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Dynamic von Karman equations with viscous damping

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In this paper we are interested to the dynamic von Karman equations coupled with viscous damping and without rotational forces, ($\alpha = 0$) [Chueshov I., Lasiecka I. (2010)], this problem describes the buckling and flexible phenomenon of small nonlinear vibration of vertical displacement to the elastic plates. Our fundamental goal is to establish the existence and the uniqueness to the weak solution for the so-called global energy, under assumption $F_0 \in H^{3+\varepsilon}(\omega)$. Finally for illustrate our theoretical results we use the finite difference method.

Keywords: *von Karman equation; nonlinear plates; viscous damping; rotational inertia; non-coupled method; finite difference method.*

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1. Introduction

Dynamic von Karman equations without rotational forces, ($\alpha = 0$) [1], describe the buckling phenomenon nonlinear vibration to the vertical displacement of elastic plates. There is considered the case when the plate is acted with an active damping in nonlinear thermoelastic plate interaction. From physical point of view the main peculiarities of the model are the possibility of small deflections of the plate and small changes of the temperature near the reference configuration of the plate. For displacement u, the Airy stress function ϕ and the thermal function θ, can be formulated by the following system, see for instance [1].

Find $(u, \phi) \in L^2([0,T], (H_0^2(\omega))^2)$ such that

$$(\mathbb{P}) \begin{cases} u_{tt} + d_0(x) g_0(u_t) + \Delta^2 u - [\phi + F_0, u] = p(x) & \text{in } \omega \times [0,T], \\ \Delta^2 \phi + [u,u] = 0, & \text{in } \omega \times [0,T], \\ u\big|_{t=0} = u_0,\ (u_t)\big|_{t=0} = u^1, & \text{in } \omega, \\ u = \partial_\nu u = 0,\ \phi = 0,\ \partial_\nu \phi = 0 & \text{on } \Gamma \times [0,T], \end{cases}$$

where ω is the surface plate, u_0, u_1 and θ_0 are the initial data and $[\cdot,\cdot]$ is the so-called Monge–Ampère operator defined by [2]

$$[\phi, u] = \partial_{11}\phi \partial_{22} u + \partial_{11} u \partial_{22}\phi - 2\partial_{12}\phi\partial_{12} u. \tag{1}$$

The case $\alpha > 0$ corresponds to the equation with rotational term. But the parameters μ, η are positive fixed real numbers and the parameter k measures the capacity of heat/thermal to the model of thermoelastic problem [1]. Now, in the case $k = 0$ and $\alpha = 0$, if we substitute $\Delta\theta$ from the second equation in the model of thermoelastic, [1] without rotational inertia can be decoupled. The first equation becomes just a model of dynamic von Karman equations with internal viscous damping [1].

The plate is subjected to the internal force $F_0 \in H^{3+\varepsilon}(\omega)$ and the external force p_0.

Our fundamental target in this paper is to give a condition verified by the external/internal loads and the initial datums to have a uniqueness weak solution of the von Karman evolution without rotational term and nor clamped boundary conditions subject to active contracting damping operator $d_0(\cdot)g_0(\cdot)$. Our approach is based on an iterative problem $(P_n)_{n\geqslant 0}$ when sequence-solution $(u_n, \phi_n)_{n\geqslant 0}$ converges to the uniqueness solution of the problem under consideration.

This paper will be organized as follows. After this introduction, Section 2 contains some basic results, Section 3 is devoted to describe the mathematical structure of the considered model by using an iterative method for establishing the uniqueness weak solution. Finally Section 4 display a numerical simulation of our initial problem.

2. Preliminary results

In this paper, ω denotes a nonempty bounded domain in $\mathbb{R}^2$, with regular boundary $\Gamma = \partial\omega$.

Let $p \geqslant 1$ be a real number and $m \geqslant 1$ be an integer. We denote by $|\cdot|_{p,\omega}$ the classical norm of $L^p(\omega)$ and by $\|\cdot\|_{m,\omega}$ that of $H^m(\omega)$. For $u \in H_0^2(\omega)$, we set $\|u\| = |\Delta u|_{2,\omega}$ for the sake of simplicity.

We also denote

$$\|u\|_* = \|u\|^2 + |u_t|_{2,\omega}^2. \tag{2}$$

Theorem 1 (Refs. [2,3]). *Let $f \in L^2(\omega)$. Then the following problem:*

$$\begin{cases} \Delta^2 v = f & \text{in } \omega, \\ v = 0 & \text{on } \Gamma, \\ \partial_\nu v = 0 & \text{on } \Gamma \end{cases}$$

has one and only one solution $v \in H_0^2(\omega) \cap H^4(\omega)$ satisfying

$$\|v\| \leqslant c_0 |f|_{1,\omega}$$

for some constant $c_0 > 0$ depending only on $\mathrm{mes}(\omega)$.

Theorem 2 (Ref. [3]). *Let $f \in L^2\big([0,T], H^2(\omega)\big)$, $u_0 \in L^2(\omega)$ and $k \geqslant 0$, $0 \leqslant \eta \leqslant \mu$ are non negative reals. Then the following problem:*

$$(\mathbb{D}) \begin{cases} k u_t - \eta \Delta u = -\mu \Delta f & \text{in } \omega \times [0,T], \\ u_{|t=0} = u_0 & \text{in } \omega, \\ u = 0 & \text{on } \Gamma \times [0,T] \end{cases}$$

has one and only one solution $u \in C([0,T]; H^2(\omega) \cap H_0^1(\omega)) \cap C^1([0,T]; L^2(\omega))$, satisfies the following inequality:

$$\forall 0 \leqslant t \leqslant T, \quad k|u|_{2,\omega}^2 + \eta \int_0^t |\nabla u|_{2,\omega}^2 \leqslant k|u_0|_{2,\omega}^2 + \mu \int_0^t |\nabla f|_{2,\omega}^2. \tag{3}$$

Theorem 3 (Ref. [1]). *Let $p \in L^2(\omega)$ and $(u_0, u^1) \in H_0^2(\omega) \times H_0^1(\omega)$, the problem*

$$(\mathbb{S}) \begin{cases} u_{tt} + \Delta^2 u = p & \text{in } \omega \times [0,T], \\ u = \partial_\nu u = 0 & \text{on } \Gamma \times [0,T], \\ u\big|_{t=0} = u_0, \ (u_t)\big|_{t=0} = u^1 & \text{in } \omega \end{cases}$$

has one and only one solution $(u, u_t) \in C([0,T], H_0^2(\omega) \times H_0^1(\omega))$.

Now, let us put

$$F_1(u,\phi) = [\phi + F_0, u] - d_0(x) g_0(u_t). \tag{4}$$

Proposition 2. Let $(u,v) \in (H_0^2(\omega))^2$ and $F_0 \in H^{3+\varepsilon}(\omega)$ be with small norms. Let $\varphi, \psi \in H_0^2(\omega)$ be the solutions of the following two problems:

$$\Delta^2 \phi = -[u,u] \ \text{ and } \ \Delta^2 \psi = -[v,v],$$

and $d_0(\cdot) g_0(\cdot)$ is a contracting non linear operator on u_t.

Then the following estimations:

$$\big|[u,\phi] - [v,\psi]\big|_{2,\omega} \leqslant c_1 \|u - v\|$$

and

$$\big|F_1(u,\phi) - F_1(v,\psi)\big|_{2,\omega} \leqslant c_2 \big(\|u - v\| + |u_t - v_t|_{2,\omega}\big)$$

hold for some $0 < c_1 < 1$ and $0 < c_2 < 1$.

Proof. Following [1],
$$\big|[u,\phi]-[v,\psi]\big|_{2,\omega} \leqslant c_0\big(\|u\|^2+\|v\|^2\big)\|u-v\|,$$
for some $c_0>0$. Let $c>0$ be small enough such that $\|u\|\leqslant c$ and $\|v\|\leqslant c$. We have
$$\big|[u,\phi]-[v,\psi]\big|_{2,\omega} \leqslant 2c_0c^2\|u-v\| \leqslant c_1\|u-v\|$$
with $0<c_1=2c_0c^2<1$ and there exists $0<\alpha<1$ so
$$\begin{aligned}\big|F_1(u,\phi)-F_1(v,\psi)\big|_{2,\omega} &\leqslant \big|[\phi+F_0,u]-[\psi+F_0,v]\big|_{2,\omega}+\big|d_0(x)g_0(u_t)-d_0(x)g_0(v_t)\big|_{2,\omega}\\ &\leqslant \big|[\phi,u]-[\psi,v]\big|_{2,\omega}+\big|[F_0,u-v]\big|_{2,\omega}+\alpha\big|u_t-v_t\big|_{2,\omega}\\ &\leqslant \big(2c_0c^2+4\|F_0\|_{3+\varepsilon,\omega}\big)\|u-v\|+\alpha|u_t-v_t|_{2,\omega}.\end{aligned}$$
If we choose
$$\|F_0\|_{3+\varepsilon,\omega}<\frac{1}{4} \text{ and } 0<c<\sqrt{\frac{1-4\|F_0\|_{3+\varepsilon,\omega}}{2c_0}},$$
then
$$0<c_2=\max\big(2c_0c^2+4\|F_0\|_{3+\varepsilon,\omega},\alpha\big)<1,$$
and
$$\big|F_1(u,\phi)-F_1(v,\psi)\big|_{2,\omega} \leqslant c_2\big(\|u-v\|+|u_t-v_t|_{2,\omega}\big),$$
so we conclude the proof. ■

Proposition 3. Assume that for $p\in L^2([0,T],L^2(\omega))$, $\theta_0\in H_0^1(\omega)$ and $(u_0,u^1)\in H_0^2(\omega)\times H_0^1(\omega)$. The following problem:
$$(\mathbb{S}_1)\begin{cases} u_{tt}+\Delta^2u+\mu\Delta\theta=p & \text{in } \omega\times[0,T],\\ k\theta_t-\eta\Delta\theta=\mu\Delta u_t & \text{in } \omega\times[0,T],\\ u=\partial_\nu u=\theta=0 & \text{on } \Gamma\times[0,T],\\ (u)\big|_{t=0}=u_0,\ (u_t)\big|_{t=0}=u^1,\ (\theta)\big|_{t=0}=\theta_0 & \text{in } \omega\end{cases}$$
has one and only one solution $(u,\theta)\in L^2([0,T],H_0^2(\omega)\times H_0^1(\omega))$ and $u_t\in L^2([0,T],H_0^1(\omega))$ satisfying that
$$\|u\|_*+k|\theta|_{2,\omega}^2+2\eta\int_0^t|\nabla\theta|_{2,\omega}^2 \leqslant e^T\left(\|u_0\|^2+|u^1|_{2,\omega}^2+k|\theta_0|_{2,\omega}^2+\int_0^T|p|_{2,\omega}^2\right). \tag{5}$$

Proof. For establishing the existence and uniqueness solution of the problem under consideration, we will study the problem $(\mathbb{S}_1)$ by considering the n-order approximate solution and use the variational problem, see [4]. ■

3. The main results

For establishing the uniqueness solution to the problem $(\mathbb{P})$, we use the following iterative approach.

Let $n\geqslant 2$ and let $0\neq u_1\in H_0^2(\omega)$ be given.
$$(\mathbb{P}_n)\begin{cases} (u_n)_{tt}+\Delta^2u_n=G(u_{n-1},\phi_{n-1}) & \text{in } \omega\times[0,T],\\ \Delta^2\phi_{n-1}=-[u_{n-1},u_{n-1}] & \text{in } \omega\times[0,T],\\ u_n=\partial_\nu u_n=0 & \text{on } \Gamma\times[0,T],\\ (u_n)\big|_{t=0}=u_0,\ ((u_n)_t)\big|_{t=0}=u^1 & \text{in } \omega,\\ \phi_{n-1}=\partial_\nu\phi_{n-1}=0 & \text{on } \Gamma\times[0,T],\end{cases}$$
where
$$G(u,\phi)=F_1(u,\phi)+p,$$
and F_1 is defined by (4).

Theorem 4. *For $p \in L^2(\omega)$, $(u_0, u^1) \in H_0^2(\omega) \times H_0^1(\omega)$ and the following quantities:*

$$\|F_0\|_{3+\varepsilon,\omega} \text{ and } |p|_{2,\omega}, \; \|u_0\|^2 + |u^1|^2_{2,\omega}$$

are small with $d_0(\cdot)g_0(\cdot)$ is a contracting operator. The problem $(\mathbb{P})$ has one and only one weak solution (u, ϕ) in $L^2([0,T], H_0^2(\omega) \times H_0^2(\omega))$, $u_t \in L^2([0,T], H_0^1(\omega))$ and $u_{tt} \in L^2([0,T], L^2(\omega))$.

Proof. We divide the proof into three steps.

Step 1: Let us consider the problem $(\mathbb{P}_n)$, where $0 \neq u_1$ does not depend on t.

Throughout this proof we use the next notation defined by (2).

According to Proposition 2 and Theorem 1, there exists a constant $c_0 > 0$.

Now, for $\|F_0\|_{3+\varepsilon,\omega} < \frac{1}{4}$, we can choose $c := c(\|F_0\|_{3+\varepsilon,\omega}, c_0, T) > 0$ such that

$$0 < 4c_0c^2 < 1, \quad 0 < c < \sqrt{\frac{1 - 4\|F_0\|_{3+\varepsilon,\omega}}{2c_0}} \;\text{ and }\; \|u_1\| < c < 1.$$

By mathematical induction on $n \geqslant 1$, we will prove that the following two inequalities:

$$\|u\|_* = \|u_n\|^2 + |(u_n)_t|^2_{2,\omega} \leqslant \|u_1\|^2 \;\text{ and }\; \|\phi_n\| \leqslant \|u_1\|$$

hold for all $n \geqslant 1$ and any $0 \leqslant t \leqslant T$.

For $n = 1$,

$$\|u_1\|_* = \|u_1\|^2 + |(u_1)_t|^2_{2,\omega} = \|u_1\|^2$$

since u_1 does not depend on t. Otherwise, for ϕ_1 being the solution of the problem $\Delta^2\phi_1 = -[u_1, u_1]$, Theorem 1 ensures that there exists $c_0 > 0$ such that

$$\|\phi_1\| \leqslant c_0\big|[u_1, u_1]\big|_{1,\omega},$$

using the proof of Proposition 2 with $\|u_1\| < c$ and $0 < 4c_0c < 1$, we can deduce that

$$\|\phi_1\| \leqslant 4c_0\|u_1\|^2 \leqslant 4c_0c\|u_1\| \leqslant \|u_1\|.$$

The desired inequalities are true for $n = 1$.

Suppose that for $k = 2, \dots, n$ and $0 \leqslant t \leqslant T$, we have

$$\|u_k\|_* \leqslant \|u_1\|^2 \;\text{ and }\; \|\phi_k\| \leqslant \|u_1\|.$$

According to Proposition 2 and Theorem 1,

$$\|\phi_n\| \leqslant c_0|[u_n, u_n]|_{1,\omega} \leqslant 4c_0\|u_n\|^2 \leqslant 4c_0c\|u_n\| \leqslant \|u_n\|.$$

Since u_{n+1} is a solution of $(\mathbb{P}_{n+1})$, Proposition 3 with $k = \mu = \eta = 0$, Proposition 2 and Theorem 1 imply that there exists $0 < c_2 = 2c_0c^2 + 4\|F_0\|_{4,\omega} < 1$ such that

$$\begin{aligned}\|u_{n+1}\|_* &\leqslant e^T\left(\|u_0\|^2 + |u^1|^2_{2,\omega} + \int_0^T |G(u_n, \phi_n)|^2_{2,\omega}\right)\\ &\leqslant e^T\left(\|u_0\|^2 + |u^1|^2_{2,\omega} + 2\int_0^T \left(|F_1(u_n, \phi_n)|^2_{2,\omega} + |p|^2_{2,\omega}\right)\right),\end{aligned}$$

it follows that

$$\begin{aligned}\|u_{n+1}\|_* &\leqslant e^T\left(\|u_0\|^2 + |u^1|^2_{2,\omega} + 2\int_0^T c_2^2(\|u_n\|^2 + |(u_n)_t|^2_{2,\omega}) + 2T|p|^2_{2,\omega}\right)\\ &\leqslant e^T\left(\|u_0\|^2 + |u^1|^2_{2,\omega} + 2\int_0^T c_2^2(\|u_n\|^2 + |(u_n)_t|^2_{2,\omega}) + 2T|p|^2_{2,\omega}\right)\\ &\leqslant e^T\left(\|u_0\|^2 + |u^1|^2_{2,\omega} + 2\int_0^T c_2^2(\|u_n\|^2 + |(u_n)_t|^2_{2,\omega}) + 2T|p|^2_{2,\omega}\right)\\ &\leqslant e^T\left(\|u_0\|^2 + |u^1|^2_{2,\omega} + 4Tc_2^2\|u^1\|^2 + 2T|p|^2_{2,\omega}\right).\end{aligned}$$

If we choose $c > 0$ sufficiently small, then $0 < c_1 < 1$ and $0 < c_3 := 4e^T c_2^2 < 1$, we have

$$\|u_{n+1}\|_* \leqslant e^T(\|u_0\|^2 + |u^1|^2_{2,\omega} + 2T|p|^2_{2,\omega}) + c_3\|u_1\|^2,$$

now we can choose

$$\|u_0\|^2 + |u^1|^2_{2,\omega} + 2T|p|^2_{2,\omega} \leqslant \frac{(1-c_3)}{e^T}\|u_1\|^2.$$

It follows that

$$\begin{aligned}\|u_{n+1}\|_* &\leqslant e^T(\|u_0\|^2 + |u^1|^2_{2,\omega} + 2T|p|^2_{2,\omega}) + c_3\|u_1\|^2,\\ &\leqslant e^T\frac{(1-c_3)}{e^T}\|u_1\|^2 + c_3\|u_1\|^2 = \|u_1\|^2.\end{aligned}$$

Further,

$$\|\phi_{n+1}\| \leqslant c_0\big|[u_{n+1}, u_{n+1}]\big|_{1,\omega},$$

with $\|u_1\| < c$ and $0 < 4c_0c < 1$, immediately yields

$$\|\phi_{n+1}\| \leqslant 4c_0\|u_{n+1}\|^2 \leqslant 4c_0\|u_1\|^2 \leqslant 4c_0c\|u_1\| \leqslant \|u_1\|.$$

Summarizing, we have proved that, for all $n \geqslant 1$ and any $\forall 0 \leqslant t \leqslant T$,

$$\|u_n\|_* \leqslant \|u_1\|^2 \ \text{ and } \ \|\phi_n\| \leqslant \|u_1\|.$$

Step 2: For $n \geqslant 2$, let (u_n, ϕ_n) be a solution of the problem $(\mathbb{P}_n)$.

For $2 \leqslant m \leqslant n$, then it is easy to see that $u_n - u_m$ is a solution of the following problem, with ϕ_n is a uniqueness solution to the problem $\Delta^2\phi_n = -[u_n, u_n]$,

$$\begin{cases} (u_n - u_m)_{tt} + \Delta^2(u_n - u_m) = G(u_{n-1}, \phi_{n-1}) - G(u_{m-1}, \phi_{m-1}) & \text{in } \omega \times [0,T],\\ u_n - u_m = \partial_\nu(u_n - u_m) = 0 & \text{on } \Gamma \times [0,T],\\ (u_n - u_m)\big|_{t=0} = ((u_n)_t - (u_m)_t)\big|_{t=0} = 0 & \text{in } \omega.\end{cases}$$

According to Proposition 2 and Theorem 1 we deduce, for all $0 \leqslant t \leqslant T$,

$$\|\phi_{n-1} - \phi_{m-1}\| \leqslant 8c_0c\|u_{n-1} - u_{m-1}\|.$$

Using Proposition 3 with $k = \mu = \eta = 0$ and Proposition 2, again we have, with $0 < c_3 = Te^T c_2^2 < 1$,

$$\begin{aligned}\|u_n - u_m\|_* &\leqslant e^T \int_0^T \big|G(u_{n-1}, \phi_{n-1}) - G(u_{m-1}, \phi_{m-1})\big|^2_{2,\omega}\\ &\leqslant e^T \int_0^T c_2^2\big(\|u_{n-1} - u_{m-1}\|^2 + |(u_{n-1})_t - (u_{m-1})_t|^2_{2,\omega}\big).\end{aligned}$$

It follows that

$$\begin{aligned}\|u_n - u_m\|_* &\leqslant e^T c_2^2 \int_0^T \|u_{n-1} - u_{m-1}\|_*\\ &\leqslant (e^T c_2^2)^{m-2} \int_0^T \cdots \int_0^T \big(\|u_{n-m+2} - u_1\|_*\big)\\ &\leqslant (c_3)^{m-2} \sum_{k=0}^{n-m+1} (e^T c_2^2)^k \int_0^T \cdots \int_0^T \|u_2 - u_1\|_*\\ &\leqslant (c_3)^{m-2} \sum_{k=0}^{n-m+1} (e^T c_2^2)^k \int_0^T \cdots \int_0^T \big(\|u_2\|_* + \|u_1\|_*\big)\\ &\leqslant (c_3)^{m-2} \sum_{k=0}^{n-m+1} (c_3)^k\big(2\|u_1\|^2\big).\end{aligned}$$

And

$$\int_0^T \|u_n - u_m\|^2_* \leqslant T(c_3)^{m-2} \sum_{k=0}^{n-m+1} (c_3)^k\big(2\|u_1\|^2\big).$$

And so we have

$$\|\phi_n - \phi_m\| \leqslant 8c_0 c\|u_n - u_m\|.$$

Then, the sequence $(u_n, \phi_n)_{n\geqslant 2}$ is Cauchy sequence in $L^2([0,T],(H_0^2(\omega))^2)$. It follows that (u_n,ϕ_n) converges to (u,ϕ) in $L^2([0,T],(H_0^2(\omega))^2)$ and $(u_n)_t$ converges to u_t in $L^2([0,T],L^2(\omega))$.

By Proposition 2 $G(u_{n-1},\phi_{n-1})$ converges to $G(u,\phi)$ in $L^2([0,T],L^2(\omega))$.

Due to Theorem 3, $(u_n,(u_n)_t) \in C([0,T],H_0^2(\omega)\times H_0^1(\omega))$ with $(u_n)\big|_{t=0} = u_0$, $((u_n)_t)\big|_{t=0} = u_1$, which implies that $(u)\big|_{t=0} = u_0$, $((u)_t)\big|_{t=0} = u^1$.

By the assumption $(u_0,u^1) \in H_0^2(\omega)\times H_0^1(\omega)$, $u_n \in C([0,T],H_0^2(\omega))$, $(u_n)_{n\geqslant 2}$ converges to u in $L^2([0,T],H_0^2(\omega))$.

Let $v \in L^2\big([0,T],H_0^2(\omega)\big)$ be such that $v_t \in L^2\big([0,T],L^2(\omega)\big)$, $v_{tt}+\Delta^2 v \in L^2\big([0,T],H^{-2}(\omega)\big)$, $v(x_1,x_2,T)=0$ and $v_t(x_1,x_2,T)=0$.

Since u_n is a solution of (P_n), by virtue of the transposition theorem, see [3,5], we deduce that

$$\int_0^T\int_\omega u_n(v_{tt}+\Delta^2 v) = \int_0^T\int_\omega G(u_{n-1},\phi_{n-1})v + \int_\omega u^1 v(0) - \int_\omega u_0 v_t(0).$$

We have u_n converges to u in $L^2([0,T],H_0^2(\omega))$, then

$$\int_0^T\int_\omega u_n(v_{tt}+\Delta^2 v) \text{ converges to } \int_0^T\int_\omega u(v_{tt}+\Delta^2 v),$$

and using Proposition 2 and

$$\int_0^T\int_\omega G(u_{n-1},\phi_{n-1}) = \int_0^T\int_\omega F_1(u_{n-1},\phi_{n-1}) + p,$$

we deduce that

$$\int_0^T\int_\omega G(u_{n-1},\phi_{n-1})v \text{ converges to } \int_0^T\int_\omega G(u,\phi)v,$$

and so

$$\int_0^T\int_\omega u(v_{tt}+\Delta^2 v) = \int_0^T\int_\omega G(u,\phi)v + \int_\omega u^1 v(0) - \int_\omega u_0 v_t(0).$$

By the transposition theorem, we obtained that u is a solution of the problem $(\mathbb{P})$.

In summary, we have proved that (u,ϕ) is a solution of the thermoelastic von Karman evolution.

Step 3: We now prove the uniqueness. Assume that there exist two solutions (u^1,ϕ^1) and (u^2,ϕ^2) in $L^2([0,T],(H_0^2(\omega))^2\times H_0^1(\omega))$.

For some $c>0$ being sufficiently small and according to step 1, we have that

$$\|u^1\| \leqslant c, \text{ and } \|u^2\| \leqslant c.$$

This implies that u^1-u^2 and $(\phi^1-\phi^2)$ satisfies the following problem:

$$(\mathbb{P}_3)\begin{cases} (u^1-u^2)_{tt} + \Delta^2(u^1-u^2) = G(u^1,\phi^1) - G(u^2,\phi^2) & \text{in } \omega\times[0,T],\\ \Delta^2(\phi^1-\phi^2) = -[u^1,u^1] + [u^2,u^2] & \text{in } \omega\times[0,T],\\ u^1-u^2 = \partial_\nu(u^1-u^2) = \phi^1-\phi^2 = \partial_\nu(\phi^1-\phi^2) = 0 & \text{on } \Gamma\times[0,T],\\ u^1(x_1,x_2,0) - u^2(x_1,x_2,0) = 0 & \text{in } \omega\\ (u^1)_t(x_1,x_2,0) - (u^2)_t(x_1,x_2,0) = 0 & \text{in } \omega, \end{cases}$$

which means that $(u^1-u^2,\phi^1-\phi^2)$ is a solution of the problem $(\mathbb{P}_3)$.

Proposition 2 with $k=\mu=\eta=0$, Proposition 3 and Theorem 1 ensure that there exists $0<c_2<1$ such that

$$\begin{aligned}\|u^1-u^2\|_* &\leqslant e^T\int_0^T \big|F_1(u^1,\phi^1) - F_1(u^2,\phi^2)\big|_{2,\omega}^2 \\ &\leqslant e^T\int_0^T c_1^2\big(\|u^1-u^2\|^2 + |u_t^1-u_t^2|_{2,\omega}^2\big) \leqslant e^T c_2^2\int_0^T \|u^1-u^2\|_*.\end{aligned}$$

Since the constant $c > 0$ is small and thus $0 < c_3 = Te^T c_2^2 < 1$, it follows that

$$\int_0^T \|u^1 - u^2\|_* \leqslant c_3 \int_0^T \|u^1 - u^2\|_*,$$

which, with $0 < c_3 < 1$, immediately yields $\forall 0 \prec t \prec T$ $u^1 = u^2$ in ω and $\phi^1 = \phi^2$ in ω.

After the variational problem of ($\mathbb{P}$) and ($\mathbb{P}_n$), we have for all $v \in L^2([0,T], H_0^2(\omega))$

$$\int_0^T \int_\omega ((u_n)_{tt} - u_{tt})v + \int_0^T \int_\omega \Delta(u_n - u)v - \int_0^T \int_\omega (F_1(u_{n-1}, \phi_{n-1}) - F_1(u, \phi))v = 0. \tag{6}$$

Now, we can pass with the limit $n \longrightarrow +\infty$ in (6) we find that

$$(u_n)_{tt} \rightharpoonup u_{tt} \quad \text{weakly in} \quad L^2\big([0,T], L^2(\omega)\big),$$

then $u_{tt} \in L^2([0,T], L^2(\omega))$.

We conclude that the dynamic von Karman equation with viscous damping, without rotational inertia, has one and only one weak solution (u, ϕ) in $L^2\big([0,T], (H_0^2(\omega))^2\big)$.

The proof of the theorem is completed. ■

4. Numerical application

Let ω be defined by

$$\omega =]0,1[\times]0,1[\subset \mathbb{R}^2$$

and $T > 0$. In order to solve numerically the problem ($\mathbb{P}$), we introduce an uniform mesh of width h. Let ω_h be the set of all mesh points inside ω with the internal points

$$x_i = ih, \quad y_j = jh, \quad i,j = 1, \ldots, N-1, \quad h = \frac{1}{N+1}, \quad \Delta t = \frac{1}{T}.$$

Let $\overline{\omega}_h$ be the set of boundary mesh points and u_h be the finite-difference approximation of u.

For approaching the weak unique solution of the dynamic nonlinear plate coupled, we then use the following discrete model of von Karman evolution developed by Bilbao and Pereira in [4, 6, 7]:

$$(*)\begin{cases} \delta_t^2 u_{ij}^n + d_0(x_{ij}) g_0(\delta_t u_{ij}^n) + \Delta_h^2 u_{ij}^n = [u_{ij}^n\, v_{ij}^n + F_{ij}] + p_{ij} & \text{in} \quad \omega_h, \\ \Delta_h^2 v_{ij}^n = -[u_{ij}^n\, u_{ij}^n] & \text{in} \quad \omega_h, \\ u_{ij}^0 = (\varphi_0)_{ij}, \quad \delta_t u_{ij}^0 = (\varphi_1)_{ij}, & \text{in} \quad \omega_h, \\ u_{ij}^n = v_{ij}^n = 0 & \text{on} \quad \overline{\omega_h}, \\ \partial_\nu u_{ij}^n = \partial_\nu v_{ij}^n = 0 & \text{on} \quad \overline{\omega_h} \end{cases}$$

with the following discrete differential operators:

$$\delta_t^2 u_{ij}^n = \frac{u_{ij}^{n+1} - 2u_{ij}^n + u_{ij}^{n-1}}{(\Delta t)^2}, \quad \delta_t u_{ij}^n = \frac{u_{ij}^{n+1} - u_{ij}^n}{\Delta t},$$

$$\Delta_h^2 u_{ij}^n = h^{-4}\big[u_{ij-2} + u_{ij+2} + u_{i-2j} + u_{i+2j} - 8(u_{ij-1} + u_{ij+1} + u_{i-1j} + u_{i+1j}) \\ + 2(u_{i-1j-1} + u_{i-1j+1} + u_{i+1j-1} + u_{i+1j+1}) - 20u_{ij}\big],$$

$$\delta_x^2 u_{ij}^n = \frac{u_{i+1j}^n - 2u_{ij}^n + u_{i-1j}^n}{(h)^2}, \quad \delta_y^2 u_{ij}^n = \frac{u_{ij+1}^n - 2u_{ij}^n + u_{ij-1}^n}{(h)^2},$$

$$\delta_{xy}^2 u_{ij}^n = \frac{u_{i+1j+1}^n - u_{i+1j-1}^n - u_{i-1j+1}^n + u_{i-1j-1}^n}{(2h)^2},$$

$$\big[u_{ij}^n, v_{ij}^n\big] = \delta_x^2 u_{ij}^n \delta_y^2 v_{ij}^n - 2\delta_{xy}^2 u_{ij}^n \delta_{xy}^2 v_{ij}^n + \delta_y^2 u_{ij}^n \delta_x^2 v_{ij}^n.$$

We have transformed the above problem to the numerical resolution in two steps itemized as follows.

First step: We use the numerical procedure of 13-point formula of finite difference developed by Gubta in [8] for illustrating the weak solution of the following biharmonic problem:

$$\begin{cases} \Delta^2 v = f_1 & \text{in } \omega, \\ v = g_1 & \text{on } \Gamma, \\ \partial_\nu v = g_2 & \text{on } \Gamma. \end{cases}$$

Second step: According to the first and second steps, we use the discrete model of von Karman evolution $(*)$ for illustrating the unique solution of the structural interaction model coupled with the dynamic von Karman evolution.

4.1. Non-coupled approach

In [8], Gubta presented a numerical analysis of finite-difference method for solving the biharmonic equation. Such method is known as the non-coupled method of 13-point formula of finite difference.

Proposition 4 (Ref. [8]). The 13-point approximation of the biharmonic equation for approaching the unique solution v of the problem $(\mathbb{P})$ is defined by

$$(1)\begin{cases} L_h v_{ij} = h^{-4}\left[v_{ij-2} + v_{ij+2} + v_{i-2j} + v_{i+2j} - 8(v_{ij-1} + v_{ij+1} + v_{i-1j} + v_{i+1j})\right. \\ \qquad \left. +2(v_{i-1j-1} + v_{i-1j+1} + v_{i+1j-1} + v_{i+1j+1}) - 20v_{ij}\right] = f_1(x_i, y_j) \end{cases}$$

for $i, j = 1, 2, \ldots, N-1$, where we set $v_{ij} = v(x_i, y_j)$.

Remark 1. When the mesh point (x_i, y_j) is adjacent to the boundary $\overline{\omega}_h$, then the undefined values of v_h are conventionally calculated by the following approximation of $\partial_\nu v$:

$$\begin{aligned} v_{i-2,j} &= \frac{1}{2}v_{i+1,j} - v_{ij} + \frac{3}{2}v_{i-1,j} - h(\partial_x v)_{i-1,j}, \\ v_{i,j-2} &= \frac{1}{2}v_{i,j+1} - v_{ij} + \frac{3}{2}v_{i,j-1} - h(\partial_y v)_{i,j-1}, \\ v_{i+2,j} &= \frac{1}{2}v_{i+1,j} - v_{ij} + \frac{3}{2}v_{i-1,j} - h(\partial_x v)_{i+1,j}, \\ v_{i,j+2} &= \frac{1}{2}v_{i,j+1} - v_{ij} + \frac{3}{2}v_{i,j-1} - h(\partial_y v)_{i,j+1}. \end{aligned}$$

4.2. Example

Let the analytical body force and lateral forces:

$$F_0(x,y) = ye^{-x^2-y^2}, \qquad \varphi_0 = 1510^{-6}x^2y^2(x-y-1)^2(y-1)^2e^{-x^2-y^2},$$

$$p(x,y) = 0.01x(x-y)e^{-x^2-y^2}, \qquad \varphi_1 = 1510^{-6}(\sin(\pi x)\sin(\pi y))^2,$$

$$d_0(x)g_0(u_t) = 10^{-3}e^{-x^2-y^2}u_t.$$

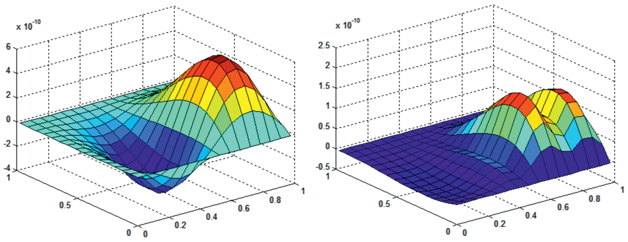

Fig. 1. Displacement of plate, $t_1 = 0.2\,\text{s}$ and $t_7 = 60\,\text{s}$.

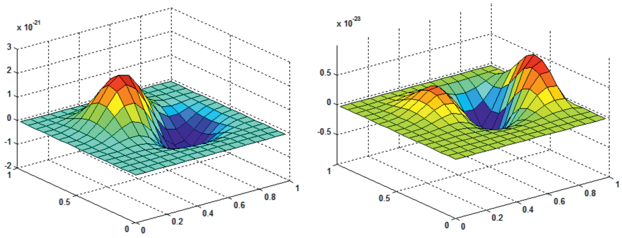

Fig. 2. The Airy stress function, $t_1 = 0.2\,\text{s}$ and $t_6 = 60\,\text{s}$.

5. Conclusion

The dynamic von Karman equations about a flexible phenomenon of small displacement play an interesting place in nonlinear oscillation of elastic plate. When the plaque is acted with an active damping. In this paper, we describe an iterative method for establishing the existence and the uniqueness of the weak solution for the von Karman equations plate. The original idea of this technique is based on the construction of a sequence-solution which approximates in a certain sense the solution of our initial problem. Our approach is in fact a good tool, simple and is practical, for illustrating the solution of our problem in the numerical point of view.

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Динамічні рівняння фон Кармана з в'язким загасанням

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У цій статті цікавимося динамічними рівняннями фон Кармана в поєднанні з в'язким демпфуванням і без обертальних сил, ($\alpha = 0$) [Chueshov I., Lasiecka I. (2010)], ця задача описує явище вигину та гнучкості малих нелінійних коливань вертикального зміщення пружних пластин. Наша фундаментальна мета полягає в тому, щоб встановити існування та єдиність слабкого рішення для так званої глобальної енергії за припущення $F_0 \in H^{3+\varepsilon}(\omega)$. Накінець, для ілюстрації теоретичних результатів використано метод скінченних різниць.

Ключові слова: *рівняння фон Кармана; нелінійні пластини; в'язке демпфування; інерція обертання; незв'язаний метод; метод скінченних різниць.*