^-}}{\alpha(t+\frac{1}{n})^{\theta+1}}dt, \quad H_\infty(s) = \int_0^s \frac{(1+t)^{\gamma^-}}{\alpha t^\theta}dt, \quad n \in \mathbb{N}^* \tag{16}$$

and for $t \geqslant 0$

$$H_{\frac{1}{n}}(t) = \int_0^t \frac{B(1+s)^{\gamma^-}}{\alpha(s+\frac{1}{n})^\theta}ds, \quad H_0(t) = \int_0^t \frac{B(1+s)^{\gamma^-}}{\alpha s^\theta}ds, \quad n \in \mathbb{N}^*. \tag{17}$$

Observe that H is well-defined, since $\theta < 1$. We will also use the following functions, for $\lambda > 0$

$$\psi_\lambda(t) = \begin{cases} 1, & 0 \leqslant t < 1, \\ -\frac{1}{\lambda}(t-1-\lambda), & 1 \leqslant t < \lambda+1, \\ 0, & \lambda+1 \leqslant t, \end{cases} \tag{18}$$

and for $k \in \mathbb{N}$

$$R_k(s) = \begin{cases} 1, & s \leqslant k, \\ k+1-s, & k \leqslant s \leqslant k+1, \\ 0, & s > k+1. \end{cases} \tag{19}$$

We will denote by C several constants whose value may change from line to line and, sometimes, on the same line. These values will only depend on the data (for instance C can depend on Ω, p, γ, θ, N, ...) but will never depend on the indexes of the sequences we will often introduce.

4. Approximate solution and a priori estimates

We consider the approximate problem

$$\begin{cases} -\operatorname{div}\left(a(x,T_n(u_n))\widehat{a}(x,u_n,\nabla u_n)\right) + B\frac{u_n|\nabla u_n|^{p(x)}}{\left(|u_n|+\frac{1}{n}\right)^{\theta+1}} = f_n, & \text{in } \Omega, \\ u_n = 0, & \text{on } \partial\Omega. \end{cases} \tag{20}$$

The problem (20) admits at least one solution $u_n \in W_0^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$ (see [19–21]). Using as a test function $u_n^- = \min\{u_n, 0\}$, one has $u_n \geqslant 0$ almost everywhere in Ω.

4.1. A priori estimates

The following lemma gives a control of the lower order term.

Lemma 2. *Let u_n be the solutions to problems (20). Then it results*

$$B\int_\Omega \frac{u_n|\nabla u_n|^{p(x)}}{\left(u_n+\frac{1}{n}\right)^{\theta+1}}dx \leqslant C. \tag{21}$$

Proof. For any fixed $h > 0$, let us consider $\frac{T_h(u_n)}{h}$ as a test function in (20) and dropping the nonnegative first term, we obtain

$$B\int_\Omega \frac{u_n|\nabla u_n|^{p(x)}}{\left(u_n+\frac{1}{n}\right)^{\theta+1}}\frac{T_h(u_n)}{h}dx \leqslant \int_\Omega f_n\frac{T_h(u_n)}{h}dx.$$

Letting h tend to 0 in this estimate, by Fatou's lemma, we obtain the result. ∎

Lemma 3. *Let $f \in L^1(\Omega)$ be a positive function. Then, there exists a constant C independent on n, such that for all continuous function η on $\overline{\Omega}$ as in (13); we have*

$$\|u_n\|_{W_0^{1,\eta(\cdot)}(\Omega)} \leqslant C, \tag{22}$$

$$\|T_k(u_n)\|_{W_0^{1,p(\cdot)}(\Omega)} \leqslant C, \quad \forall k > 0. \tag{23}$$

Proof. Case (a). In the first step, let η^+ be a constant satisfying

$$1 \leqslant \eta^+ < \frac{N(p^- - \theta)}{N-\theta} < p^-. \tag{24}$$

The estimate (21) and that $u_n(u_n+\frac{1}{n})^{-\theta-1} \geqslant 2^{-\theta-1}u_n^{-\theta}$ on the set $\{u_n \geqslant 1\}$, we have

$$B\int_{\{u_n\geqslant 1\}} \frac{|\nabla u_n|^{p(x)}}{u_n^\theta}dx \leqslant C. \tag{25}$$

By Young's inequality (with the exponents $\frac{p(x)}{p^-}$ and $\frac{p(x)}{p(x)-p^-}$), (25) and the fact that $u_n^\theta \geqslant 1$, we get

$$\int_{\{u_n\geqslant 1\}} \frac{|\nabla u_n|^{p^-}}{u_n^\theta}dx \leqslant \int_{\{u_n\geqslant 1\}} \frac{|\nabla u_n|^{p(x)}}{u_n^\theta}dx + \int_{\{u_n\geqslant 1\}} \frac{1}{u_n^\theta}dx \leqslant \int_{\{u_n\geqslant 1\}} \frac{|\nabla u_n|^{p(x)}}{u_n^\theta}dx + |\Omega| \leqslant C. \quad (26)$$

Using Hölder inequality (since $\eta^+ < p^-$), (26) and that $u_n \leqslant G_1(u_n) + 1$,

$$\begin{aligned} \int_\Omega |\nabla G_1(u_n)|^{\eta^+} dx &\leqslant \left(\int_{\{u_n\geqslant 1\}} \frac{|\nabla u_n|^{p^-}}{u_n^\theta}dx\right)^{\frac{\eta^+}{p^-}} \left(\int_{\{u_n\geqslant 1\}} u_n^{\frac{\theta\eta^+}{p^- - \eta^+}} dx\right)^{\frac{p^- - \eta^+}{p^-}} \\ &\leqslant C\left(\int_{\{u_n\geqslant 1\}} u_n^{\frac{\theta\eta^+}{p^- - \eta^+}} dx\right)^{\frac{p^- - \eta^+}{p^-}} \\ &\leqslant C\left(\int_{\{u_n\geqslant 1\}} (G_1(u_n))^{\frac{\theta\eta^+}{p^- - \eta^+}} dx\right)^{\frac{p^- - \eta^+}{p^-}} + C. \end{aligned} \quad (27)$$

Inequality (13) implies that $\frac{\eta^+\theta}{p^- - \eta^+} < {\eta^+}^*$. By Sobolev's inequality and (27),

$$\begin{aligned} \left(\int_{\{u_n\geqslant 1\}} (G_1(u_n))^{{\eta^+}^*} dx\right)^{\frac{\eta^+}{{\eta^+}^*}} &\leqslant C\int_{\{u_n\geqslant 1\}} |\nabla G_1(u_n)|^{\eta^+} dx \\ &\leqslant C\left(\int_{\{u_n\geqslant 1\}} (G_1(u_n))^{{\eta^+}^*} dx\right)^{\frac{p^- - \eta^+}{p^-}} + C. \end{aligned} \quad (28)$$

Since $\frac{p^- - \eta^+}{p^-} < \frac{\eta^+}{{\eta^+}^*}$ (since $p^- < N$), by inequality (27) and (28),

$$\int_{\{u_n\geqslant 1\}} |\nabla G_1(u_n)|^{\eta^+} dx \leqslant C. \quad (29)$$

Now, using $T_k(u_n)$ as test function in (20),

$$\int_{\{u_n\leqslant k\}} |\nabla T_k(u_n)|^{p(x)} dx \leqslant Ck(k+1)^{\gamma^+} \quad \forall n \in \mathbb{N}, \quad (30)$$

taking $k = 1$ in (30), we deduce that $T_1(u_n)$ is bounded in $W_0^{1,p(x)}(\Omega)$ hence in $W_0^{1,\eta(x)}(\Omega)$. Using (29) and the fact that $u_n = G_1(u_n) + T_1(u_n)$, we deduce that (22). Moreover (30) implies that (23).

Case (b). Now let us consider a continuous variable exponent $\eta(\cdot)$ on $\overline{\Omega}$ satisfying the pointwise estimate (13) and

$$\eta^+ \geqslant \frac{N(p^- - \theta)}{N - \theta}.$$

By the continuity of $\eta(\cdot)$ and $p(\cdot)$ on $\overline{\Omega}$ there exists a constant $\delta > 0$ such that

$$\max_{t\in Q(x,\delta)\cap\Omega} \eta(t) < \min_{t\in Q(x,\delta)\cap\Omega} \frac{N(p(t) - \theta)}{N - \theta}, \quad (31)$$

where $Q(x,\delta)$ is a cube with center x and diameter δ. Note that $\overline{\Omega}$ is compact and, therefore, we can cover it with a finite number of cubes $(Q_j)_{j=1,\ldots,k}$ with edges parallel to the coordinate axes. Moreover, there exists a constant $\nu > 0$ such that $\delta > |\Omega_j| > \nu$, $\Omega_j = Q_j \cap \Omega$ for all $j = 1, \ldots, k$. We denote by η_j^+ (respectivel p_j^-) the local maximum of $\eta(\cdot)$ on $\overline{\Omega}_j$ (respectively the local minimum of $p(\cdot)$ on $\overline{\Omega}_j$), such that

$$\eta_j^+ < \frac{N(p_j^- - \theta)}{N - \theta}, \quad \text{for all} \quad j = 1, \ldots, k.$$

Using now the same arguments as before locally, we see that the inequality (27) holds on Ω_j, so

$$\int_{\Omega_j} |\nabla G_1(u_n)|^{\eta_j^+}\,dx \leqslant C\left(\int_{\{x\in\Omega_j\,:\,u_n(x)\geqslant 1\}} (G_1(u_n))^{\frac{\theta\eta_j^+}{p_j^- - \eta_j^+}}\,dx\right)^{\frac{p_j^- - \eta_j^+}{p_j^-}} + C. \tag{32}$$

Denote by $\widetilde{G_1}(u_n)$ the average of $G_1(u_n)$ over Ω_j

$$\widetilde{G_1}(u_n) = \frac{1}{\operatorname{meas}(\Omega_j)} \int_{\Omega_j} G_1(u_n)\,dx.$$

By Poincaré–Wirtinger inequality, we obtain

$$\|G_1(u_n) - \widetilde{G_1}(u_n)\|_{L^{\eta_j^{+*}}(\Omega_j)} \leqslant C\|\nabla G_1(u_n)\|_{L^{\eta_j^+}(\Omega_j)}, \quad \text{for all} \quad j = 1,\dots,k. \tag{33}$$

Using (28) (since $\eta^+ > 1$) and (33),

$$\begin{aligned}\|G_1(u_n)\|_{L^{\eta_j^{+*}}(\Omega_j)} &\leqslant \|G_1(u_n) - \widetilde{G_1}(u_n)\|_{L^{\eta_j^{+*}}(\Omega_j)} + \|\widetilde{G_1}(u_n)\|_{L^{\eta_j^{+*}}(\Omega_j)} \\ &\leqslant C\|\nabla G_1(u_n)\|_{L^{\eta_j^+}(\Omega_j)} + C, \quad \text{for all} \quad j = 1,\dots,k.\end{aligned} \tag{34}$$

We deduce from (32) and (34)

$$\int_{\{x\in\Omega_j\,:\,u_n(x)\geqslant 1\}} |\nabla G_1(u_n)|^{\eta_j^+}\,dx \leqslant C, \quad \text{for all} \quad j = 1,\dots,k. \tag{35}$$

Knowing that $\eta(x) \leqslant \eta_j^+$ for all $x \in \Omega_j$, and all $j = 1,\dots,k$, by (29) and (35), we conclude that

$$\int_\Omega |\nabla u_n|^{\eta(x)} dx \leqslant \sum_{j=1}^k \int_{\Omega_j} |\nabla u_n|^{\eta(x)} dx \leqslant C.$$

This finishes the proof of the Lemma 3. ∎

Lemma 4. *Suppose that the hypotheses of Theorem 2 are satisfied, then the solution* (u_n) *to problems* (1) *are uniformly bounded in* $W_0^{1,q(\cdot)}(\Omega)$*, where* q *is a continuous function on* $\overline{\Omega}$ *satisfying* (14).

Proof. Take $\phi = (u_n + 1)^{\theta + p^- s} - 1$, with

$$s = \frac{p^{-*} - \theta m'}{p^- m' - p^{-*}}, \tag{36}$$

as a test function in problems (20), (note that $s < 0$ and $\theta + p^- s > 0$). Using (3), (4), (21), $f_n \leqslant f$, and dropping the nonnegative first term, we obtain

$$B\int_\Omega \frac{u_n\,|\nabla u_n|^{p(x)}}{\left(u_n + \frac{1}{n}\right)^{\theta+1}}(u_n+1)^{\theta+p^- s}dx \leqslant \int_\Omega f(u_n+1)^{\theta+p^- s}dx + C.$$

Remark that $|\nabla u_n|^{p^-} \leqslant |\nabla u_n|^{p(x)} + 1$, it is not difficult to prove that

$$\int_{\{u_n\geqslant 1\}} \frac{u_n\,|\nabla u_n|^{p^-}}{\left(u_n + \frac{1}{n}\right)^{\theta+1}}(u_n+1)^{\theta+p^- s}dx < \int_{\{u_n\geqslant 1\}} \frac{u_n\,|\nabla u_n|^{p(x)}}{\left(u_n + \frac{1}{n}\right)^{\theta+1}}(u_n+1)^{\theta+p^- s}dx + C.$$

Using the fact that $2u_n(u_n+1)^\theta \geqslant \left(u_n + \frac{1}{n}\right)^{\theta+1}$ on the set $\{u_n \geqslant 1\}$, then we can write

$$\begin{aligned}\int_{\{u_n\geqslant 1\}} \left|\nabla\left((u_n+1)^{s+1} - 2^{s+1}\right)\right|^{p^-} dx &\leqslant C\int_{\{u_n\geqslant 1\}} |\nabla u_n|^{p^-}(u_n+1)^{p^- s}dx \\ &\leqslant C\int_{\{u_n\geqslant 1\}} f(u_n+1)^{\theta+p^- s}dx + C \\ &\leqslant C\left(\int_{\{u_n\geqslant 1\}} (u_n+1)^{(p^- s+\theta)m'}dx\right)^{\frac{1}{m'}} + C.\end{aligned} \tag{37}$$

Using (37) and the Sobolev inequality on the left one imply

$$\left(\int_{\{u_n\geqslant 1\}} \left|(u_n+1)^{s+1}-2^{s+1}\right|^{p^{-*}}\cdot dx\right)^{\frac{p^-}{p^{-*}}} \leqslant C\int_{\{u_n\geqslant 1\}} \left|\nabla\left((u_n+1)^{s+1}-2^{s+1}\right)\right|^{p^-} dx$$

$$\leqslant C\left(\int_{\{u_n\geqslant 1\}} (u_n+1)^{(p^-s+\theta)m'} dx\right)^{\frac{1}{m'}} + C. \tag{38}$$

We remark that (36) is equivalent to require $(s+1)p^{-*} = (p^-s+\theta)\,m'$, by (38), we have

$$\left(\int_{\{u_n\geqslant 1\}} \left|(u_n+1)^{s+1}-2^{s+1}\right|^{p^{-*}} dx\right)^{\frac{p^-}{p^{-*}}} \leqslant C\left(\int_{\{u_n\geqslant 1\}} (u_n+1)^{(s+1)p^{-*}} dx\right)^{\frac{1}{m'}} + C.$$

Since $\frac{p-}{p-*} > \frac{1}{m'}$, due to the hypotheses on m and θ the previous inequality gives

$$\int_{\{u_n\geqslant 1\}} (u_n+1)^{(s+1)p^{-*}} dx \leqslant C. \tag{39}$$

By (39), and using the same techniques as in the proof of Lemma 3, we prove that (u_n) is uniformly bounded in $W_0^{1,q(x)}(\Omega)$. Here we consider the case $q^+ < \frac{mN(p^- - \theta)}{N-\theta m} < p^-$ and the opposite case. ■

Lemma 5. *Let m, p be restricted as in Theorem 3. Then, there exists a constant C independent on n, such that*

$$\|u_n\|_{W_0^{1,p(\cdot)}(\Omega)} \leqslant C. \tag{40}$$

Proof. Testing (20) with $\phi = (u_n+1)^\theta - 1$, using (3), (4), (21), $f_n \leqslant f$, and dropping the nonnegative first term, we obtain

$$B\int_\Omega \frac{u_n|\nabla u_n|^{p(x)}}{(u_n+\frac{1}{n})^{\theta+1}}(u_n+1)^\theta dx \leqslant C\int_\Omega f u_n^\theta dx + C. \tag{41}$$

By (41), and the fact that $\frac{1}{2}(u_n+\frac{1}{n})^{\theta+1} \leqslant u_n(u_n+1)^\theta$ on the set $A_1 = \{x\in\Omega\colon u_n(x) > 1\}$, we deduce that

$$\frac{B}{2}\int_{A_1} |\nabla u_n|^{p(x)} dx \leqslant C\int_{A_1} f(u_n-1)^\theta dx + C. \tag{42}$$

Using Sobolev's inequality (with exponent $p(x)$ on the left-hand side in (42)),

$$\begin{aligned}
\int_{A_1} |\nabla u_n|^{p(x)} dx &= \int_{A_1} |\nabla (u_n-1)|^{p(x)} dx \\
&\geqslant \|\nabla(u_n-1)\|_{L^{p(\cdot)}(A_1)} - 1 \\
&\geqslant C\|u_n-1\|_{L^{p^*(\cdot)}(A_1)} - C \\
&\geqslant C\min\left(\left(\rho_{p^*(\cdot),A_1}(u_n-1)\right)^{\frac{1}{(p^*)^+}}, \left(\rho_{p^*(\cdot),A_1}(u_n-1)\right)^{\frac{1}{(p^*)^-}}\right) - C \\
&\geqslant C\left(\int_{A_1} (u_n-1)^{p^*(x)}\right)^{\frac{1}{\tau}} dx - C,
\end{aligned} \tag{43}$$

where

$$\tau = \begin{cases} (p^*)^+, & \rho_{p^*(\cdot),A_1}(v) \geqslant 1, \\ (p^*)^-, & \rho_{p^*(\cdot),A_1}(v) \leqslant 1, \end{cases} \qquad v = u_n - 1.$$

By Hölder's inequality (on the right-hand side in (42)),

$$\int_{A_1} f(u_n-1)^\theta dx \leqslant C\|f\|_{L^{\frac{p^*(\cdot)}{p^*(\cdot)-\theta}}(A_1)} \|(u_n-1)^\theta\|_{L^{\frac{p^*(\cdot)}{\theta}}(A_1)}$$

$$\leqslant C\|f\|_{L^{m(\cdot)}(A_1)} \max\left((\rho_{\frac{p^*(\cdot)}{\theta},A_1}(u_n-1)^\theta)^{\frac{1}{(\frac{p^*}{\theta})^+}}, (\rho_{\frac{p^*(\cdot)}{\theta},A_1}(u_n-1)^\theta)^{\frac{1}{(\frac{p^*}{\theta})^-}} \right)$$

$$\leqslant C\left(\int_{A_1}(u_n-1)^{p^*(x)}dx\right)^{\frac{1}{\tau_1}}, \tag{44}$$

where

$$\tau_1 = \begin{cases} (\frac{p^*}{\theta})^+, & \rho_{\frac{p^*(\cdot)}{\theta},A_1}(v) \geqslant 1, \\ (\frac{p^*}{\theta})^-, & \rho_{\frac{p^*(\cdot)}{\theta},A_1}(v) \leqslant 1, \end{cases} \qquad v = (u_n-1)^\theta.$$

Using (42)–(44) and that $\frac{1}{\tau} > \frac{1}{\tau_1}$, it flows that

$$\int_{A_1}(u_n-1)^{p^*(x)}dx \leqslant C, \tag{45}$$

inequality (42) and (45) imply

$$\int_{A_1}|\nabla u_n|^{p(x)}dx \leqslant C. \tag{46}$$

Taking $T_1(u_n)$ as a test function in (20), using (3), (4), $f_n \leqslant f$ and dropping the non-negative lower order term, we get

$$\int_{\{u_n\geqslant 1\}}|\nabla T_1(u_n)|^{p(x)} \leqslant C2^{\gamma_+^+}. \tag{47}$$

Thus, by (46) and (47) the statement of the Lemma 5 is proved. ■

5. Proof of theorems

Because the proofs of Theorem 3 and 2 are similar to that of Theorem 1, here we only give the proof of Theorem 1. By Lemma 3, the sequence $(u_n)_n$ is bounded in $W_0^{1,\eta(\cdot)}(\Omega)$. Therefore, there exists a function $u \in W_0^{1,\eta(\cdot)}(\Omega)$ such that (up to a subsequence)

$$\begin{cases} u_n \rightharpoonup u \ \text{ in } \ W_0^{1,\eta(\cdot)}(\Omega), \\ u_n \to u \ \text{ a.e. in } \ \Omega. \end{cases} \tag{48}$$

Now, we are going to prove

$$\nabla u_n \to \nabla u \ \text{ a.e. in } \ \Omega. \tag{49}$$

Let $h,k>0$. We use $T_h(u_n - T_k(u))$ as a test function in (20), by hypothesis (3) and estimate (21),

$$\int_\Omega \frac{\alpha}{(1+T_n(u_n))^{\gamma(x)}}\widehat{a}(x,u_n,\nabla u_n)\nabla T_h(u_n-T_k(u))dx \leqslant Ch. \tag{50}$$

This gives, by $\frac{1}{(1+u_n)^{\gamma(x)}} \leqslant \frac{1}{(1+T_n(u_n))^{\gamma(x)}}$, (6), and (50),

$$\begin{aligned} \int_{A_{k,h}^n}\frac{|\nabla(u_n-T_k(u))|^{p(x)}}{(1+u_n)^{\gamma(x)}}dx &\leqslant \int_{A_{k,h}^n} C\frac{(\widehat{a}(x,u_n,\nabla u_n)-\widehat{a}(x,u_n,\nabla T_k(u)))\nabla T_h(u_n-T_k(u))}{(1+T_n(u_n))^{\gamma(x)}}dx \\ &\leqslant C\frac{Ch}{\alpha} - C\int_{A_{k,h}^n}\frac{\widehat{a}(x,u_n,\nabla T_k(u))\nabla T_h(u_n-u)}{(1+T_n(u_n))^{\gamma(x)}}dx \\ &= C\frac{Ch}{\alpha} - C\int_{A_{k,h}^n}\frac{\widehat{a}(x,T_{h+k}(u_n),\nabla T_k(u))\nabla T_h(u_n-u)}{(1+T_n(u_n))^{\gamma(x)}}dx, \end{aligned} \tag{51}$$

where $A_{k,h}^n = \{|u_n - T_k(u)| \leqslant h, |u| \leqslant k\}$. Combining (6), (23), (48), we obtain

$$\begin{cases} \nabla T_h(u_n-u) \rightharpoonup 0 \ \text{ in } \ L^{p(\cdot)}(\Omega), \\ \widehat{a}(x,T_{k+h}(u_n),\nabla T_k(u)) \to \widehat{a}(x,u,\nabla T_k(u)) \ \text{ in } \ L^{p'(\cdot)}(\Omega). \end{cases} \tag{52}$$

According (51), (52) and passing to the limit for $n\to+\infty$, we get

$$\limsup_{n\to+\infty}\int_{A^n_{k,h}} |\nabla(u_n-T_k(u))|^{p(x)} \leqslant Ch(1+k+h)^{\gamma^+}. \tag{53}$$

Let now $\tau(x)$ be such that $1<\tau(x)<\eta(x)<p(x)$. It is clear that

$$\int_\Omega |\nabla(u_n-u)|^{\tau(x)} \leqslant \int_{B^n_{k,h}} |\nabla(u_n-u)|^{\tau(x)} + \int_{C^n_{k,h}} |\nabla(u_n-u)|^{\tau(x)} + \int_{D^n_h} |\nabla(u_n-u)|^{\tau(x)},$$

where $B^n_{k,h}=\{|u_n-u|\leqslant h, |u|\leqslant k\}$, $C^n_{k,h}=\{|u_n-u|\leqslant h, |u|>k\}$ and $D^n_h=\{|u_n-u|>h\}$. By Hölder's inequality, and that u_n is uniformly bounded in $W^{1,\eta(\cdot)}_0(\Omega)$, we can write

$$\begin{aligned}\int_\Omega |\nabla(u_n-u)|^{\tau(x)}dx \leqslant C\max\left(\left(\int_{B^n_{k,h}}|\nabla(u_n-u)|^{p(x)}dx\right)^{\frac{1}{(\frac{p}{\tau})^+}}, \left(\int_{B^n_{k,h}}|\nabla(u_n-u)|^{p(x)}dx\right)^{\frac{1}{(\frac{p}{\tau})^-}}\right) \\ + C\max\left(\mu\left(\{|u|>k\}\right)^{\frac{1}{(\frac{\eta}{\eta-\tau})^+}}, \mu(\{|u|>k\})^{\frac{1}{(\frac{\eta}{\eta-\tau})^-}}\right) \\ + C\max\left(\mu(\{|u_n-u|>h\})^{\frac{1}{(\frac{\eta}{\eta-\tau})^+}}, \mu(\{|u_n-u|>h\})^{\frac{1}{(\frac{\eta}{\eta-\tau})^-}}\right).\end{aligned} \tag{54}$$

Thus, we deduce from (53) and (54), that

$$\limsup_{h\to0}\limsup_{n\to\infty}\int_\Omega |\nabla(u_n-u)|^{\tau(x)}dx \leqslant C\max\left(\mu(\{|u|>k\})^{\frac{1}{(\frac{\eta}{\eta-\tau})^+}}, \mu(\{|u|>k\})^{\frac{1}{(\frac{\eta}{\eta-\tau})^-}}\right). \tag{55}$$

At the limit as $k\to+\infty$, $\mu(\{|u|>k\})$ converges to 0. Therefore (up to subsequences), $\nabla u_n\to\nabla u$ a.e. in Ω.

Now, we are going to prove the strict positivity of the weak limit u of the sequence of approximated solutions u_n.

Lemma 6. *Let $0<\theta<1$. Let u_n and u be as in* (49). *Then $u>0$.*

Proof. Let $\phi\in W^{1,p(\cdot)}_0(\Omega)\cap L^\infty(\Omega)$ is a positive function. We choose $e^{-BH_n(u_n)}\phi$ (H_n given by (16)) as a test function in (20), we get

$$\begin{aligned} -B\int_\Omega a(x,T_n(u_n))\widehat{a}(x,u_n,\nabla u_n)\nabla u_n\frac{u_n(1+T_n(u_n))^{\gamma^-}}{\alpha\left(u_n+\frac1n\right)^{\theta+1}}e^{-BH_n(u_n)}\phi dx \\ +\int_\Omega a(x,T_n(u_n))\widehat{a}(x,u_n,\nabla u_n)e^{-BH_n(u_n)}\nabla\phi dx \\ +B\int_\Omega \frac{u_n|\nabla u_n|^{p(x)}}{\left(u_n+\frac1n\right)^{\theta+1}}e^{-BH_n(u_n)}\phi dx = \int_\Omega f_n e^{-BH_n(u_n)}\phi dx.\end{aligned}$$

Using hypothesis (3), (4), $f_n\geqslant T_1(f)$, $\forall n\geqslant1$,

$$\beta\int_\Omega \widehat{a}(x,u_n,\nabla u_n)\cdot\nabla\phi e^{-BH_n(u_n)}dx \geqslant \int_\Omega T_1(f)e^{-BH_n(u_n)}\phi\,dx. \tag{56}$$

Let φ in $W^{1,p(\cdot)}_0(\Omega)\cap L^\infty(\Omega)$, with $\varphi\geqslant0$. Taking $\phi=\psi_\lambda(u_n)\varphi$ (ψ_λ given by (18)) in (56), by using (4) and that $\psi'_\lambda(t)\leqslant0$, we obtain

$$\int_\Omega T_1(f)\varphi\psi_\lambda(u_n)e^{-BH_n(u_n)}\chi_{\{0\leqslant u_n\leqslant\lambda+1\}}dx \leqslant \beta\int_\Omega \widehat{a}(x,u_n,\nabla u_n)\cdot\nabla\varphi\psi_\lambda(u_n)e^{-BH_n(u_n)}\chi_{\{0\leqslant u_n\leqslant\lambda+1\}}dx. \tag{57}$$

By (57) and that $\psi_\lambda(u_n)\chi_{\{0\leqslant u_n\leqslant\lambda+1\}}\to\chi_{\{0\leqslant u_n\leqslant1\}}$ as $\lambda\to0$, we have

$$\beta \int_{\{0\leqslant u_n\leqslant 1\}} \widehat{a}(x,T_1(u_n),\nabla T_1(u_n))\cdot\nabla\varphi e^{-BH_n(T_1(u_n))}dx \geqslant \int_{\{0\leqslant u_n\leqslant 1\}} T_1(f)\varphi e^{-BH_n(T_1(u_n))}dx. \quad (58)$$

According to (5), (30), (48), and (49), yielding

$$\widehat{a}(x,T_1(u_n),\nabla T_1(u_n)) \rightharpoonup \widehat{a}(x,T_1(u),\nabla T_1(u)) \quad \text{in} \quad L^{p'(\cdot)}(\Omega). \quad (59)$$

Now, we pass to the limit as $n\to+\infty$ in (58), we deduce from (59) that

$$\int_\Omega \widehat{a}(x,T_1(u),\nabla T_1(u))\cdot\nabla\varphi e^{-BH_\infty(T_1(u))}dx \geqslant \frac{1}{\beta}\int_{\{0\leqslant u\leqslant 1\}} T_1(f)\varphi e^{-BH_\infty(T_1(u))}dx, \quad (60)$$

for all φ in $W_0^{1,p(\cdot)}(\Omega)\cap L^\infty(\Omega)$, with $\varphi\geqslant 0$, and then, by density, for every nonnegative φ in $W_0^{1,p(\cdot)}(\Omega)$. Now, we define the function

$$P(t)=\int_0^t e^{-BH_\infty(s)}ds,$$

the inequality (60) is equivalent to

$$\int_\Omega M(x,v,\nabla v)\cdot\nabla\varphi dx \geqslant \frac{1}{\beta}\int_\Omega g(x)\varphi dx,$$

where

$$\begin{cases} M(x,s,\xi)=e^{-BH_\infty(T_1(u))}\widehat{a}\left(x, sT_1(u)\left[\int_0^{T_1(u)} e^{-BH_\infty(s)}ds\right]^{-1}, \frac{\xi}{e^{-BH_\infty(T_1(u))}}\right), \\ g(x)=\frac{1}{\beta}T_1(f)e^{-BH_\infty(1)}\chi_{\{0\leqslant u(x)\leqslant 1\}}, \quad v=P(T_1(u)). \end{cases}$$

The comparison principle in $W_0^{1,p(\cdot)}(\Omega)$ says that $v(x)\geqslant z(x)$ (see [22]), where z is the bounded weak solution of

$$\begin{cases} z\in W_0^{1,p(\cdot)}(\Omega), \\ -\operatorname{div}(M(x,z,\nabla z))=g(x). \end{cases}$$

Indeed, by using (4)–(6), M satisfies, for almost every $x\in\Omega$, for every ξ,ξ' in $\mathbb{R}^N$

$$\begin{aligned} &M(x,s,\xi)\cdot\xi\geqslant|\xi|^{p(x)}, \\ &|M(x,s,\xi)|\leqslant C_1|s|^{p(x)-1}+C_2|\xi|^{p(x)-1}+C_3, \\ &(\widehat{a}(x,s,\xi)-\widehat{a}(x,s,\xi'))(\xi-\xi')\geqslant 0. \end{aligned}$$

Since g is nonnegative and not identically zero, and $p(x)$ verifies (2), (9). Then, the weak Harnack inequality (see [23]) yields $z>0$ in Ω and so $v>0$. Since $T_1(u)\geqslant v$, we conclude that $T_1(u)>0$ in Ω, which then implies that $u>0$ in Ω. ■

Corollary 1. *Let $0<\theta<1$. We have $\frac{|\nabla u|^{p(x)}}{u^\theta}\in L^1(\Omega)$.*

Proof. In fact, by passing to the limit in (21), we deduce from (48), Lemma 6, and Fatou's lemma, that

$$B\int_\Omega \frac{|\nabla u|^{p(x)}}{u^\theta}dx \leqslant \int_\Omega f dx. \quad (61)$$

■

Passage to the limit. We consider $v=e^{-H_{\frac{1}{n}}(u_n)}e^{H_{\frac{1}{j}}(T_j(u))}R_k(u_n)\varphi$ ($H_{\frac{1}{n}}$ and R_k given by (17) and (19)), where $j\in\mathbb{N}$ and φ is a positive $C_0^1(\Omega)$ function, as a test function in (20), then

$$\int_\Omega a(x,T_n(u_n))\widehat{a}(x,T_{k+1}(u_n),\nabla T_{k+1}(u_n))\cdot\nabla\varphi e^{-H_{\frac{1}{n}}(u_n)}e^{H_{\frac{1}{j}}(T_j(u))}R_k(u_n)\,dx$$

$$+\frac{B}{\alpha}\int_\Omega a(x,T_n(u_n))\widehat{a}(x,T_{k+1}(u_n),\nabla T_{k+1}(u_n))\nabla T_j(u)\frac{(1+T_j(u))^{\gamma^-}}{(T_j(u)+\frac{1}{j})^\theta}$$

$$
\begin{aligned}
&\times e^{-H_{\frac{1}{n}}(u_n)} e^{H_{\frac{1}{j}}(T_j(u))} R_k(u_n)\varphi\,dx \\
&= \frac{B}{\alpha}\int_\Omega \left[a(x,T_n(u_n))\widehat{a}(x,T_{k+1}(u_n),\nabla T_{k+1}(u_n))\cdot\nabla u_n \frac{(1+u_n)^{\gamma^-}}{(u_n+\frac{1}{n})^\theta} - \alpha\frac{u_n|\nabla u_n|^{p(x)}}{(u_n+\frac{1}{n})^{\theta+1}}\right] \\
&\quad\times e^{-H_{\frac{1}{n}}(u_n)} e^{H_{\frac{1}{j}}(T_j(u))} R_k(u_n)\varphi\,dx + \int_\Omega T_n(f) e^{-H_{\frac{1}{n}}(u_n)} e^{H_{\frac{1}{j}}(T_j(u))} R_k(u_n)\varphi\,dx \\
&\quad - \int_\Omega a(x,T_n(u_n))\widehat{a}(x,u_n,\nabla u_n)\cdot\nabla u_n e^{-H_{\frac{1}{n}}(u_n)} e^{H_{\frac{1}{j}}(T_j(u))} R'_k(u_n)\varphi\,dx. \qquad (62)
\end{aligned}
$$

Using (3), (4) and that $R'_k(s) \leqslant 0$, we have

$$
-\int_\Omega a(x,T_n(u_n))\widehat{a}(x,u_n,\nabla u_n)\cdot\nabla u_n e^{-H_{\frac{1}{n}}(u_n)} e^{H_{\frac{1}{j}}(T_j(u))} R'_k(u_n)\varphi\,dx \geqslant 0. \qquad (63)
$$

Combining (3), (4),

$$
\left[a(x,T_n(u_n))\widehat{a}(x,u_n,\nabla u_n)\cdot\nabla u_n \frac{(1+u_n)^{\gamma^-}}{(u_n+\frac{1}{n})^\theta} - \alpha\frac{u_n|\nabla u_n|^{p(x)}}{(u_n+\frac{1}{n})^{\theta+1}}\right] \geqslant 0. \qquad (64)
$$

Letting $n\to+\infty$, by using (30), (62), (63), (64), and Fatou's lemma,

$$
\begin{aligned}
&\int_\Omega a(x,u)\widehat{a}(x,u,\nabla u)\cdot\nabla\varphi e^{-H_0(u)} e^{H_{\frac{1}{j}}(T_j(u))} R_k(u)\,dx \\
&\qquad + \frac{B}{\alpha}\int_\Omega a(x,u)\widehat{a}(x,u,\nabla u)\nabla T_j(u)\frac{(1+T_j(u))^{\gamma^-}}{(T_j(u)+\frac{1}{j})^\theta} e^{-H_0(u)} e^{H_{\frac{1}{j}}(T_j(u))} R_k(u)\varphi\,dx \\
&\qquad \geqslant \frac{B}{\alpha}\int_\Omega a(x,u)\widehat{a}(x,u,\nabla u)\cdot\nabla u\frac{(1+u)^{\gamma^-}}{u^\theta} e^{-H_0(u)} e^{H_{\frac{1}{j}}(T_j(u))} R_k(u)\varphi\,dx \\
&\qquad - B\int_\Omega \frac{|\nabla u|^{p(x)}}{u^\theta} e^{-H_0(u)} e^{H_{\frac{1}{j}}(T_j(u))} R_k(u)\varphi\,dx + \int_\Omega f e^{-H_0(u)} e^{H_{\frac{1}{j}}(T_j(u))} R_k(u)\varphi\,dx. \qquad (65)
\end{aligned}
$$

Let $j > k+1$. Using (3), (5), (61) and $R_k(u) = 0$ on $\{u > k+1\}$,

$$
\begin{aligned}
&\left|a(x,u)\widehat{a}(x,u,\nabla u)\nabla T_j(u)\frac{(1+T_j(u))^{\gamma^-}}{(T_j(u)+\frac{1}{j})^\theta} e^{-H_0(u)} e^{H_{\frac{1}{j}}(T_j(u))} R_k(u)\varphi\right| \\
&\leqslant \beta\left(C|u|^{\frac{\theta(p(x)-1)}{p(x)}} + C'|\nabla u|^{p(x)-1}\right)|\nabla u|\frac{(1+u)^{\gamma^-}}{(u+\frac{1}{j})^\theta} R_k(u)\varphi \\
&\leqslant \beta\left(C|u|^\theta + C'|\nabla u|^{p(x)-1}\right)|\nabla u|\frac{(1+u)^{\gamma^-}}{u^\theta} R_k(u)\varphi \\
&\leqslant \beta\left(C|\nabla u| + C'\frac{|\nabla u|^{p(x)}}{u^\theta}\right)(1+u)^{\gamma^-} R_k(u)\varphi \in L^1(\Omega).
\end{aligned} \qquad (66)
$$

We pass to the limit as $j\to\infty$ in (65), using that $e^{-H_0(u)} e^{H_{\frac{1}{j}}(T_j(u))} \leqslant 1$ (since $H_{\frac{1}{j}}(T_j(u)) \leqslant H_{\frac{1}{j}}(u) \leqslant H_0(u)$), (66) and Lebesgue's theorem, and then to the limit as $k\to+\infty$. We obtain

$$
\int_\Omega a(x,u)\widehat{a}(x,u,\nabla u)\nabla\varphi\,dx + B\int_\Omega \frac{|\nabla u|^{p(x)}}{u^\theta}\varphi\,dx \geqslant \int_\Omega f\varphi\,dx. \qquad (67)
$$

To prove the opposite inequality, we choose $\varphi\in C_0^1(\Omega)$ with $\varphi\geqslant 0$, as test function in (20), to obtain

$$
\int_\Omega a(x,T_n(u_n))\widehat{a}(x,u_n,\nabla u_n)\nabla\varphi\,dx + B\int_\Omega \frac{u_n|\nabla u_n|^{p(x)}}{(u_n+\frac{1}{n})^{\theta+1}}\varphi\,dx = \int_\Omega f_n\varphi\,dx. \qquad (68)
$$

From (5), (48), (49), Lemma 3,

$$\widehat{a}(x,u_n,\nabla u_n)\to\widehat{a}(x,u,\nabla u)\quad\text{in}\quad L^{q(x)}(\Omega),\quad \forall q(x)\in\left(1,\frac{\eta(x)}{p(x)-1}\right). \tag{69}$$

Therefore (68), (69) and Fatou's lemma imply

$$\int_\Omega a(x,u)\widehat{a}(x,u,\nabla u)\nabla\varphi\,dx + B\int_\Omega \frac{|\nabla u|^{p(x)}}{u^\theta}\varphi\,dx \leqslant \int_\Omega f\varphi\,dx. \tag{70}$$

Combining (67) and (70), we deduce that

$$\int_\Omega a(x,u)\widehat{a}(x,u,\nabla u)\nabla\varphi\,dx + \int_\Omega \frac{|\nabla u|^{p(x)}}{u^\theta}\varphi\,dx = \int_\Omega f\varphi\,dx, \tag{71}$$

for every φ in $C_0^1(\Omega)$, with $\varphi\geqslant 0$.

Now, let φ any $C_0^1(\Omega)$ function and $\varepsilon>0$. We define $\varphi_\pm^\varepsilon=\rho^\varepsilon*\varphi_\pm$ as the convolution of a mollifier ρ^ε with $\varphi_\pm$. Then $\varphi_\pm^\varepsilon$ is a positive $C_0^1(\Omega)$ function, for ε sufficiently small. By (71), we have

$$\int_\Omega a(x,u)\widehat{a}(x,u,\nabla u)\nabla(\varphi_-^\varepsilon-\varphi_+^\varepsilon)\,dx + \int_\Omega \frac{|\nabla u|^{p(x)}}{u^\theta}(\varphi_-^\varepsilon-\varphi_+^\varepsilon)\,dx = \int_\Omega f(\varphi_-^\varepsilon-\varphi_+^\varepsilon)\,dx.$$

Since $\varphi_-^\varepsilon-\varphi_+^\varepsilon\to\varphi$ uniformly in Ω and in $W_0^{1,\eta(x)}(\Omega)$ for every $\varepsilon\to 0$, the results follows.

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Вироджена еліптична задача зі сингулярним градієнтом нижчого порядку та змінним показником

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У цій статті доводиться існування та регулярність слабких розв'язків для класу нелінійних еліптичних рівнянь із виродженою коерцитивною силою та сингулярними членами нижчого порядку з природним зростанням відносно за градієнтом і $L^{m(\cdot)}$ $(m(x) \geqslant 1)$ даними. Функціональна постановка включає простори Лебега–Соболева зі змінними показниками.

Ключові слова: *вироджена задача; сингулярний член; регулярний розв'язок; принцип порівняння; нерівність Гарнака.*