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Discrete mathematical modeling and optimal control of the marital status: Islamic polygamous marriage model case

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In this paper, we discuss a discrete mathematical model of Islamic polygamy and the social position of Muslims. In eleven compartments we explain the social situation and give an explanation of the marital status of each Males and females in Islamic societies that allow polygamy. In order to controlling and reducing the number of virgins men and women, divorced men and women we implement two control variables. The first control characterizes the benefits of an awareness campaign to educate virgin men and women about the benefits marriage to the individual and society, and the second control is about the legal procedures, administrative complexities and the grave financial and social implications of divorce. After that, we applied the optimal control theory to describe such an optimal strategies and finally a numerical simulation was performed to verify the theoretical analysis using a progressive-regressive discrete schema that converges following a convenient test related to the Forward–Backward Sweep Method (FBSM).

Keywords: *discrete marital status model; polygamous marriage; Pontryagin's maximum principle; optimal control.*

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1. Introduction

Polygamy for humanity is a topic that has been found in many cultures and civilizations over the centuries and in our article we will focus on polygamy in Islam. Polygamy is a term in Islam that denotes that it is permissible for a man to have more than one wife in his relation marital, provided that they do not exceed four.

The issue of polygamy is always the subject of renewed controversy and disagreement within Islamic societies, between a defender who considers polygamy to be God's law that must be implemented and an opponent of the idea of polygamy as it negatively affects the wife family and children, as they put it. Opinions usually abound and vary about polygamy, which is the right that Islamic law has permitted for men and some religious scholars excluded it due to the difficulty of achieving its conditions mentioned in the Quran, while some Arab and Islamic countries criminalized it and others restricted it. In most Arab countries there are no conditions or restrictions on polygamy and the law does not deal with it, and Arab governments allow it in accordance with the principles and conditions of Islamic law only, including: justice, physical ability, and the number of wives not exceeding four per man.

Polygamy is legal in 58 of the 200 sovereign states, the vast majority of which are Muslim-majority states located in Africa and Asia. Some Arab and Islamic countries allow polygamy, but with strict conditions and restrictions. Some countries require a judge's permission, after he verifies the conditions set by the state to approve polygamy. As for Morocco, it sets harsh conditions against a man married to another in its law issued since 2003, a law that has succeeded in reducing polygamy, to less than 1000 cases annually. However, in some Islamic countries, such in other countries, polygamy is prohibited on their lands and criminalized, and it is punishable by imprisonment for a term, a fine, or one of two penalties. A large number of Western countries prohibit polygamy. In some of them, prison sentences are imposed on the husband whose marriage to another is proven, and this prohibition applies to Muslim immigrants as well. The husband who combines two wives is considered to have committed a crime in the law at the proportions of the states, in the event that the first does not divorce, and the courts even obligate him to pay a fine, in addition to imprisonment, in addition to the first wife obtaining the right of custody of the children, and the state also provides her with social assistance. As for other countries, they have enacted a law that applies against refugee Muslim preachers and imams, in case they are proven to incite polygamy among Muslims.

Recently, the controversy has renewed in Morocco about polygamy, as a result of the country's recording of an increase in the rate of polygamy, nearly 20 years after the adoption of the Family Code. In [1] the latest reports of the Central Agency for Public Mobilization and Statistics revealed the size of the phenomenon of polygamy recently. Where it was found that the number of marriage contracts for previously married people amounted to 1128.7 thousand contracts during 2019, including 4 percent of marriage contracts for a man who has more than one wife, and the statistics report indicates that the number of marriage contracts for those who have more than one wife. The number of marriage contracts between two wives reached 41.2 thousand contracts during the period, in addition to 2363 marriage contracts for the husband with 3 wives, and 390 marriage contracts for the one who has 4 wives.

In recent years, a great deal of reluctance to marry has been observed. Divorce has also increased, and this is an unusual phenomenon. For this reason, we propose in our article a mathematical model of the relationships between the population dynamics of the age group over eighteen years. We are interested in changes in the rates of marriage and divorce, and then we try to create functions to control and reduce the number of spinsters and reduce divorce rates.

In many studies, topics close to the topic of polygamy were discussed, in other fields, the topic of polygamy was discussed in statistics, religion, and law [2–7]. M. Lhous in [8] have discussed Discrete mathematical modeling and optimal control of the marital status for the monogamous marriage case. And in [9] M. Lhous have develop a new mathematical model to study, analyze, and control the family status in several regions and the impact of the connectivity of regions and the mobility of residents on the marital status of the family, by adopting a multi-region discrete-time model. The civil status of a person in islamic family and society can be classified for women into one of four categories: virgin, married, divorced or widowed. For males in seven compartments virgin, married (by one, two, three or four wives), divorced or widowed. The model VMDW to be studied classifies the marital status of the family dynamics of a population into eleven compartments: virgin men V^M, virgin women V^W, married men by one wife M^{M_1}, married men by two wives M^{M_2}, married men by three wives M^{M_3}, married men by four wives M^{M_4}, married women M^W, divorced men D^M, divorced women D^W, widowed men W^M and widowed women W^W.

For our mathematical system, there are two signs of delay functions that we will include in the equations, they will be considered through the conditions that must be respected for marriage. For those who want to marry a widow, they must wait for a period of no less than four months and ten days after the death of the husband. As for those who want to marry a divorced woman, there are two cases. The first case is that if the divorced woman is not pregnant, she must wait four months and ten days. The pregnant woman, she must wait for her to give birth to the fetus or its miscarriage. These conditions are according to the teachings of the Islamic religion.

In this paper, we study an approach that determines an optimal control relative to a discrete marital status model which allows to inoculate controls, that allows to reduce the virgin individuals and to increase the number of married individuals with a minimal cost. The optimal control problem was subject of an optimization criterion represented by the minimization of an objective function. There are lot studies that explain the method of optimal control [1, 10–13]. The optimality system is solved based on an iterative discrete schema that converges following an appropriate test similar to one related to the Forward–Backward Sweep Method (FBSM).

The paper is organized as following: in Section 2, the model VMDW is described for the polygamies marriage case. In Section 3, we give some results concerning the existence of the optimal control and we use Pontryagin's maximum principle to study analysis of control strategies and to determine the necessary condition for the optimal control. Numerical simulations are given in Section 4. Finally, we conclude the paper in Section 5.

2. Model description

We used an adaptive mathematical model VMDW to describe social relations in Islamic societies between men and women (marriage, polygamy, divorce and widows).

The system of equations associated with the schematic diagram that governs the model are:

$$V_{i+1}^{M} = V_i^M - \frac{\alpha_1 V_i^W + \beta_1 D_{i-\tau_1}^W + \alpha_3 W_{i-\tau_2}^W}{N_i} V_i^M, \tag{1}$$

$$V_{i+1}^{W} = V_i^W - \frac{\gamma_1 M_i^{M_1} + \gamma_2 M_i^{M_2} + \gamma_3 M_i^{M_3}}{N_i} V_i^W - \frac{\alpha_1 V_i^M + \beta_2 D_i^M + \alpha_2 W_i^M}{N_i} V_i^W, \tag{2}$$

$$\begin{aligned} M_{i+1}^{M_1} = M_i^{M_1} &+ \frac{\alpha_1 V_i^W + \beta_1 D_{i-\tau_1}^W + \alpha_3 W_{i-\tau_2}^W}{N_i} V_i^M + \frac{\beta_2 V_i^W + \beta_3 D_{i-\tau_1}^W + \beta_5 W_{i-\tau_2}^W}{N_i} D_i^M \\ &+ \frac{\alpha_2 V_i^W + \beta_4 D_{i-\tau_1}^W + \alpha_4 W_{i-\tau_2}^W}{N_i} W_i^M - (\lambda_1 + \delta_1 + \mu_1) M_i^{M_1} + (\lambda_2 + \mu_2) M_i^{M_2}, \end{aligned} \tag{3}$$

$$M_{i+1}^{M_2} = M_i^{M_2} + \frac{\gamma_1 M_i^{M_1} V_i^W}{N_i} - (\lambda_2 + \delta_2 + \mu_2) M_i^{M_2} + (\lambda_3 + \mu_3) M_i^{M_3}, \tag{4}$$

$$M_{i+1}^{M_3} = M_i^{M_3} + \frac{\gamma_2 M_i^{M_2} V_i^W}{N_i} - (\lambda_3 + \delta_3 + \mu_3) M_i^{M_3} + (\lambda_4 + \mu_4) M_i^{M_4}, \tag{5}$$

$$M_{i+1}^{M_4} = M_i^{M_4} + \frac{\gamma_3 M_i^{M_3} V_i^W}{N_i} - (\lambda_4 + \delta_4 + \mu_4) M_i^{M_4}, \tag{6}$$

$$\begin{aligned} M_{i+1}^{W} = M_i^W &+ \frac{\alpha_1 V_i^W + \beta_1 D_{i-\tau_1}^W + \alpha_3 W_{i-\tau_2}^W}{N_i} V_i^M + \frac{\beta_2 V_i^W + \beta_3 D_{i-\tau_1}^W + \beta_5 W_{i-\tau_2}^W}{N_i} D_i^M \\ &+ \frac{\alpha_2 V_i^W + \beta_4 D_{i-\tau_1}^W + \alpha_4 W_{i-\tau_2}^W}{N_i} W_i^M + \frac{\gamma_1 M_i^{M_1} + \gamma_2 M_i^{M_2} + \gamma_3 M_i^{M_3}}{N_i} V_i^W \\ &- (\lambda_1 + \delta_1 + \mu_1) M_i^{M_1} - (\lambda_2 + \delta_2 + \mu_2) M_i^{M_2} - (\lambda_3 + \delta_3 + \mu_3) M_i^{M_3} \\ &- (\lambda_3 + \delta_3 + \mu_3) M_i^{M_3} - (\lambda_4 + \delta_4 + \mu_4) M_i^{M_4}, \end{aligned} \tag{7}$$

$$D_{i+1}^{M} = D_i^M - \frac{\beta_2 V_i^W + \beta_3 D_{i-\tau_1}^W + \beta_5 W_{i-\tau_2}^W}{N_i} D_i^M + \delta_1 M_i^{M_1}, \tag{8}$$

$$\begin{aligned} D_{i+1}^{W} = D_i^W &- \frac{\beta_1 V_i^M + \beta_3 D_i^M + \beta_4 W_i^M}{N_i} D_{i-\tau_1}^W + \delta_1 M_i^{M_1} + \delta_2 M_i^{M_2} \\ &+ \delta_3 M_i^{M_3} + \delta_4 M_i^{M_4}, \end{aligned} \tag{9}$$

$$W_{i+1}^{M} = W_i^M - \frac{\alpha_2 V_i^W + \beta_4 D_{i-\tau_1}^W + \alpha_4 W_{i-\tau_2}^W}{N_i} W_i^M + \mu_1 M_i^{M_1}, \tag{10}$$

$$\begin{aligned} W_{i+1}^{W} = W_i^W &- \frac{\alpha_3 V_i^M + \beta_5 D_i^M + \alpha_4 W_i^M}{N_i} W_{i-\tau_2}^W + \lambda_1 M_i^{M_1} + \lambda_2 M_i^{M_2} \\ &+ \lambda_3 M_i^{M_3} + \lambda_4 M_i^{M_4}. \end{aligned} \tag{11}$$

The application area Ω of this study represents a country, a city, a town, or a small domain.

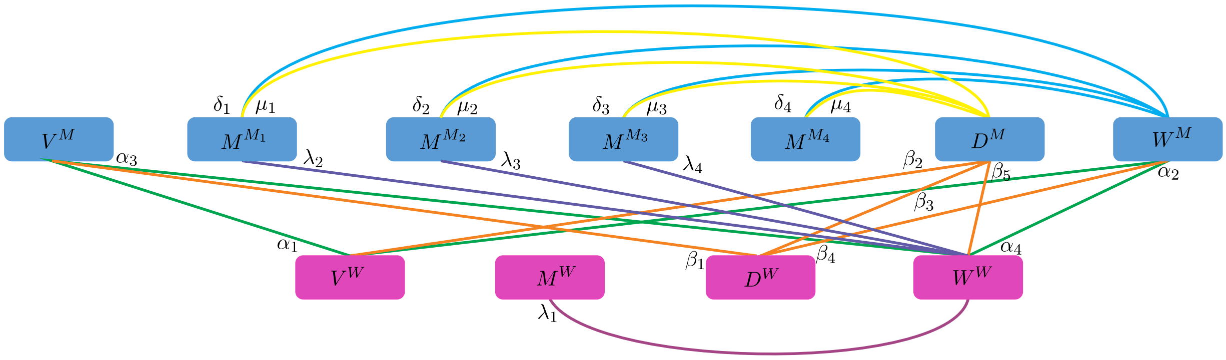

Fig. 1. The flow between the eight compartments VMDW.

The unit of i can correspond to days, months or years, it depends on the frequency of the survey and demographic studies as needed. However demographic statistics are generally done annually so the unit of i can be considered as years, where V_0^M, V_0^W, $M_0^{M_1}$, $M_0^{M_2}$, $M_0^{M_3}$, $M_0^{M_4}$, M_0^W, D_0^M, D_0^W, W_0^M, and W_0^W are the given initial state. In equations (1)–(11), all parameters are non-negative and defined in Table 1.

Table 1. The description of parameters used for the definition of discrete time systems (1)–(11).

Parameter	Description
α_1	Marriage rate of virgin man to a virgin woman
α_2	Marriage rate of widower man to a virgin woman
α_3	Marriage rate of virgin man to a widow
α_4	Marriage rate of widower man to a widow
β_1	Marriage rate of virgin man to a divorced woman
β_2	Marriage rate of divorced man to a virgin woman
β_3	Marriage rate of divorced man to a divorced woman
β_4	Marriage rate of widower man to a divorced woman
β_5	Marriage rate of divorced man to a widow
λ_1	rate of women who lost their husband for a man marries one woman
λ_2	rate of women who lost their husband for a man marries two women
λ_3	rate of women who lost their husband for a man marries three women
λ_4	rate of women who lost their husband for a man marries four women
δ_1	Divorce rate of married man
δ_2	Divorce rate of married men with two women
δ_3	Divorce rate of married men with three women
δ_4	Divorce rate of married men with four women
μ_1	rate of men married one woman who lost their wives
μ_2	rate of men married two women who lost their wives
μ_3	rate of men married three women who lost their wives
μ_4	rate of men married four women who lost their wives
γ_1	Marriage rate with the second
γ_2	Marriage rate with the third
γ_3	Marriage rate with the fourth
τ_1	The waiting period after a woman's divorce
τ_2	The waiting period for the widow

For the equation (1) of the model, virgin men can contact a virgin, divorced or widowed woman with an α_1, β_1, and α_3 rates respectively, and this contact can result in a marriage. Thus the number of virgin men decreases and the number of virgins at the instant i is substituted for the number $\alpha_1 \frac{V_i^M V_i^W}{N_i} + \beta_1 \frac{V_i^M D_{i-\tau_1}^W}{N_i} + \alpha_3 \frac{V_i^M W_{i-\tau_2}^W}{N_i}$ and this number is added to the number of men marrying with one, two and three woman at the time $i+1$.

Similarly in equation (2) the number of virgin women decreases at the instant $i+1$ by substituting the number of virgin women at the instant i the number $\alpha_1 \frac{V_i^M V_i^W}{N_i} + \beta_2 \frac{D_i^M V_i^W}{N_i} + \alpha_2 \frac{W_i^M V_i^W}{N_i}$, which represents the number of married women after the contact with a virgin, divorced or widowed woman with the α_1, β_2, and α_2 rates respectively.

In equation (8), a divorced man can contact a virgin, divorced or widowed woman with β_2, β_3, and β_5 rates respectively, so this contact can result in marriage. Then the number $\beta_2\frac{D_i^M V_i^W}{N_i}+\beta_3\frac{D_i^M D_{i-\tau_1}^W}{N_i}+\beta_5\frac{D_i^M W_{i-\tau_2}^W}{N_i}$ is decreased by the number of divorced men at the time $i+1$ and added to the number of married men with one, two and three women. In addition a married man with one woman, two women, three or four women can divorce with a rate of λ_1, λ_2, λ_3, and λ_4 respectively and as a result the number of divorced men with one woman, two women, three or four women increases from $\lambda_1 M_i^{M_1}$, $\lambda_2 M_i^{M_2}$, $\lambda_3 M_i^{M_3}$, and $\lambda_4 M_i^{M_4}$ respectively. The same principle applies to equations (9), (10) and (11).

For the equation (3) the number of married men increases at the instant $i+1$ by the number of virgin, divorced, and widowed men who are married by contacting virgin, divorced or widowed women and decreases the natural mortality with a δ_1 and divorce rate with a λ_1 rate. The same principle can be applied to equations (7).

In the equation (4) the number of married men with the second woman increases at the instant $i+1$ by the number of virgin, married men with one woman, divorced and widowed men who are married by contacting virgin, divorced or widowed women and decreases the natural mortality with δ_2 and divorce rate with λ_2 rate. The same principle can be applied to equations (5) and (6).

3. An optimal control problem

We also know that Morocco and many countries are still suffering at this moment from the phenomenon of reluctance to marry from both parties, young men and girls, and divorce has increased. For this we proposed a control variable (u_i, v_i) that characterizes the benefits of an awareness campaign to educate virgin men and women about the benefits of marriage for the individual and for the society, especially the legal procedures, administrative complications, the heavy financial and social consequences of divorces respectively in the above mentioned model (1)–(11). So, the mathematical system with time delay in state and control system of variables is given by the nonlinear retarded system of differential equations:

$$V_{i+1}^M = V_i^M - \frac{\alpha_1 V_i^W + \beta_1 D_{i-\tau_1}^W + \alpha_3 W_{i-\tau_2}^W}{N_i} V_i^M - u_i V_i^M, \tag{12}$$

$$V_{i+1}^W = V_i^W - \frac{\gamma_1 M_i^{M_1} + \gamma_2 M_i^{M_2} + \gamma_3 M_i^{M_3}}{N_i} V_i^W - \frac{\alpha_1 V_i^M + \beta_2 D_i^M + \alpha_2 W_i^M}{N_i} V_i^W - u_i V_i^W, \tag{13}$$

$$\begin{aligned} M_{i+1}^{M_1} = M_i^{M_1} &+ \frac{\alpha_1 V_i^W + \beta_1 D_{i-\tau_1}^W + \alpha_3 W_{i-\tau_2}^W}{N_i} V_i^M + \frac{\beta_2 V_i^W + \beta_3 D_{i-\tau_1}^W + \beta_5 W_{i-\tau_2}^W}{N_i} D_i^M \\ &+ \frac{\alpha_2 V_i^W + \beta_4 D_{i-\tau_1}^W + \alpha_4 W_{i-\tau_2}^W}{N_i} W_i^M - (\lambda_1+\delta_1+\mu_1) M_i^{M_1} + (\lambda_2+\mu_2) M_i^{M_2} \\ &+ u_i V_i^M + r_1 v_i D_i^M, \end{aligned} \tag{14}$$

$$M_{i+1}^{M_2} = M_i^{M_2} + \frac{\gamma_1 M_i^{M_1} V_i^W}{N_i} - (\lambda_2+\delta_2+\mu_2) M_i^{M_2} + (\lambda_3+\mu_3) M_i^{M_3} + r_2 v_i D_i^M, \tag{15}$$

$$M_{i+1}^{M_3} = M_i^{M_3} + \frac{\gamma_2 M_i^{M_2} V_i^W}{N_i} - (\lambda_3+\delta_3+\mu_3) M_i^{M_3} + (\lambda_4+\mu_4) M_i^{M_4} + r_3 v_i D_i^M, \tag{16}$$

$$M_{i+1}^{M_4} = M_i^{M_4} + \frac{\gamma_3 M_i^{M_3} V_i^W}{N_i} - (\lambda_4+\delta_4+\mu_4) M_i^{M_4} + r_4 v_i D_i^M, \tag{17}$$

$$\begin{aligned} M_{i+1}^{W} = M_i^{W} &+ \frac{\alpha_1 V_i^W + \beta_1 D_{i-\tau_1}^W + \alpha_3 W_{i-\tau_2}^W}{N_i} V_i^M + \frac{\beta_2 V_i^W + \beta_3 D_{i-\tau_1}^W + \beta_5 W_{i-\tau_2}^W}{N_i} D_i^M \\ &+ \frac{\alpha_2 V_i^W + \beta_4 D_{i-\tau_1}^W + \alpha_4 W_{i-\tau_2}^W}{N_i} W_i^M + \frac{\gamma_1 M_i^{M_1} + \gamma_2 M_i^{M_2} + \gamma_3 M_i^{M_3}}{N_i} V_i^W \end{aligned}$$

$$
\begin{aligned}
&- (\lambda_1+\delta_1+\mu_1)M_i^{M_1} - (\lambda_2+\delta_2+\mu_2)M_i^{M_2} - (\lambda_3+\delta_3+\mu_3)M_i^{M_3} \\
&- (\lambda_4+\delta_4+\mu_4)M_i^{M_4} + u_i V_i^W + v_i D_i^W, \qquad (18)
\end{aligned}
$$

$$D_{i+1}^M = D_i^M - \frac{\beta_2 V_i^W + \beta_3 D_{i-\tau_1}^W + \beta_5 W_{i-\tau_2}^W}{N_i} D_i^M + \delta_1 M_i^{M_1} - v_i D_i^M, \qquad (19)$$

$$
\begin{aligned}
D_{i+1}^W = D_i^W - \frac{\beta_1 V_i^M + \beta_3 D_i^M + \beta_4 W_i^M}{N_i} D_{i-\tau_1}^W + \delta_1 M_i^{M_1} + \delta_2 M_i^{M_2} + \delta_3 M_i^{M_3} \\
+ \delta_4 M_i^{M_4} - v_i D_i^W, \qquad (20)
\end{aligned}
$$

$$W_{i+1}^M = W_i^M - \frac{\alpha_2 V_i^W + \beta_4 D_{i-\tau_1}^W + \alpha_4 W_{i-\tau_2}^W}{N_i} W_i^M + \mu_1 M_i^{M_1}, \qquad (21)$$

$$
\begin{aligned}
W_{i+1}^W = W_i^W - \frac{\alpha_3 V_i^M + \beta_5 D_i^M + \alpha_4 W_i^M}{N_i} W_{i-\tau_2}^W + \lambda_1 M_i^{M_1} + \lambda_2 M_i^{M_2} \\
+ \lambda_3 M_i^{M_3} + \lambda_4 M_i^{M_4}. \qquad (22)
\end{aligned}
$$

The model (12)–(22) fulfils the constant population through time constraint, i.e.:

$$N_i = V_i^M + V_i^W + M_i^{M_1} + M_i^{M_2} + M_i^{M_3} + M_i^{M_4} + M_i^W + D_i^M + D_i^W + W_i^M + W_i^W = N > 0.$$

And $r_1 + r_2 + r_3 + r_4 = 1$.

After applying the control u_i to reduce the number of divorced men we have r_1 represent percentage of divorcees who remarried, r_2 mean the percentage of men with two wives who divorced one in order to remarry and remarry the second, r_3 is the percentage of men with three wives who had divorced one in order to remarry and remarry the third and r_4 is the percentage of men who have four wives who have divorced one of the wives to remarry the fourth wife.

Note this assertion proves that the constant population through time is independent of the control strategy.

Characterization of the optimal control

For an initial state $(V_0^M, V_0^W, M_0^{M_1}, M_0^{M_2}, M_0^{M_3}, M_0^{M_4}, M_0^W, D_0^M, D_0^W, W_0^M, W_0^W)$, we consider an optimization criterion defined by the following objective function

$$J(u,v) = \sum_{i=0}^{N} (A_1 V_i^W + A_2 D_i^W - A_3 M_i^W) + \sum_{i=0}^{N-1} \left(\frac{\tau_1'}{2}(u_i)^2 + \frac{\tau_2'}{2}(v_i)^2 \right) \qquad (23)$$

subject to system (12)–(22) here A_1, A_2 and A_3 are positive constants to keep a balance in the size of V_i^W, D_i^W and M_i^W, respectively. In the objective functional, τ_1' and τ_2' are the positive weight parameters which are associated with the controls u_i and v_i.

In other words, we seek the optimal controls (u^*, v^*) such that

$$J(u^*, v^*) = \min\{J(u,v) \mid (u,v) \in \mathcal{U}_{ad}\}, \qquad (24)$$

where $\mathcal{U}_{ad}$ is the set of admissible controls defined by

$$\mathcal{U}_{ad} = \{(u,v) \mid u^{\min} \leqslant u_i \leqslant u^{\max}, v^{\min} \leqslant v_i \leqslant v^{\max}, i \in \{0,\dots,N-1\}\},$$

where $(u^{\min}, u^{\max}, v^{\min}, v^{\max}) \in]0,1[^4$.

The sufficient condition for existence of optimal control (u^*, v^*) for the problem (23) coming from the following theorem.

Theorem 1. *There exist an optimal control (u^*, v^*) such that*

$$J(u^*, v^*) = \min_{(u,v)\in\mathcal{U}_{ad}} J(u,v)$$

subject to the control system (12)–(22) with initial conditions.

Proof. There is a finite number of time steps, $V^M = (V_0^M, V_1^M, \dots, V_N^M)$, $V^W = (V_0^W, V_1^W, \dots, V_N^W)$, $\dots$, $W^W = (W_0^W, W_1^W, \dots, W_N^W)$ are uniformly bounded for all $(u,v) \in \mathcal{U}_{ad}$. Thus $J(u,v)$ is uniformly bounded for all (u,v) in the control set $\mathcal{U}_{ad}$. Since $J(u,v)$ is bounded, $\inf\limits_{(u,v)\in\mathcal{U}_{ad}} J(u,v)$ is finite, and there exist a sequence $(u^j,v^j)\in\mathcal{U}_{ad}$ such that $\lim\limits_{j\longrightarrow\infty} J(u^j,v^j)=\inf\limits_{(u,v)\in\mathcal{U}_{ad}} J(u,v)$ and corresponding sequences of states V^{Mj}, V^{Wj}, $\ldots$, W^{Wj}. Since there is a finite number of uniformly bounded sequences, there exists $(u^*,v^*)\in\mathcal{U}_{ad}$ and V^{Mj}, $\ldots$, $W^{Wj}\in\mathbb{R}^{N+1}$ such that on a subsequence, $u^j\longrightarrow u^*, v^j\longrightarrow v^*, V^{Mj}\longrightarrow V^{M^*},\ldots,W^{Wj}\longrightarrow W^{W^*}$. Finally, due to the finite dimensional structure of the system (12)–(22) and the objective function $J(u,v)$, (u^*,v^*) is an optimal control with corresponding states V^{M^*}, V^{W^*}, $\ldots$, W^{W^*}. Therefore $\inf\limits_{(u,v)\in\mathcal{U}_{ad}} J(u,v)$ is achieved. ■

In the following, we shall use the placeholder variables y_1 and y_2 for the retarded state variable, i.e. $y_1=D^W_{i-\tau_1}$ and $y_2=W^W_{i-\tau_2}$.

In order to find an optimal solution, first we find the Hamiltonian for the optimal control problem (24). In fact, the Hamiltonian H of the optimal problem is given by

$$\begin{aligned} H &= A_1V_i^W+A_2D_i^W-A_3M_i^W+\frac{\tau_1'}{2}u_i^2+\frac{\tau_2'}{2}v_i^2\\ &\qquad +\sum_{k=1}^{11}\zeta_{k,i+1}f_k\big(V_k^M,V_k^W,M_k^{M_1},M_k^{M_2},M_k^{M_3},M_k^{M_4},M_k^W,D_k^M,D_k^W,W_k^M,W_k^W,y_1,y_2\big).\end{aligned}$$

Where ζ_1, ζ_2, $\ldots$, ζ_8 are the adjoint functions to be determined suitably.

And f_k is the right side of the difference equation of the k^{th} state variable at the time step.

At the same time by using Pontryagin's maximum principle [14], we derive necessary conditions for our optimal control. We obtain the following theorem.

Theorem 2 (Necessary Conditions). *Let V^{M^*}, V^{W^*}, M^{M_1*}, M^{M_2*}, M^{M_3*}, M^{M_4*}, M^{W^*}, D^{M^*}, D^{W^*}, W^{M^*} and W^{W^*} be optimal state solutions with associated optimal control (u^*,v^*) for the optimal control problem (24). Then, there exist adjoint variables ζ_1, ζ_2, $\ldots$, ζ_8, that satisfy*

$$\begin{aligned}\Delta\zeta_{1,i} = -\bigg[&\zeta_{1,i+1}+\frac{\alpha_1V_i^W}{N_i}(\zeta_{1,i+1}+\zeta_{2,i+1}-\zeta_{3,i+1}-\zeta_{7,i+1})\\ &+\frac{\alpha_3W^W_{i-\tau_2}}{N_i}(\zeta_{1,i+1}-\zeta_{3,i+1}-\zeta_{7,i+1}+\zeta_{11,i+1})\\ &+\frac{\beta_1D^W_{i-\tau_1}}{N_i}(\zeta_{1,i+1}-\zeta_{3,i+1}-\zeta_{7,i+1}+\zeta_{9,i+1})+u_i(\zeta_{1,i+1}-\zeta_{3,i+1})\bigg],\\ \Delta\zeta_{2,i} = -\bigg[&A_1+\zeta_{2,i+1}-\frac{\alpha_1V_i^M}{N_i}(\zeta_{1,i+1}+\zeta_{2,i+1}-\zeta_{3,i+1}-\zeta_{7,i+1})\\ &+\frac{\alpha_2W_i^M}{N_i}(-\zeta_{2,i+1}+\zeta_{3,i+1}+\zeta_{7,i+1}-\zeta_{10,i+1})+\frac{\beta_2D_i^M}{N_i}(-\zeta_{2,i+1}+\zeta_{3,i+1}+\zeta_{7,i+1}-\zeta_{8,i+1})\\ &+\frac{\gamma_1M_i^{M_1}}{N_i}(-\zeta_{2,i+1}+\zeta_{4,i+1}+\zeta_{7,i+1})+\frac{\gamma_2M_i^{M_2}}{N_i}(-\zeta_{2,i+1}+\zeta_{5,i+1}+\zeta_{7,i+1})\\ &+\frac{\gamma_3M_i^{M_3}}{N_i}(-\zeta_{2,i+1}+\zeta_{6,i+1}+\zeta_{7,i+1})+u_i(-\zeta_{2,i+1}+\zeta_{7,i+1})\bigg],\\ \Delta\zeta_{3,i} = -\bigg[&-\zeta_{3,i+1}+\frac{\gamma_1V_i^W}{N_i}(-\zeta_{2,i+1}+\zeta_{4,i+1}+\zeta_{7,i+1})+\lambda_1(\zeta_{3,i+1}-\zeta_{7,i+1}+\zeta_{11,i+1})\\ &+\mu_1(\zeta_{3,i+1}-\zeta_{7,i+1}+\zeta_{10,i+1})+\delta_1(\zeta_{3,i+1}-\zeta_{7,i+1}+\zeta_{8,i+1}+\zeta_{9,i+1})\bigg],\\ \Delta\zeta_{4,i} = -\bigg[&\zeta_{4,i+1}+\frac{\gamma_2V_i^W}{N_i}(\zeta_{2,i+1}-\zeta_{5,i+1}-\zeta_{7,i+1})-\lambda_2(\zeta_{3,i+1}+\zeta_{4,i+1}+\zeta_{7,i+1}-\zeta_{11,i+1})\\ &+\mu_2(\zeta_{3,i+1}+\zeta_{4,i+1}+\zeta_{7,i+1})-\delta_2(\zeta_{4,i+1}+\zeta_{7,i+1}-\zeta_{9,i+1})\bigg],\end{aligned}$$

$$
\begin{aligned}
\Delta\zeta_{5,i} = &-\Bigg[\zeta_{5,i+1} + \frac{-\gamma_3 V_i^W}{N_i}(-\zeta_{2,i+1}+\zeta_{6,i+1}+\zeta_{7,i+1}) + \lambda_3(\zeta_{4,i+1}-\zeta_{5,i+1}-\zeta_{7,i+1}+\zeta_{11,i+1}) \\
&+ \mu_3(\zeta_{4,i+1}-\zeta_{5,i+1}-\zeta_{7,i+1}) + \delta_3(-\zeta_{5,i+1}-\zeta_{7,i+1}+\zeta_{9,i+1})\Bigg], \\
\Delta\zeta_{6,i} = &-\big[\zeta_{6,i+1} + \lambda_4(\zeta_{5,i+1}-\zeta_{6,i+1}-\zeta_{7,i+1}+\zeta_{11,i+1}) \\
&+ \mu_4(\zeta_{5,i+1}-\zeta_{6,i+1}-\zeta_{7,i+1}) + \delta_4(-\zeta_{6,i+1}-\zeta_{7,i+1}+\zeta_{9,i+1})\big], \\
\Delta\zeta_{7,i} = &\ A_3 - \zeta_{7,i+1}, \\
\Delta\zeta_{8,i} = &-\Bigg[\zeta_{8,i+1} + \frac{\beta_2 V_i^W}{N_i}(-\zeta_{2,i+1}+\zeta_{3,i+1}+\zeta_{7,i+1}-\zeta_{8,i+1}) \\
&+ \frac{\beta_3 D_{i-\tau_1}^W}{N_i}(\zeta_{3,i+1}+\zeta_{7,i+1}-\zeta_{8,i+1}-\zeta_{9,i+1}) + \frac{\beta_5 W_{i-\tau_2}^W}{N_i}(\zeta_{3,i+1}+\zeta_{7,i+1}-\zeta_{8,i+1}-\zeta_{11,i+1}) \\
&+ v_i(r_1\zeta_{3,i+1}+r_2\zeta_{4,i+1}+r_3\zeta_{5,i+1}+r_4\zeta_{6,i+1}-\zeta_{8,i+1})\Bigg], \\
\Delta\zeta_{9,i} = &-\Bigg[A_2 + \zeta_{9,i+1+\tau_1} + (\zeta_{7,i+1+\tau_1}-\zeta_{9,i+1+\tau_1})v_{i+\tau_1} - \\
&- \chi_{\{0,\dots,N-\tau_1\}}\Bigg(\frac{\beta_1 V_{i+\tau_1}^M}{N_{i+\tau_1}}(-\zeta_{1,i+1+\tau_1}+\zeta_{3,i+1+\tau_1}+\zeta_{7,i+1+\tau_1}-\zeta_{9,i+1+\tau_1}) \\
&+ \frac{\beta_3 D_{i+\tau_1}^M}{N_{i+\tau_1}}(\zeta_{3,i+1+\tau_1}+\zeta_{7,i+1+\tau_1}-\zeta_{8,i+1+\tau_1}-\zeta_{9,i+1+\tau_1}) \\
&+ \frac{\beta_4 W_{i+\tau_1}^M}{N_{i+\tau_1}}(\zeta_{3,i+1+\tau_1}+\zeta_{7,i+1+\tau_1}-\zeta_{9,i+1+\tau_1}-\zeta_{10,i+1+\tau_1})\Bigg)\Bigg], \\
\Delta\zeta_{10,i} = &-\Bigg[\zeta_{10,i+1} + \frac{\alpha_2 V_i^W}{N_i}(-\zeta_{2,i+1}+\zeta_{3,i+1}+\zeta_{7,i+1}-\zeta_{10,i+1}) \\
&+ \frac{\alpha_4 W_{i-\tau_2}^W}{N_i}(\zeta_{3,i+1}+\zeta_{7,i+1}-\zeta_{10,i+1}-\zeta_{11,i+1}) + \frac{\beta_4 D_{i-\tau_2}^W}{N_i}(\zeta_{3,i+1}+\zeta_{7,i+1}-\zeta_{9,i+1}-\zeta_{10,i+1})\Bigg], \\
\Delta\zeta_{11,i} = &-\Bigg[\zeta_{11,i+1+\tau_2} + \chi_{\{0,\dots,N-\tau_2\}}\Big[\frac{\alpha_3 V_{i+\tau_2}^M}{N_{i+\tau_2}}(-\zeta_{1,i+1+\tau_2}+\zeta_{3,i+1+\tau_2}+\zeta_{7,i+1+\tau_2} \\
&- \zeta_{11,i+1+\tau_2}) + \frac{\beta_5 D_{i+\tau_2}^M}{N_{i+\tau_2}}(\zeta_{3,i+1+\tau_2}+\zeta_{7,i+1+\tau_2}-\zeta_{8,i+1+\tau_2}-\zeta_{11,i+1+\tau_2}) \\
&+ \frac{\alpha_4 W_{i+\tau_2}^M}{N_{i+\tau_2}}(\zeta_{3,i+1+\tau_2}+\zeta_{7,i+1+\tau_2}-\zeta_{10,i+1+\tau_2}-\zeta_{11,i+1+\tau_2})\Big]\Bigg],
\end{aligned}
$$

with for $n=\{1,2\}$, $\chi_{\{0,\dots,i-\tau_n\}=I}(k) = \begin{cases} 1, & \textit{if} \quad k\in I, \\ 0, & \textit{else} \end{cases}$ *with transversality conditions*

$$\zeta_{1,N}=0, \quad \zeta_{2,N}=-A_1, \quad \zeta_{3,N}=0, \quad \zeta_{4,N}=0, \quad \zeta_{5,N}=0, \quad \zeta_{6,N}=0,$$

$$\zeta_{7,N}=A_3, \quad \zeta_{8,N}=0, \quad \zeta_{9,N}=-A_2, \quad \zeta_{10,N}=0 \quad \textit{and} \quad \zeta_{11,N}=0.$$

Furthermore, the optimal control $(u_i{}^*, v_i{}^*)$ *is given by*

$$u_i{}^* = \min\left\{\max\left\{\frac{V_i^M(\zeta_{1,i+1}-\zeta_{3,i+1}) + V_i^W(\zeta_{2,i+1}-\zeta_{7,i+1})}{\tau_1'}, u_{\min}\right\}, u_{\max}\right\}, \tag{25}$$

$$L = \frac{D_i^M(\zeta_{8,i+1}-r_1\zeta_{3,i+1}-r_2\zeta_{4,i+1}-r_3\zeta_{5,i+1}-r_4\zeta_{6,i+1}) + D_i^W(\zeta_{9,i+1}-\zeta_{7,i+1})}{\tau_2'},$$

$$v_i{}^* = \min\big\{\max\{L, u_{\min}\}, u_{\max}\big\}. \tag{26}$$

for $i=0,\dots,N-1$.

Proof. In order to derive the necessary condition for optimal control, the Pontryagins maximum principle in discrete time given in [8, 9, 14, 15] was used. We obtain the following adjoint equations:

$$\begin{aligned}
\Delta\zeta_{1,i} &= -\frac{\partial H_i}{\partial V_i^M}\\
&= -\Bigg[\zeta_{1,i+1}\bigg(1 - \frac{\alpha_1 V_i^W - \alpha_3 W_{i-\tau_2}^W - \beta_1 D_{i-\tau_1}^W}{N_i} - u_i\bigg) - \zeta_{2,i+1}\frac{\alpha_1 V_i^W}{N_i}\\
&\quad + \zeta_{3,i+1}\bigg(\frac{\alpha_1 V_i^W + \beta_1 D_{i-\tau_1}^W + \alpha_3 W_{i-\tau_2}^W}{N_i} + u_i\bigg)\\
&\quad + \zeta_{7,i+1}\frac{\alpha_1 V_i^W + \beta_1 D_{i-\tau_1}^W + \alpha_3 W_{i-\tau_2}^W}{N_i} - \zeta_{9,i+1}\frac{\beta_1 D_{i-\tau_1}^W}{N_i} - \zeta_{11,i+1}\frac{\alpha_3 W_{i-\tau_2}^W}{N_i}\Bigg]\\
&= -\Bigg[\zeta_{1,i+1} + \frac{\alpha_1 V_i^W}{N_i}\left(\zeta_{1,i+1} + \zeta_{2,i+1} - \zeta_{3,i+1} - \zeta_{7,i+1}\right)\\
&\quad + \frac{\alpha_3 W_{i-\tau_2}^W}{N_i}\left(\zeta_{1,i+1} - \zeta_{3,i+1} - \zeta_{7,i+1} + \zeta_{11,i+1}\right)\\
&\quad + \frac{\beta_1 D_{i-\tau_1}^W}{N_i}(\zeta_{1,i+1} - \zeta_{3,i+1} - \zeta_{7,i+1} + \zeta_{9,i+1}) + u_i(\zeta_{1,i+1} - \zeta_{3,i+1})\Bigg],\\
\Delta\zeta_{2,i} &= -\frac{\partial H_i}{\partial V_i^W}\\
&= -\Bigg[A_1 - \zeta_{1,i+1}\frac{\alpha_1 V_i^M}{N_i}\\
&\quad + \zeta_{2,i+1}\bigg(1 - \frac{\alpha_1 V_i^M + \beta_2 D_i^M + \alpha_2 W_i^M}{N_i} - \frac{\gamma_1 M_i^{M_1} + \gamma_2 M_i^{M_2} + \gamma_3 M_i^{M_3}}{N_i} - u_i\bigg)\\
&\quad + \zeta_{3,i+1}\frac{\alpha_1 V_i^M + \beta_2 D_i^M + \alpha_2 W_i^M}{N_i} + \zeta_{4,i+1}\frac{\gamma_1 M_i^{M_1}}{N_i} + \zeta_{5,i+1}\frac{\gamma_2 M_i^{M_2}}{N_i} + \zeta_{6,i+1}\frac{\gamma_3 M_i^{M_3}}{N_i}\\
&\quad + \zeta_{7,i+1}\bigg(\frac{\alpha_1 V_i^M + \beta_2 D_i^M + \alpha_2 W_i^M}{N_i} + \frac{\gamma_1 M_i^{M_1} + \gamma_2 M_i^{M_2} + \gamma_3 M_i^{M_3}}{N_i} + u_i\bigg)\\
&\quad - \zeta_{8,i+1}\frac{\beta_2 D_i^M}{N_i} - \zeta_{10,i+1}\frac{\alpha_2 W_i^M}{N_i}\Bigg]\\
&= -\Bigg[A_1 + \zeta_{2,i+1} - \frac{\alpha_1 V_i^M}{N_i}(\zeta_{1,i+1} + \zeta_{2,i+1} - \zeta_{3,i+1} - \zeta_{7,i+1})\\
&\quad + \frac{\alpha_2 W_i^M}{N_i}(-\zeta_{2,i+1} + \zeta_{3,i+1} + \zeta_{7,i+1} - \zeta_{10,i+1}) + \frac{\beta_2 D_i^M}{N_i}(-\zeta_{2,i+1} + \zeta_{3,i+1} + \zeta_{7,i+1} - \zeta_{8,i+1})\\
&\quad + \frac{\gamma_1 M_i^{M_1}}{N_i}(-\zeta_{2,i+1} + \zeta_{4,i+1} + \zeta_{7,i+1}) + \frac{\gamma_2 M_i^{M_2}}{N_i}(-\zeta_{2,i+1} + \zeta_{5,i+1} + \zeta_{7,i+1})\\
&\quad + \frac{\gamma_3 M_i^{M_3}}{N_i}(-\zeta_{2,i+1} + \zeta_{6,i+1} + \zeta_{7,i+1}) + u_i(-\zeta_{2,i+1} + \zeta_{7,i+1})\Bigg],\\
\Delta\zeta_{3,i} &= -\frac{\partial H_i}{\partial M_i^{M_1}}\\
&= -\Bigg[-\zeta_{2,i+1}\frac{\gamma_1 V_i^W}{N_i} - \zeta_{3,i+1}(1 - (\lambda_1 + \delta_1 + \mu_1)) + \zeta_{4,i+1}\frac{\gamma_1 V_i^M}{N_i}\\
&\quad + \zeta_{7,i+1}\bigg(\frac{\gamma_1 V_i^W}{N_i} - (\lambda_1 + \delta_1 + \mu_1)\bigg) + \zeta_{8,i+1}\delta_1 + \zeta_{9,i+1}\delta_1 + \zeta_{10,i+1}\mu_1 + \zeta_{11,i+1}\lambda_1\Bigg],\\
&= -\Bigg[-\zeta_{3,i+1} + \frac{\gamma_1 V_i^W}{N_i}(-\zeta_{2,i+1} + \zeta_{4,i+1} + \zeta_{7,i+1}) + \lambda_1(\zeta_{3,i+1} - \zeta_{7,i+1} + \zeta_{11,i+1})
\end{aligned}$$

$$
\begin{aligned}
&\quad + \mu_1(\zeta_{3,i+1} - \zeta_{7,i+1} + \zeta_{10,i+1}) + \delta_1(\zeta_{3,i+1} - \zeta_{7,i+1} + \zeta_{8,i+1} + \zeta_{9,i+1})\Big],\\
\Delta\zeta_{4,i} &= -\frac{\partial H_i}{\partial M_i^{M_2}}\\
&= -\Big[-\zeta_{2,i+1}\frac{\gamma_2 V_i^W}{N_i} + \zeta_{3,i+1}\frac{\lambda_2+\mu_2}{N_i} + \zeta_{4,i+1}\frac{1-(\lambda_2+\delta_2+\mu_2)}{N_i} + \zeta_{5,i+1}\frac{\gamma_2 V_i^W}{N_i}\\
&\quad + \zeta_{7,i+1}\left(\frac{\gamma_2 V_i^W}{N_i} - (\lambda_2+\delta_2+\mu_2)\right) + \zeta_{9,i+1}\delta_2 + \zeta_{11,i+1}\lambda_2\Big]\\
&= -\Big[\zeta_{4,i+1} + \frac{\gamma_2 V_i^W}{N_i}(\zeta_{2,i+1} - \zeta_{5,i+1} - \zeta_{7,i+1}) - \lambda_2(\zeta_{3,i+1} + \zeta_{4,i+1} + \zeta_{7,i+1} - \zeta_{11,i+1})\\
&\quad + \mu_2(\zeta_{3,i+1} + \zeta_{4,i+1} + \zeta_{7,i+1}) - \delta_2(\zeta_{4,i+1} + \zeta_{7,i+1} - \zeta_{9,i+1})\Big],\\
\Delta\zeta_{5,i} &= -\frac{\partial H_i}{\partial M_i^{M_3}}\\
&= -\Big[\zeta_{2,i+1}\frac{-\gamma_3 V_i^W}{N_i} + \zeta_{4,i+1}(\lambda_3+\mu_3) + \zeta_{5,i+1}(1-(\lambda_3+\delta_3+\mu_3))\\
&\quad + \zeta_{6,i+1}\frac{\gamma_3 V_i^W}{N_i} + \zeta_{7,i+1}\left(\frac{\gamma_3 V_i^W}{N_i} - (\lambda_3+\delta_3+\mu_3)\right) + \zeta_{9,i+1}\delta_3 + \zeta_{11,i+1}\lambda_3\Big]\\
&= -\Big[\zeta_{5,i+1} + \frac{-\gamma_3 V_i^W}{N_i}(-\zeta_{2,i+1} + \zeta_{6,i+1} + \zeta_{7,i+1}) + \lambda_3(\zeta_{4,i+1} - \zeta_{5,i+1} - \zeta_{7,i+1} + \zeta_{11,i+1})\\
&\quad + \mu_3(\zeta_{4,i+1} - \zeta_{5,i+1} - \zeta_{7,i+1}) + \delta_3(-\zeta_{5,i+1} - \zeta_{7,i+1} + \zeta_{9,i+1})\Big],\\
\Delta\zeta_{6,i} &= -\frac{\partial H_i}{\partial M_i^{M_4}}\\
&= -\big[\zeta_{5,i+1}(\lambda_4+\mu_4) + \zeta_{6,i+1}(1-(\lambda_4+\delta_4+\mu_4)) + \zeta_{7,i+1}(-(\lambda_4+\delta_4+\mu_4)) + \zeta_{9,i+1}\delta_4 + \zeta_{11,i+1}\lambda_4\big]\\
&= -\big[\zeta_{6,i+1} + \lambda_4(\zeta_{5,i+1} - \zeta_{6,i+1} - \zeta_{7,i+1} + \zeta_{11,i+1})\\
&\quad + \mu_4(\zeta_{5,i+1} - \zeta_{6,i+1} - \zeta_{7,i+1}) + \delta_4(-\zeta_{6,i+1} - \zeta_{7,i+1} + \zeta_{9,i+1})\big],\\
\Delta\zeta_{7,i} &= -\frac{\partial H_i}{\partial M_i^{W}}\\
&= -\left[-A_3 + \zeta_{7,i+1}\right]\\
&= A_3 - \zeta_{7,i+1},\\
\Delta\zeta_{8,i} &= -\frac{\partial H_i}{\partial D_i^{M}}\\
&= -\Big[-\zeta_{2,i+1}\frac{\beta_2 V_i^W}{N_i} + \zeta_{3,i+1}\left(\frac{\beta_2 V_i^W + \beta_3 D_{i-\tau_1}^W + \beta_5 W_{i-\tau_2}^W}{N_i} + r_1 v_i\right)\\
&\quad + \zeta_{4,i+1} r_2 v_i + \zeta_{5,i+1} r_3 v_i + \zeta_{6,i+1} r_4 v_i + \zeta_{7,i+1}\left(\frac{\beta_2 V_i^W + \beta_3 D_{i-\tau_1}^W + \beta_5 W_{i-\tau_2}^W}{N_i}\right)\\
&\quad + \zeta_{8,i+1}\left(1 - \frac{\beta_2 V_i^W + \beta_3 D_{i-\tau_1}^W + \beta_5 W_{i-\tau_2}^W}{N_i} - v_i\right) - \zeta_{9,i+1}\left(\frac{\beta_3 D_{i-\tau_1}^W}{N_i}\right) - \zeta_{11,i+1}\frac{\beta_5 W_{i-\tau_2}^W}{N_i}\Big]\\
&= -\Big[\zeta_{8,i+1} + \frac{\beta_2 V_i^W}{N_i}(-\zeta_{2,i+1} + \zeta_{3,i+1} + \zeta_{7,i+1} - \zeta_{8,i+1})\\
&\quad + \frac{\beta_3 D_{i-\tau_1}^W}{N_i}(\zeta_{3,i+1} + \zeta_{7,i+1} - \zeta_{8,i+1} - \zeta_{9,i+1}) + \frac{\beta_5 W_{i-\tau_2}^W}{N_i}(\zeta_{3,i+1} + \zeta_{7,i+1} - \zeta_{8,i+1} - \zeta_{11,i+1})\\
&\quad + v_i(r_1\zeta_{3,i+1} + r_2\zeta_{4,i+1} + r_3\zeta_{5,i+1} + r_4\zeta_{6,i+1} - \zeta_{8,i+1})\Big],
\end{aligned}
$$

$$
\begin{aligned}
\Delta\zeta_{9,i} &= -\frac{\partial H_i}{\partial D_i^W} - \chi_{\{0,\ldots,N-\tau_1\}} \frac{\partial H_{i+\tau_1}}{y_1} \\
&= -\Bigg[A_2 + \zeta_{9,i+1} + \zeta_{7,i+1} v_i - \zeta_{9,i+1} v_i \\
&\quad + \chi_{\{0,\ldots,N-\tau_1\}} \Bigg(\zeta_{1,i+1+\tau_1} \frac{-\beta_1 V_{i+\tau_1}^M}{N_{i+\tau_1}} + \zeta_{3,i+1+\tau_1} \frac{\beta_1 V_{i+\tau_1}^M + \beta_3 D_{i+\tau_1}^M + \beta_4 W_{i+\tau_1}^M}{N_{i+\tau_1}} \\
&\quad + \zeta_{7,i+1+\tau_1} \frac{\beta_1 V_{i+\tau_1}^M + \beta_3 D_{i+\tau_1}^M + \beta_4 W_{i+\tau_1}^M}{N_{i+\tau_1}} - \zeta_{8,i+1+\tau_1} \frac{\beta_3 D_{i+\tau_1}^M}{N_{i+\tau_1}} \\
&\quad - \zeta_{9,i+1+\tau_1} \frac{\beta_1 V_{i+\tau_1}^M + \beta_3 D_{i+\tau_1}^M + \beta_4 W_{i+\tau_1}^M}{N_{i+\tau_1}} - \zeta_{10,i+1+\tau_1} \frac{\beta_4 W_{i+\tau_1}^M}{N_{i+\tau_1}} \Bigg) \Bigg] \\
&= -\Bigg[A_2 + \zeta_{9,i+1+\tau_1} + (\zeta_{7,i+1+\tau_1} - \zeta_{9,i+1+\tau_1}) v_{i+\tau_1} \\
&\quad - \chi_{\{0,\ldots,N-\tau_1\}} \Bigg(\frac{\beta_1 V_{i+\tau_1}^M}{N_{i+\tau_1}} \big(-\zeta_{1,i+1+\tau_1} + \zeta_{3,i+1+\tau_1} + \zeta_{7,i+1+\tau_1} - \zeta_{9,i+1+\tau_1} \big) \\
&\quad + \frac{\beta_3 D_{i+\tau_1}^M}{N_{i+\tau_1}} (\zeta_{3,i+1+\tau_1} + \zeta_{7,i+1+\tau_1} - \zeta_{8,i+1+\tau_1} - \zeta_{9,i+1+\tau_1}) \\
&\quad + \frac{\beta_4 W_{i+\tau_1}^M}{N_{i+\tau_1}} (\zeta_{3,i+1+\tau_1} + \zeta_{7,i+1+\tau_1} - \zeta_{9,i+1+\tau_1} - \zeta_{10,i+1+\tau_1}) \Bigg) \Bigg], \\
\Delta\zeta_{10,i} &= -\frac{\partial H_i}{\partial W_i^M} \\
&= -\Bigg[-\zeta_{2,i+1} \frac{\alpha_2 V_i^W}{N_i} + \zeta_{3,i+1} \frac{\alpha_2 V_i^W + \beta_4 D_{i-\tau_1}^W + \alpha_4 W_{i-\tau 2}^W}{N_i} - \zeta_{9,i+1} \frac{\beta_4 D_{i-\tau_2}^W}{N_i} \\
&\quad + \zeta_{10,i+1} \Bigg(1 - \frac{(\alpha_2 V_i^W + \beta_4 D_{i-\tau_1}^W + \alpha_4 W_{i-\tau_2}^W)}{N_i} \Bigg) - \zeta_{11,i+1} \frac{\alpha_4 W_{i-\tau_2}^W}{N_i} \Bigg] \\
&= -\Bigg[\zeta_{10,i+1} + \frac{\alpha_2 V_i^W}{N_i} (-\zeta_{2,i+1} + \zeta_{3,i+1} + \zeta_{7,i+1} - \zeta_{10,i+1}) \\
&\quad + \frac{\alpha_4 W_{i-\tau_2}^W}{N_i} (\zeta_{3,i+1} + \zeta_{7,i+1} - \zeta_{10,i+1} - \zeta_{11,i+1}) \\
&\quad + \frac{\beta_4 D_{i-\tau_2}^W}{N_i} (\zeta_{3,i+1} + \zeta_{7,i+1} - \zeta_{9,i+1} - \zeta_{10,i+1}) \Bigg], \\
\Delta\zeta_{11,i} &= -\frac{\partial H_i}{\partial W_i^W} - \chi_{\{0,\ldots,N-\tau_2\}} \frac{\partial H_{i+\tau_2}}{y_2} \\
&= -\Bigg[\zeta_{11,i+1} + \chi_{\{0,\ldots,N-\tau_2\}} \Bigg(-\zeta_{1,i+1+\tau_2} \frac{\alpha_3 V_{i+\tau_2}^M}{N_{i+\tau_2}} + \zeta_{3,i+1+\tau_2} \frac{\alpha_3 V_{i+\tau_2}^M + \beta_5 D_{i+\tau_2}^M + \alpha_4 W_{i+\tau_2}^M}{N_{i+\tau_2}} \\
&\quad + \zeta_{7,i+1+\tau_2} \frac{\alpha_3 V_{i+\tau_2}^M + \beta_5 D_{i+\tau_2}^M + \alpha_4 W_{i+\tau_2}^M}{N_{i+\tau_2}} - \zeta_{8,i+1+\tau_2} \frac{\beta_5 D_{i+\tau_2}^M}{N_{i+\tau_2}} \\
&\quad - \zeta_{10,i+1+\tau_2} \frac{\alpha_4 W_{i+\tau_2}^M}{N_{i+\tau_2}} - \zeta_{11,i+1+\tau_2} \frac{\alpha_3 V_{i+\tau_2}^M + \beta_5 D_{i+\tau_2}^M + \alpha_4 W_{i+\tau_2}^M}{N_{i+\tau_2}} \Bigg] \\
&= -\Bigg[\zeta_{11,i+1+\tau_2} + \chi_{\{0,\ldots,N-\tau_2\}} \Bigg[\frac{\alpha_3 V_{i+\tau_2}^M}{N_{i+\tau_2}} (-\zeta_{1,i+1+\tau_2} + \zeta_{3,i+1+\tau_2} + \zeta_{7,i+1+\tau_2} - \zeta_{11,i+1+\tau_2}) \\
&\quad + \frac{\beta_5 D_{i+\tau_2}^M}{N_{i+\tau_2}} (\zeta_{3,i+1+\tau_2} + \zeta_{7,i+1+\tau_2} - \zeta_{8,i+1+\tau_2} - \zeta_{11,i+1+\tau_2}) \\
&\quad + \frac{\alpha_4 W_{i+\tau_2}^M}{N_{i+\tau_2}} (\zeta_{3,i+1+\tau_2} + \zeta_{7,i+1+\tau_2} - \zeta_{10,i+1+\tau_2} - \zeta_{11,i+1+\tau_2}) \Bigg] \Bigg],
\end{aligned}
$$

with transversality conditions

$$\zeta_{1,N}=0,\quad \zeta_{2,N}=-A_1,\quad \zeta_{3,N}=0,\quad \zeta_{4,N}=0,\quad \zeta_{5,N}=0,\quad \zeta_{6,N}=0,$$
$$\zeta_{7,N}=A_3,\quad \zeta_{8,N}=0,\quad \zeta_{9,N}=-A_2,\quad \zeta_{10,N}=0,\quad \zeta_{11,N}=0.$$

To obtain the optimality conditions we take the variation with respect to control (u_i^*, v_i^*) and set it equal to zero

$$\frac{\partial H}{\partial u_i}=\tau_1' u_i-\zeta_{1,i+1}V_i^M-\zeta_{2,i+1}V_i^W+\zeta_{3,i+1}V_i^M\zeta_{7,i+1}V_i^W=0,$$
$$\frac{\partial H}{\partial v_i}=\tau_2' v_i+\zeta_{3,i+1}r_1D_i^M+\zeta_{4,i+1}r_2D_i^M+\zeta_{5,i+1}r_3,$$
$$D_i^M+\zeta_{6,i+1}r_4D_i^M+\zeta_{7,i+1}D_i^W-\zeta_{8,i+1}D_i^M-\zeta_{9,i+1}D_i^W=0.$$

Then we obtain the optimal control

$$u_i{}^*=\frac{V_i^M(\zeta_{1,i+1}-\zeta_{3,i+1})+V_i^W(\zeta_{2,i+1}-\zeta_{7,i+1})}{\tau_1'},$$
$$v_i{}^*=\frac{D_i^M(\zeta_{8,i+1}-r_1\zeta_{3,i+1}-r_2\zeta_{4,i+1}-r_3\zeta_{5,i+1}-r_4\zeta_{6,i+1})+D_i^W(\zeta_{9,i+1}-\zeta_{7,i+1})}{\tau_2'}.$$

By the bounds in $\mathcal{U}_{ad}$, it is easy to obtain $(u_i{}^*, v_i{}^*)$ in the following form

$$u_i{}^*=\min\left\{\max\left\{\frac{V_i^M(\zeta_{1,i+1}-\zeta_{3,i+1})+V_i^W(\zeta_{2,i+1}-\zeta_{7,i+1})}{\tau_1'},u_{\min}\right\},u_{\max}\right\},$$
$$L=\frac{D_i^M(\zeta_{8,i+1}-r_1\zeta_{3,i+1}-r_2\zeta_{4,i+1}-r_3\zeta_{5,i+1}-r_4\zeta_{6,i+1})+D_i^W(\zeta_{9,i+1}-\zeta_{7,i+1})}{\tau_2'}$$
$$v_i{}^*=\min\left\{\max\{L,u_{\min}\},u_{max}\right\},$$

for $i=0,\ldots,N-1$. ■

4. Numerical simulation

In this section, we will present some percentages showing the great spread and the alarming increase of the phenomenon of spinsterhood and divorce in our Arab society.

The Lebanese women ranked first in the ranking of the Arab women who were the most late in the age of marriage, with 85 percent of the percentage of females over the age of 35 years. Tunisian women advanced in the second place, with the percentage of spinsterhood rising to about 81 percent, while Iraqi women ranked third, with an estimated rate of between 70 to 85 percent. Emirati women ranked fourth, with an estimated rate of 70 to 75 percent, and Syrian women ranked fifth, with 70 percent. Moroccan women ranked sixth (60 percent), with an increase of about 20 percent over the past decade. Jordanian women ranked seventh with 55 percent, with an increase of about 10 percent over the past decade, while Algerian women ranked eighth with 51 percent as governorates, compared to the same percentage that was in 2013. Egyptian women ranked ninth with 48 percent, with Recorded an increase of about 8 percent since 2013.

Divorce rates have increased in Arab countries. The British newspaper, The Independent, confirmed that divorce rates in Arab countries have increased 10 times over the past 50 years, while official statistics confirms the occurrence of two divorce cases every minute in some countries such as Egypt and seven cases per hour in other countries such as Saudi Arabia.

We present some numerical solutions of system (1)–(11) for different values of the parameters. We will start by presenting the numerical simulation of eleven equations, using parameters, some of which are estimated and some of which are real. The goal is to see clear that the data showing changes that can be observed for the rapid increases in the development and increase in the number of single men and women, as well as divorce rates, and widows can be added as well.

We solve the system by an iterative method with forward solving of the state system followed by backward solving of the adjoint system. We start with an initial guess for the first iteration and then before the next iteration, we update the controls by using the characterization. We continue until convergence of successive iterates is achieved. A code is written and compiled in MATLAB using the following data.

Different simulations can be carried out using various values of parameters. In the present numerical approach, we use the following parameters values taken from [9]: $\alpha_1 = 0.0023$, $\alpha_2 = 0.0015$, $\alpha_3 = 0.0018$, $\alpha_4 = 0.00013$, $\beta_1 = 0.000022$, $\beta_2 = 0.000026$, $\beta_3 = 0.000027$, $\beta_4 = 0.000024$, $\beta_5 = 0.000025$, $\lambda_1 = 0.00003$, $\lambda_2 = 0.00002$, $\lambda_3 = 0.00018$, $\lambda_4 = 0.00004$, $\delta_1 = 0.00002$, $\delta_2 = 0.00002$, $\delta_3 = 0.00022$, $\delta_4 = 0.000028$, $\mu_1 = 0.0002$, $\mu_2 = 0.0003$, $\mu_3 = 0.0004$, $\mu_4 = 0.0004$, $\gamma_1 = 0.00035$, $\gamma_2 = 0.0001$, $\gamma_3 = 0.0001$, $\tau_1 = 2000$, and $\tau_2 = 3290$.

The population values used (reel values in Morocco in 2014) in numerical simulation: $V^M(0) = 9102000$, $V^W(0) = 10500000$, $M^{M_1}(0) = 5869830$, $M^{M_2}(0) = 863000$, $M^{M_3}(0) = 431500$, $M^{M_4}(0) = 287667$, $M^W(0) = 7452000$, $D^M(0) = 215000$, $D^W(0) = 361000$, $W^M(0) = 573000$, and $W^W(0) = 831000$.

The graphs represent the changes that occur in each group in society over a period of one year.

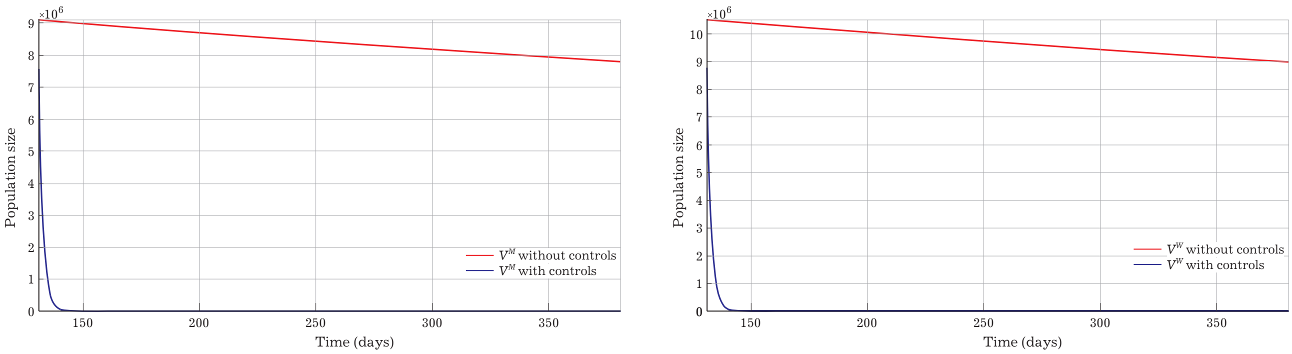

Fig. 2. The results of simulating the evolution of the numbers of the virgin man, the virgin woman, the married man and the married woman, with and without control.

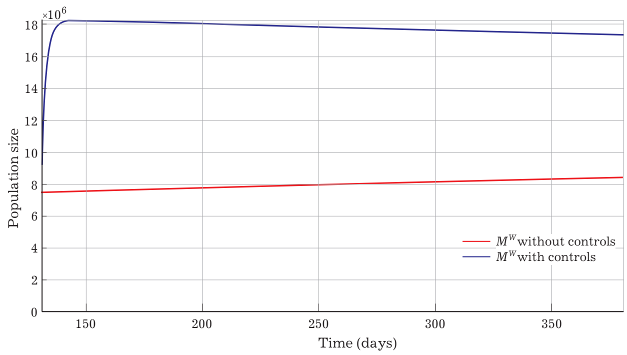

Fig. 3. The results of simulating the evolution of the numbers of the virgin man, the virgin woman, the married man and the married woman, with and without control.

In Figure 2 we have the variation over time of virgin men and virgin women, we can see the number decreasing so slowly, for virgin men starting at 9102000 to 7805940 and for virgin women starting at 10500000 to 8986230. But after applying the u_i control, we see very clearly the number of virgin men and virgin women decreasing as quickly after 3 months, for the first starting from the initial number at 910200 and for the second starting from the initial number at 37897.

In Figure 3 discussing the variation of married women starting with 7452000 and after one year we have a small increase to 841690, and after applying the control we have a good increase from 7452000 to 18233500 then 17320500 a slight decrease due to widowed women occurred during this period.

Figure 4 shows that we start from men married with one wife. We have 5869830 married men and after a year we have a small increase to 6912590. After applying the control we have a good increase from 5869830 to 15085200 then 14363100 a slight decrease due to widowed men occurred during this period. For married men with two, three and four wives we can see the effect of control v_i gives an increase of 967006, 8104 and 7636 respectively.

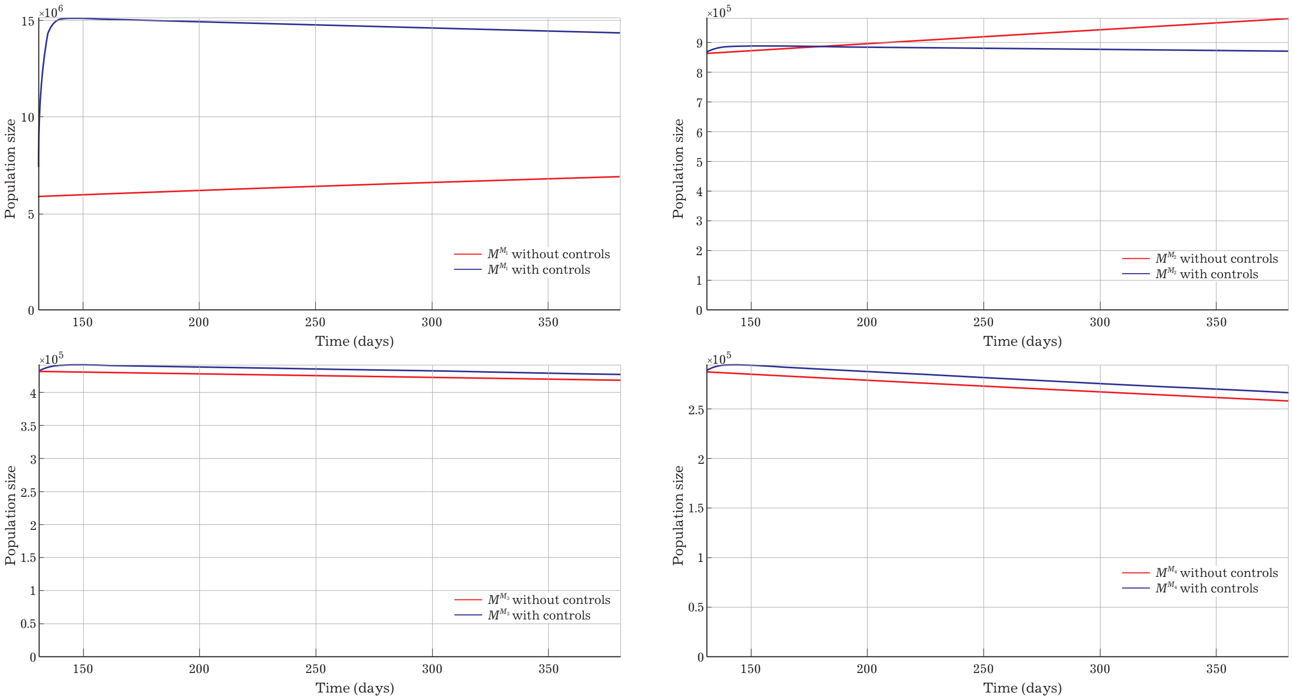

Fig. 4. The results of simulating the evolution of the married men with one wife, two wives, three wives and four wives, with and without control.

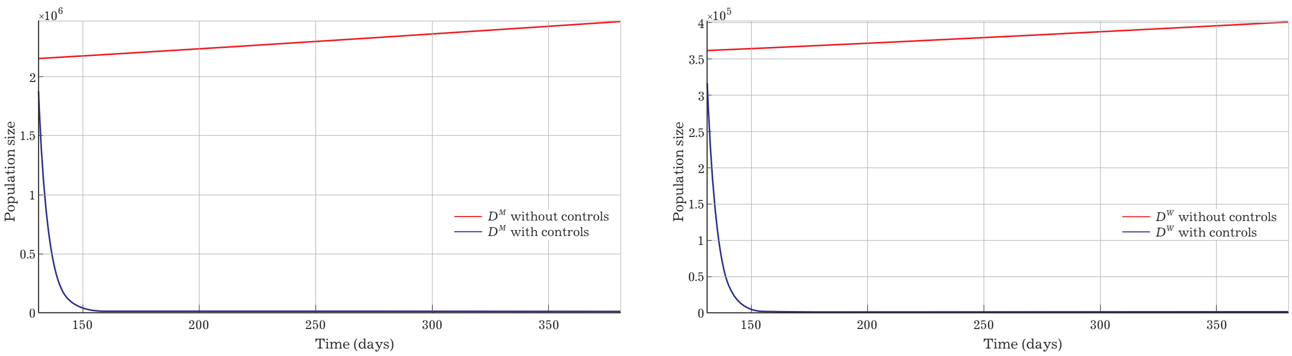

Fig. 5. The results of simulating the evolution of the divorces men and divorces women, with and without control.

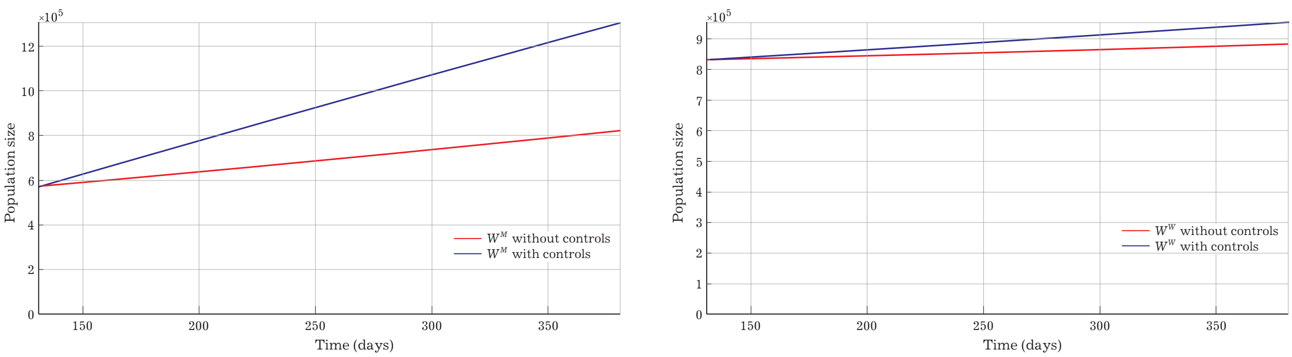

Fig. 6. The results of simulating the evolution of the widow men and widow women, with and without control.

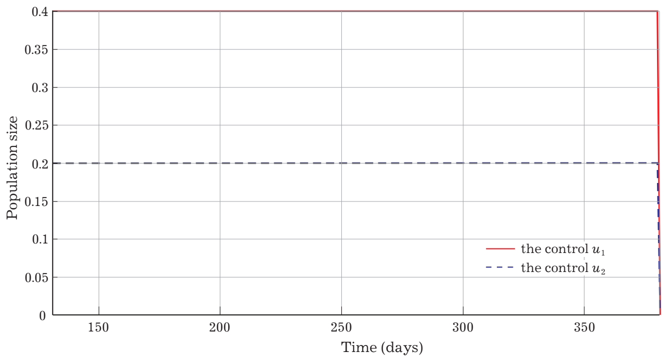

Fig. 7. Simulation results of the controls u_i and v_i.

Figure 5 shows the bad increase in male divorces, in one year we have more than 43792 cases. But after applying the control, we can see the decrease from 215116 to 4571 cases. Women divorces in one year we have more than 53815 cases. But after applying the control, we have a decrease from 361 149 to 9 752 cases.

In Figure 6 male and female widows increase automatically due to increase in natural deaths of married males and females in one year.

5. Conclusions

In this paper, we have formulated a mathematical model of polygamous marriage. In order to reduce the number of virgin men and women, divorced men and women, two controls have been applied. The first control introduced is assumed the benefits of an awareness campaign to educate virgin men and virgin women about the benefits of marriage for individuals and for society, and the second control characterizes legal proceedings, administrative complications and the heavy financial and social consequences of divorces. We applied the results of the control theory to characterize the optimal controls. The numerical simulation of the results obtained showed the effectiveness of the proposed control strategies.

[1] Joshi H. R. Optimal control of an HIV immunology model. Optimal Control Applications and Methods. **23** (4), 199–213 (2002).

[2] Kenneth I., Granle Jr. Issues of Legitimacy for Children of Polygamous Marriage in US Immigration Practices. Open Journal for Legal Studies. **3** (1), 79–86 (2020).

[3] Bahari I. S., Norhayati M. N., Hazlina N. H. N., Aiman C. A. A. M. S., Arif N. A. N. M. Psychological impact of polygamous marriage on women and children: a systematic review and meta-analysis. BMC Pregnancy and Childbirth. **21**, 823 (2021).

[4] Yazid M., Fakhri A., Mohammad O., Zaini Y. Review on the Provisions of Polygamous Marriage in Malaysia. Hak cipta, Jabatan Syariah, Fakulti Pengajian Islam, UKM (2015).

[5] Berg P., Guldbrandtsen B., Rune B. L., Christian N., Storm P. The polygamic marriage problem (2014).

[6] Al-Krenawi A, Graham J., Izzeldin A. The Psychosocial Impact of Polygamous Marriages on Palestinian Women. Women & Health. **34** (1), 1–16 (2001).

[7] Al-Krenawi A., Graham J. R. The story of bedouin-arab women in a polygamous marriage. Women's Studies International Forum. **22** (5), 497–509 (1999).

[8] Lhous M., Rachik M., Laarabi H., Abdelhak A. Discrete mathematical modeling and optimal control of the marital status: the monogamous marriage case. Advances in Difference Equations. **2017**, 339 (2017).

[9] Lhous M., Zakary O., Rachik M. A Mathematical Overview of the Monogamous Marriage in a Multiregion Framework: Modelling and Control. Discrete Dynamics in Nature and Society. **2020**, 9385261 (2020).

[10] Göllmann L., Maurer H. Theory and application of optimal control problem with multiple time-delays. Journal of Industrial and Management Optimisation. **10** (2), 413–441 (2014).

[11] Kouidere A., Labzai A., Khajji B., Ferjouchia H., Balatif O., Boutayeb A., Rachik M. Optimal control strategy with multi-delay in state and control variables of a discrete mathematical modeling for the dynamics of diabetic population. Communications in Mathematical Biology and Neuroscience. **2020**, 14 (2020).

[12] Cristiana J. S., Maurer H., Torres D. F. M. Optimal cobtrol of a tuberculosis model with state and control delays. Mathematical Biosciences and Engineering. **14** (1), 321–337 (2017).

[13] Kouidere A., Kada D., Balatif O., Rachik O., Naim M. Optimal control approach of a mathematical modeling with multiple delays of the negative impact of delays in applying preventive precautions against the spread of the COVID-19 pandemic with a case study of Brazil and cost-effectiveness. Chaos, Solitons & Fractals. **142**, 110438 (2021).

[14] Pontryagin L. S., Boltyanskii V. G., Gamkrelidze R. V., Mishchenko E. F. The Mathematical Theory of Optimal Processes. Wiley, New York (1962).

[15] Sakkoum A., Lhous M., Magri El. M. A mathematical simulation and optimal control of a VIH model with different infectious level. Journal of Mathematical and Computational Science. **12**, 117 (2022).

Дискретне математичне моделювання та оптимальне керування сімейним станом: модель ісламського полігамного шлюбу

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У цій статті обговорюється дискретна математична модель ісламської полігамії та соціального становища мусульман. В одинадцяти складових пояснено соціальну ситуацію та дано пояснення щодо сімейного стану кожного чоловіка та жінки в ісламських суспільствах, які допускають полігамію. Щоб контролювати та зменшувати кількість незайманих чоловіків і жінок, розлучених чоловіків і жінок, реалізуються дві керуючі змінні. Перша характеризує переваги інформаційної кампанії для навчання незайманих чоловіків і жінок щодо переваги шлюбу для особистості та суспільства, а друга стосується юридичних процедур, адміністративних складнощів і серйозних фінансових та соціальних наслідків розлучення. Після цього застосовано теорію оптимального керування, щоб описати такі оптимальні стратегії, і, нарешті, було виконано чисельне моделювання для перевірки теоретичного аналізу з використанням прогресивно-регресивної дискретної схеми, яка збігається відповідно до зручного тесту, пов'язаного з методом прямої та зворотної прогонки.

Ключові слова: *дискретна модель сімейного стану; полігамний шлюб; принцип максимуму Понтрягіна; оптимальне керування.*