MATHEMATICAL MODELING AND COMPUTING, Vol. 10, No. 3, pp. 807–815 (2023)

A new algorithm for solving Toeplitz linear systems

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(Received 15 February 2023; Revised 20 July 2023; Accepted 21 July 2023)

In this paper, we are interested in solving the Toeplitz linear systems. By exploiting the special Toeplitz structure, we give a new decomposition form of the coefficient matrix. Based on this matrix decomposition form and combined with the Sherman–Morrison formula, we propose an efficient algorithm for solving the considered problem. A typical example is presented to illustrate the different steps of the proposed algorithm. In addition, numerical tests are given showing the efficiency of our algorithm.

Keywords: *Toeplitz matrix; Sherman–Morrison formula; decomposition method.*
2010 MSC: 15-XX, 15A09, 15A23 **DOI:** 10.23939/mmc2023.03.807

1. Introduction

We consider the solution of the following n by n system:

$$Tx = f, \tag{1}$$

where T is a Toeplitz matrix defined as:

$$T = \begin{bmatrix} t_0 & t_{-1} & \dots & \dots & t_{2-n} & t_{1-n} \\ t_1 & t_0 & t_{-1} & \ddots & \ddots & t_{2-n} \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ t_{n-2} & \ddots & \ddots & t_1 & t_0 & t_{-1} \\ t_{n-1} & t_{n-2} & \dots & \dots & t_1 & t_0 \end{bmatrix}$$

and

$$x, f \in \mathbb{R}^{n\times 1}.$$

Toeplitz matrices emerge in many applications and are encountered in many problems such as digital signal processing (numerical filtering, estimation theory, speech processing, etc.), as well as image processing, control theory, partial differential equations (elliptic or parabolic), integral equations (convolution type Volterra or Fredholm), Padé approximation and other areas of numerical analysis [1].

The linear system (1) has been the subject of several studies. In 2010 A. Chesnokov and al. presented a fast algorithm to solve Toeplitz block systems, this algorithm builds the circular transformation of the given Toeplitz system, then using the formula of Sherman–Morrison–Woodbury transforms its inverse into the inverse of the original matrix [2]. D. A. Bini used the fast Fourier transformation to invert triangular Toeplitz matrices [3]. Recently W. K. Lin and al. introduced an approximate inversion method for Toeplitz triangular matrices based on the polynomial of trigonometric interpolation, they proposed a revised algorithm of Bini method [4].

Many other methods have been proposed for solving such systems. These methods are Gaussian elimination [5], cyclic reduction [6], special LU factorization [7, 8], and Toeplitz factorization with Sherman–Morrison formula [9–11].

When the Toeplitz matrix is full, solving the system $Tx = b$ becomes more difficult and requires special methods to be solved. In this paper, by writing the considered matrix as a sum of two matrices, the first as band Toeplitz matrix and the second as the remainder which will be factorized as a product of two matrices to be specified. Thus, the inverse of the Toeplitz matrix will be reduced to the computation of the inverse of Banded Toeplitz matrix after applying the Sherman–Morrison–Woodburu formula.

The paper is organize as follows: in Section 2, we present some results concerning the computation of a banded Toeplitz matrix; Section 3 gives the different steps of our new algorithm to solve the Toeplitz linear systems. Finally, in Section 4, typical example is presents to illustrate the steps of the proposed algorithm. In addition, numerical examples are given to put in evidence the potential advantages of our method in terms of CPU-time and quality of the obtained solution.

2. Inverse of a banded Toeplitz matrix

In the following, we present an algorithm which computes the inverse of $(2k+1)$-banded Toeplitz matrix T_b of size n, given as follow:

$$T_b = \begin{bmatrix} t_0 & t_{-1} & \dots & t_{-k} & 0 & \dots & 0 & 0 \\ t_1 & t_0 & t_{-1} & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ t_k & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & t_{-k} \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & t_1 & t_0 & t_{-1} \\ 0 & 0 & \dots & 0 & t_k & \dots & t_1 & t_0 \end{bmatrix}.$$

The approach consists in embedding the banded Toeplitz matrix in a larger triangular Toeplitz matrix [12].

Specifically, we embed T_b in a lower triangular Toeplitz matrix M of size $(n+k)\times(n+k)$, where the first column of M is given by

$$r = \left(t_{-k}, \dots, t_{-1}, t_0, t_1, \dots, t_k, 0, \dots, 0\right)^T \in \mathbb{R}^{n+k,1}.$$

More precisely, if we note L a matrix of size $k\times k$ given by

$$L = \begin{bmatrix} t_{-k} & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ t_{-1} & \dots & \dots & t_{-k} \end{bmatrix}.$$

We can write M as follows

$$M = \begin{bmatrix} t_{-k} & 0 & \dots & \dots & \dots & \dots & \dots & \dots & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ t_{-1} & \dots & t_{-k} & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ t_0 & t_{-1} & \ddots & t_{-k} & 0 & \ddots & 0 & \ddots & \ddots & \vdots \\ t_1 & t_0 & t_{-1} & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 & \ddots & \ddots & \vdots \\ t_k & \ddots & \ddots & \ddots & \ddots & \ddots & t_{-k} & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & t_{-k} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & t_k & \dots & t_1 & t_0 & t_{-1} & \dots & t_{-k} \end{bmatrix} = \begin{bmatrix} L & 0 & \dots & 0 \\ & & & \vdots \\ & T_b & & 0 \\ & & & L \end{bmatrix} \tag{2}$$

To compute the inverse of matrix M, we present the following result.

Proposition 1 (Ref. [12]). Let M be a lower triangular Toeplitz matrix as defined in (2). Thus, M^{-1} is a lower triangular Toeplitz matrix defined by its first column: $\left(v_1, v_2, \ldots, v_{n+k}\right)^T$. In addition, M^{-1} can be partitioned as follows:

$$M^{-1} = \begin{bmatrix} v_1 & 0 & \dots & \dots & \dots & \dots & \dots & \dots & \dots & 0 \\ v_2 & v_1 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ v_{n-k+1} & \ddots & \ddots & v_1 & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & v_2 & v_1 & \ddots & \ddots & \ddots & \ddots & \vdots \\ v_{n-1} & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ v_n & v_{n-1} & \ddots & v_{n-k+1} & \ddots & \ddots & v_1 & \ddots & \ddots & \vdots \\ v_{n+1} & v_n & \ddots & \ddots & v_{n-k+1} & \ddots & v_2 & v_1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ v_{n+k} & \dots & v_{n+1} & v_n & v_{n-1} & \dots & v_{n-k+1} & \dots & v_2 & v_1 \end{bmatrix} = \begin{bmatrix} A & B \\ & \\ C & D \end{bmatrix},$$

where

$$A = \begin{pmatrix} v_1 & 0 & \dots & 0 \\ v_2 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ v_{n-k+1} & \ddots & \ddots & v_1 \\ \vdots & \ddots & \ddots & v_2 \\ v_{n-1} & \ddots & \ddots & \vdots \\ v_n & v_{n-1} & \dots & v_{n-k+1} \end{pmatrix}, \quad B = \begin{pmatrix} 0 & \dots & \dots & \dots & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ v_1 & 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ v_{n-k} & \dots & v_1 & 0 & \dots & 0 \end{pmatrix},$$

$$C = \begin{pmatrix} v_{n+1} & v_n & \dots & v_{n-k+2} \\ v_{n+2} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & v_n \\ v_{n+k} & \dots & v_{n+2} & v_{n+1} \end{pmatrix}, \quad D = \begin{pmatrix} v_{n-k+1} & \dots & v_2 & v_1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ v_{n-1} & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ v_n & v_{n-1} & \dots & v_{n-k+1} & \dots & v_2 & v_1 \end{pmatrix}$$

are matrices of size $n \times k$, $n \times n$, $k \times k$, $k \times n$, respectively.

Then, we investigate the structure of matrix M^{-1} to compute the inverse of the banded Toeplitz matrix T_b. This result is presented in Theorem 1.

Theorem 1 (Ref. [12]). *Let T_b be a nonsingular banded Toeplitz matrix and M its associated lower triangular matrix. Suppose that M^{-1} is partitioned as follows:*

$$M^{-1} = \begin{bmatrix} A & B \\ & \\ C & D \end{bmatrix},$$

where A, B, C, and D are matrices of size $n \times k$, $n \times n$, $k \times k$, $k \times n$, respectively. If T_b is nonsingular, then C^{-1} is also nonsingular and the Schur complement of the block C for the matrix M^{-1} is defined by

$$T_b^{-1} = B - AC^{-1}D.$$

Proof. Let

$$M = \begin{bmatrix} L_1 & 0_k \\ & \\ T_b & L_2 \end{bmatrix}$$

is a nonsingular $(n+k)\times(n+k)$ matrix, where 0_k is a $k\times k$ null matrix

$$L_1 = \begin{bmatrix} L & 0_k & \dots & 0_k \end{bmatrix}$$

and

$$L_2 = \begin{bmatrix} L \\ 0_k \\ \vdots \\ 0_k \end{bmatrix},$$

such that,

$$L = \begin{bmatrix} t_{-k} & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ t_{-1} & \dots & \dots & t_{-k} \end{bmatrix}.$$

Let

$$P = \begin{bmatrix} 0_{n,k} & I_n \\ I_k & 0_{n,k} \end{bmatrix},$$

where $O_{n,k}$ denotes an $n\times k$ null matrix, and T_b is nonsingular.

Thus,

$$PM = \begin{bmatrix} T_b & L_2 \\ L_1 & 0_k \end{bmatrix} = \begin{bmatrix} T_b & 0_{n,k} \\ L_2 & R \end{bmatrix} \times \begin{bmatrix} I_n & T_b^{-1}L_2 \\ 0_{k,n} & I_k \end{bmatrix},$$

where $R = -L_1 T_b^{-1} L_2$, if T_b is nonsingular and R is nonsingular.

Then,

$$M^{-1} = \begin{bmatrix} I_n & -T_b^{-1}L_2 \\ 0_{k,n} & I_k \end{bmatrix} \times \begin{bmatrix} T_b^{-1} & 0_{n,k} \\ -R^{-1}L_1T_b^{-1} & R^{-1} \end{bmatrix}$$
$$= \begin{bmatrix} -T_b^{-1}L_2R^{-1} & T_b^{-1} + T_b^{-1}L_2R^{-1}L_1T_b^{-1} \\ R^{-1} & R^{-1}L_1T_b^{-1} \end{bmatrix}.$$

So, we have

$$\begin{cases} A = -T_b^{-1}L_2R^{-1}, \\ B = T_b^{-1} + T_b^{-1}L_2R^{-1}L_1T_b^{-1}, \\ C = R^{-1}, \\ D = R^{-1}L_1T_b^{-1}. \end{cases}$$

Finally,

$$T_b^{-1} = B - T_b^{-1}L_2R^{-1}L_1T_b^{-1} = B - AC^{-1}D.$$

■

3. Main algorithm

In this section, we present our approach to solve the class of linear systems (1) based on a new decomposition of the Toeplitz matrix and the application of Sherman–Morrison–Woodburu formula. Indeed, the considered decomposition consists to express the matrix T in the form of banded Toepliz matrix and the rest which will be decomposed in a well-determined form. Then, the application of Sherman–Morrison–Woodburu formula reduced the computation of the inverse of T to the inverse of T_b as banded Toeplitz matrix given on the decomposition (4) defined as follows:

$$T = T_b + F,$$

where

$$T_b = \begin{bmatrix} t_0 & t_{-1} & \dots & t_{-k} & 0 & \dots & \dots & 0 \\ t_1 & t_0 & t_{-1} & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ t_k & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & t_{-k} \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & t_1 & t_0 & t_{-1} \\ 0 & \dots & \dots & 0 & t_k & \dots & t_1 & t_0 \end{bmatrix}$$

T_b is $(2k+1)$-band Toeplitz n-size matrix.

$$F = \begin{bmatrix} 0 & \dots & 0 & t_{-(k+1)} & \dots & \dots & t_{-(n-1)} \\ 0 & 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & t_{-(k+1)} \\ t_{(k+1)} & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ t_{(n-1)} & \dots & \dots & t_{(k+1)} & 0 & \dots & 0 \end{bmatrix},$$

F is matrix of rank no more than $2k$ of size n.

So, the resolution of the system (1) is equivalent to the following system:

$$(T_b + F)x = b.$$

Then, the solution is given by

$$x = (T_b + F)^{-1} b.$$

To find the inverse of matrix $(T_b + F)$, we can write $(T_b + F)$ in the form:

$$(T_b + F) = (T_b + UV^T),$$

where

$$U = \begin{bmatrix} I & O \\ & \\ O & P \end{bmatrix} \quad \text{and} \quad V^T = \begin{bmatrix} O & Q \\ & \\ I & O \end{bmatrix}, \tag{3}$$

$$P = \begin{bmatrix} t_{(k+1)} & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ t_{(n-1)} & \dots & \dots & t_{(k+1)} \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} t_{-(k+1)} & \dots & \dots & t_{-(n-1)} \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & t_{-(k+1)} \end{bmatrix}$$

The matrices P and Q are $(n-k-1, n-k-1)$ triangular matrices that reside in the lower left and upper right corners of F, respectively. It should be noted that, the size of the matrices O and I depend on the number k which must be adapted to be able to calculate the product of U and V^T.

Then, we apply the *formula of Sherman–Morrison–Woodbury* to get

$$(T_b + UV^T)^{-1} = T_b^{-1} - T_b^{-1}U(I_n + V^T T_b^{-1} U)^{-1} V^T T_b^{-1}.$$

Finally, we obtain the following solution

$$x = T_b^{-1} f - T_b^{-1}U(I_n + V^T T_b^{-1} U)^{-1} V^T T_b^{-1} f. \tag{4}$$

Algorithm 1 Solving a Toeplitz system.

1: Input T, f, k.
2: Recover T_b and F from T.
3: Recover U and V as defined in (3).
4: Give the matrix M as defined in (2).
5: Calculate the inverse M^{-1}.
6: Recover A, B, C, and D from M^{-1}.
7: Compute $T_b^{-1} = B - AC^{-1}D$.
8: Compute the solution x using

$$x = T_b^{-1} f - T_b^{-1}U(I_n + V^T T_b^{-1} U)^{-1} V^T T_b^{-1} f.$$

4. Numerical experiments

In this section, three examples are considered to show the efficiency of the proposed algorithm. The first example is a typical test of a matrix of size $(6,6)$ illustrating the different steps of the proposed algorithm. However, examples 2 and 3 are numerical tests carried out on Matlab on Symmetric and Non-symmetric Toeplitz matrices, of different sizes, implemented on an Intel(R) Core(TM) i3-3110M CPU @ with a 2.40 GHz processor and 4 GB of RAM, to show the efficiency of the algorithm in terms of CPU-time and the solution of the system considered. Since the solutions of the considered systems are unknown, we evaluate the error given by $\|Tx - f\|$ by ∞-norm, in addition to the execution (CPU) time in seconds of our method.

Example 1. In this first example, we consider a full Toeplitz matrix of size 6×6 to illustrate the different steps of the proposed algorithm.

Step 1: Consider the system:

$$\begin{pmatrix} -1 & -1 & 2 & 0 & 1 & 1 \\ -1 & -1 & -1 & 2 & 0 & 1 \\ 2 & -1 & -1 & -1 & 2 & 0 \\ 0 & 2 & -1 & -1 & -1 & 2 \\ 1 & 0 & 2 & -1 & -1 & -1 \\ 1 & 1 & 0 & 2 & -1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 0 \\ 0 \\ -3 \\ 1 \end{pmatrix}.$$

Step 2: In this step, we recover T_b and F from the matrix F.

We fix $k = 3$, so $t_k = t_3 = 2 \neq 0$ i.e. the condition on the choice of k is that is t_k not null.

Then, we obtain:

$$T_b = \begin{pmatrix} -1 & -1 & 2 & 0 & 0 & 0 \\ -1 & -1 & -1 & 2 & 0 & 0 \\ 2 & -1 & -1 & -1 & 2 & 0 \\ 0 & 2 & -1 & -1 & -1 & 2 \\ 0 & 0 & 2 & -1 & -1 & -1 \\ 0 & 0 & 0 & 2 & -1 & -1 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Step 3: To complete the decomposition, we use (3) to obtain the decomposition of $F = UV^T$,

$$U = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}, \quad V^T = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Step 4: The triangular matrix M of size 8 obtained as defined in (2) is given by

$$M = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 2 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & 2 & 0 & 0 & 0 & 0 \\ 2 & -1 & -1 & -1 & 2 & 0 & 0 & 0 \\ 0 & 2 & -1 & -1 & -1 & 2 & 0 & 0 \\ 0 & 0 & 2 & -1 & -1 & -1 & 2 & 0 \\ 0 & 0 & 0 & 2 & -1 & -1 & -1 & 2 \end{pmatrix}.$$

Step 5: In this step, we compute the inverse of M where we obtain

$$M^{-1} = \begin{pmatrix} 0.5000 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.2500 & 0.5000 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.3750 & 0.2500 & 0.5000 & 0 & 0 & 0 & 0 & 0 \\ 0.5625 & 0.3750 & 0.2500 & 0.5000 & 0 & 0 & 0 & 0 \\ 0.0938 & 0.5625 & 0.3750 & 0.2500 & 0.5000 & 0 & 0 & 0 \\ 0.2656 & 0.0938 & 0.5625 & 0.3750 & 0.2500 & 0.5000 & 0 & 0 \\ 0.0859 & 0.2656 & 0.0938 & 0.5625 & 0.3750 & 0.2500 & 0.5000 & 0 \\ -0.3398 & 0.0859 & 0.2656 & 0.0938 & 0.5625 & 0.3750 & 0.2500 & 0.5000 \end{pmatrix}.$$

Step 6: We recover the matrices A, B, C, and D from the matrix M^{-1} as defined in Theorem 1,

$$A = \begin{pmatrix} 0.5000 & 0 \\ 0.2500 & 0.5000 \\ 0.3750 & 0.2500 \\ 0.5625 & 0.3750 \\ 0.0938 & 0.5625 \\ 0.2656 & 0.0938 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0.5000 & 0 & 0 & 0 & 0 & 0 \\ 0.2500 & 0.5000 & 0 & 0 & 0 & 0 \\ 0.3750 & 0.2500 & 0.5000 & 0 & 0 & 0 \\ 0.5625 & 0.3750 & 0.2500 & 0.5000 & 0 & 0 \end{pmatrix},$$

$$C = \begin{pmatrix} 0.0859 & 0.2656 \\ -0.3398 & 0.0859 \end{pmatrix}, \quad D = \begin{pmatrix} 0.0938 & 0.5625 & 0.3750 & 0.2500 & 0.5000 & 0 \\ 0.2656 & 0.0938 & 0.5625 & 0.3750 & 0.2500 & 0.5000 \end{pmatrix}.$$

Step 7: We compute the inverse of T_b from the considered matrices in the previous step using the following formula:

$$T_b^{-1} = B - AC^{-1}D$$

to obtain,

$$T_b^{-1} = \begin{pmatrix} 0.3200 & -0.1200 & 0.6000 & 0.4000 & 0.1200 & 0.6800 \\ -0.1200 & -1.0800 & -0.6000 & -0.4000 & -0.9200 & 0.1200 \\ 0.6000 & -0.6000 & -0.0000 & -0.0000 & -0.4000 & 0.4000 \\ 0.4000 & -0.4000 & -0.0000 & -0.0000 & -0.6000 & 0.6000 \\ 0.1200 & -0.9200 & -0.4000 & -0.6000 & -1.0800 & -0.1200 \\ 0.6800 & 0.1200 & 0.4000 & 0.6000 & -0.1200 & 0.3200 \end{pmatrix}.$$

Step 8: The solution of the system $Tx = f$ is given in this step following the given formula in (4),

$$x = T_b^{-1}f - T_b^{-1}U(I_n + V^T T_b^{-1}U)^{-1}V^T T_b^{-1}f = \begin{pmatrix} -0.3533 \\ 0.5978 \\ -0.3804 \\ 0.8804 \\ 0.9022 \\ 0.1033 \end{pmatrix}.$$

Example 2. In this example, we consider the $n \times n$ symmetric Toeplitz linear system $Tx = f$ of the form

$$\begin{pmatrix} -1 & 1 & 1 & \dots & \dots & 1 & 1 \\ 1 & -1 & 1 & 1 & \ddots & \ddots & 1 \\ 1 & 1 & -1 & 1 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 1 \\ 1 & \ddots & \ddots & 1 & 1 & -1 & 1 \\ 1 & 1 & \dots & \dots & 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_{n-2} \\ x_{n-1} \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 0 \\ \vdots \\ 0 \\ -3 \\ 1 \end{pmatrix}.$$

Table 1. Numerical result for Example 2.

n	$\lVert Tx - f \rVert$	cpu-time (s)	Cond(T)
60	$2.3314e^{-15}$	$6.3600e^{-2}$	29.0000
100	$4.2188e^{-15}$	$6.2400e^{-2}$	49.0000
300	$6.6613e^{-15}$	$6.2400e^{-2}$	146.2364
500	$8.8817e^{-15}$	$6.2400e^{-2}$	243.2778
1000	$2.5535e^{-14}$	$1.0920e^{-1}$	485.8879
2000	$5.6621e{-}14$	$1.0920e^{-1}$	971.1069

Example 3. Next, we consider the $n \times n$ non-symmetric Toeplitz linear systems $Tx = f$ of the form

$$\begin{pmatrix} -4 & 1 & 1 & \dots & \dots & 1 & 1 \\ 2 & -4 & 1 & 1 & \ddots & \ddots & 1 \\ -1 & 2 & -4 & 1 & 1 & \ddots & \vdots \\ 1 & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 1 \\ 1 & \ddots & 1 & -1 & 2 & -4 & 1 \\ 1 & 1 & \dots & 1 & -1 & 2 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_{n-2} \\ x_{n-1} \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 0 \\ \vdots \\ 0 \\ -3 \\ -1 \end{pmatrix}$$

Table 2. Numerical result for Example 3.

n	$\lVert Tx - f \rVert$	cpu-time (s)	Cond(T)
60	$5.0626\,e^{-14}$	$1.5600\,e^{-2}$	18.1512
100	$2.9531\,e^{-14}$	$3.1200\,e^{-2}$	31.6792
300	$1.8496\,e^{-13}$	$3.1200\,e^{-2}$	99.2284
500	$1.5032\,e^{-13}$	$4.6800\,e^{-2}$	166.7521
1000	$3.2474\,e^{-13}$	$3.1200\,e^{-2}$	335.5481
2000	$2.8903\,e^{-12}$	$1.4040\,e^{-1}$	673.1309

From the numerical tests, we show the performance of our algorithm. In fact, the results given in Tables 1 and 2 show that, for different choices of size n, with symmetric and non-symmetric Toeplitz Full Matrices, we obtain a good approximation in terms of numerical efficiency and computational time. In addition, the computation of the Cond(T) show that the considered matrices are ill-conditioned, so our approach has a regularizing effect.

5. Conclusion

In this paper, we have presented a new algorithm to solve Toeplitz linear systems. We have broken down the matrix into the sum of two matrices. The first one is band Toeplitz matrix of size $(2k+1)$ and the second one is matrix of a rank less than $2k$. Then, we applied Sherman–Morrison formula to find the inverse of the decomposed matrix reducing the computation of the inverse of Toeplitz matrix of the initial system to the inverse of banded Toeplitz matrix. Numerical examples are presented showing the efficiency of the proposed method.

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Новий алгоритм розв'язування лінійних систем Тепліца

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У цій статті нас цікавить розв'язання лінійних систем Тепліца. Використовуючи спеціальну структуру Тепліца, даємо нову форму розкладання матриці коефіцієнтів. Базуючись на цій матричній формі декомпозиції та в поєднанні з формулою Шермана–Морісона, запропоновано ефективний алгоритм для розв'язання розглянутої проблеми. Наведено типовий приклад для ілюстрації різних кроків запропонованого алгоритму. Крім того, наведені чисельні тести, що демонструють ефективність нашого алгоритму.

Ключові слова: *матриця Тепліца; формула Шермана–Морісона; метод декомпозиції.*