MATHEMATICAL MODELING AND COMPUTING, Vol. 10, No. 3, pp. 833–840 (2023)

# Integral of an extension of the sine addition formula

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(Received 18 February 2023; Accepted 20 July 2023)

In this paper, we determine the continuous solutions of the integral functional equation of Stetkær's extension of the sine addition law $\int_G f(xyt)d\mu(t) = f(x)\chi_1(y) + \chi_2(x)f(y)$, $x, y \in G$, where $f\colon G \to \mathbb{C}$, $G$ is a locally compact Hausdorff group, $\mu$ is a regular, compactly supported, complex-valued Borel measure on $G$ and $\chi_1$, $\chi_2$ are fixed characters on $G$.

**Keywords:** *functional equation; sine addition law; character; additive function; Borel measure.*

**2010 MSC:** 39B32, 39B52 **DOI:** 10.23939/mmc2023.03.833

## 1. Notations and terminology

Throughout the paper, we consider the following notations and assumptions. Let $G$ be a locally compact Hausdorff group with neutral element $e$. The *commutator* between $x \in G$ and $y \in G$ is $[x, y] = xyx^{-1}y^{-1}$. Let $[G, G]$ denote the smallest subgroup of $G$ containing the set $\{[x, y] \mid x \in G,\ y \in G\}$. $[G, G]$ is called the *derived subgroup* of $G$. $C(G)$ denotes the algebra of continuous, complex valued functions on $G$. The set of homomorphisms $a\colon G \to (\mathbb{C}, +)$ will be called the additive maps and denoted by $\mathcal{A}(G)$.

A character $\chi$ of $G$ is a homomorphism $\chi\colon G \to \mathbb{C}^*$, where $\mathbb{C}^*$ denotes the multiplicative group of non-zero complex numbers. It is well known that the set of characters on $G$ is a linearly independent subset of the vector space of all complex-valued functions on $G$ (see [1, Corollary 3.20]).

Let $M_C(G)$ denote the space of all regular, compactly supported, complex-valued Borel measures on $G$ and $\delta_z$ the Dirac measure concentrated at $z$. For $\mu \in M_C(G)$, we use the notation

$$\mu(f) = \int_G f(t)d\mu(t),$$

for all $f \in C(G)$.

## 2. Introduction

The trigonometric addition and subtraction formulas have been studied in the context of functional equations by a number of mathematicians. The monographs by Aczél [2], by Kannappan [3], by Stetkær [1] and by Székelyhidi [4] have references and detailed discussions of the classic results.

Chung, Kannappan and Ng [5] solved on any group $G$, the functional equation

$$f(xy) = f(x)g(y) + f(y)g(x) + h(x)h(y), \quad x, y \in G.$$

Poulsen and Stetkær [6] found the complete set of continuous solutions of each of the functional equations

$$g(xy) = g(x)g(y) - f(x)f(y), \quad x, y \in G, \tag{1}$$

$$f(xy) = f(x)g(y) + f(y)g(x), \quad x, y \in G. \tag{2}$$

The following integral versions of the addition and subtraction formulas for cosine and sine:

$$\int_G g(xyt)d\mu(t) = g(x)g(y) - f(x)f(y), \quad x, y \in G,$$

 

$$\int_G f(x\sigma(y)t)d\mu(t) = f(x)g(y) \pm g(x)f(y), \quad x, y \in G,$$

where $G$ is a locally compact Hausdorff group, $\mu$ is a regular, compactly supported, complex-valued Borel measure on $G$ and $\sigma$ denotes an involution of $G$, i.e., $\sigma(xy) = \sigma(x)\sigma(y)$ and $\sigma(\sigma(x)) = x$ for all $x, y \in G$, were solved by Zeglami, Tial and Kabbaj in [7] and [8] respectively.

In the paper [9], Stetkær determined the solutions $f\colon G \to \mathbb{C}$ of the functional equation

$$f(xy) = f(x)\chi_1(y) + \chi_2(x)f(y), \quad x, y \in G, \tag{3}$$

where $\chi_1$ and $\chi_2$ are two characters on $G$ and the functional equation

$$f(xy) = g(x)h_1(y) + \chi(x)h_2(y), \quad x, y \in G, \tag{4}$$

where $f, g, h_1, h_2\colon G \to \mathbb{C}$ are the unknown functions and $\chi$ is a character on $G$.

Let $\mu \in M_C(G)$. Our main contributions in this paper are the following. First, we give an explicit description of the continuous solutions $f\colon G \to \mathbb{C}$ of the following integral version of Stetkær's extension of the sine addition law

$$\int_G f(xyt)d\mu(t) = f(x)\chi_1(y) + \chi_2(x)f(y), \ \ x, y \in G, \tag{5}$$

where $\chi_1$, $\chi_2$ are continuous fixed characters on $G$ such that $\mu(\chi_1) = \mu(\chi_2) = 1$.

In the case where $(\mu(\chi_1), \mu(\chi_2)) \neq (1, 1)$, we show that the only continuous solutions of the equation (5) is $f = 0$, except for the two cases $\chi_1 = \chi_2$, $\mu(\chi_1) = 2$ and $\mu(\chi_1) = 1$, $\mu(\chi_2) \neq 1$ where the equation (5) admits non trivial solutions.

To solve the equation (5), we reduce it to the equation (3) and the following functional equation

$$f(xy) = f(x)\chi_1(y) + \chi_2(x)f(y) - \gamma\chi_2(xy), \quad x, y \in G,$$

where $\gamma \in \mathbb{C}$.

As application, we give the continuous solutions $f\colon G \to \mathbb{C}$ of the following functional equation

$$f(xyz_0) = f(x)\chi_1(y) + \chi_2(x)f(y), \quad x, y \in G, \tag{6}$$

where $\chi_1$, $\chi_2$ are two continuous characters on $G$ such that $\chi_1(z_0) = \chi_2(z_0) = 1$ for a fixed constant $z_0 \in G$.

In the last section, we provide two examples to show that nontrivial continuous solutions of (5) occur in real life.

Results of [9] have been an inspiration for this work. We refer also to [10–12] for some contextual discussions.

## 3. The solutions of the integral of an extension of the sine addition law

The purpose of this section is, first, to give an explicit description of the continuous complex-valued solutions of the functional equation

$$f(xy) = f(x)\chi(y) + \chi(x)f(y) + \chi(xy), \quad x, y \in G, \tag{7}$$

where $\chi$ is a continuous character on $G$. And, secondly, to determine the continuous solutions $f\colon G \to \mathbb{C}$ of the functional equation (5), namely

$$\int_G f(xyt)\, d\mu(t) = f(x)\chi_1(y) + \chi_2(x)f(y), \quad x, y \in G, \tag{8}$$

where $\mu \in M_C(G)$ and $\chi_1$, $\chi_2$ are continuous characters on $G$.

In the following Proposition, we exhibit the continuous solutions of the functional equation (7).

**Proposition 5.** Let $G$ be a topological group and $\chi$ a continuous character on $G$. The function $f \in C(G)$ is a solution of the functional equation (7) if and only if $f = \chi(a - 1)$, where $a$ is a continuous additive function on $G$.

**Proof.** Dividing the right-hand and the left-hand sides of equation (7) by $\chi(xy) = \chi(x)\chi(y)$, we find

$$F(xy) = F(x) + F(y) + 1 \ \text{ for all } \ x, y \in G,$$

where $F(x)=\frac{f(x)}{\chi(x)}$ for all $x\in G$, which implies that

$$(F+1)(xy)=(F+1)(x)+(F+1)(y) \text{ for all } x,y\in G.$$

So, the function $F+1$ is additive. Then there exists a continuous additive function on $G$ such that $F(x)=a(x)-1$ for all $x\in G$. Finally, $f=\chi(a-1)$ on $G$.

Conversely, simple computations prove that the formula above for $f$ defines solutions of (7). ■

Now we are in the position to describe all continuous solutions of the functional equation (8). We begin with the case $\mu(\chi_1)=\mu(\chi_2)=1$.

**Theorem 1.** *Let $G$ be a locally compact Hausdorff group, $\mu\in M_C(G)$ and $\chi_1$, $\chi_2$ are two continuous characters on $G$ such that $\mu(\chi_1)=\mu(\chi_2)=1$. Assume that the function $f\in C(G)$ is a solution of the equation (8). Then we have the following cases:*
*i) If $\chi_1=\chi_2=\chi$ then $f$ has one of the forms:*
*a) $f=\chi a$, where $a\colon G\to\mathbb{C}$ is a continuous additive function such that $\mu(a\chi)=0$.*
*b) $f=\gamma\chi(1-a)$, where $\gamma$ is a constant in $\mathbb{C}$ and $a\colon G\to\mathbb{C}$ is a continuous additive function on $G$ such that $\mu(a\chi)=-1$.*
*ii) If $\chi_1(y_0)\neq\chi_2(y_0)$ for a fixed $y_0\in G$ then*

$$f(x)=\alpha(\chi_1(x)-\chi_2(x))+A([y_0,x])\chi_1(x),\quad x\in G,$$

*where $\alpha$ ranges over $\mathbb{C}$ and $A\colon [G,G]\to\mathbb{C}$ over the continuous additive functions with the transformation property*

$$A(xcx^{-1})=\frac{\chi_2(x)}{\chi_1(x)}A(c) \text{ for all } x\in G \text{ and } c\in[G,G], \tag{9}$$

*such that $\mu(A([y_0,\cdot])\chi_1)=0$.*

*Furthermore if $G$ is Abelian then, in the case ii), the continuous solutions of the equation (8) are the functions of the form $f=\alpha(\chi_1-\chi_2)$, where $\alpha\in\mathbb{C}$.*

*Conversely, the formulas above for $f$ define solutions of (8).*

**Proof.** Let $f$ be a solution of (8). Letting $y=e$ in (8), we get that

$$\int_G f(xt)\,d\mu(t)=f(x)+\gamma\chi_2(x),\quad x\in G, \tag{10}$$

where $\gamma=f(e)$. So, using (10), we can reformulate the form of the equation (8) as

$$f(xy)=f(x)\chi_1(y)+\chi_2(x)f(y)-\gamma\chi_2(x)\chi_2(y),\quad x,y\in G. \tag{11}$$

**Case 1.** Suppose that $\gamma=0$ then the equation (11) becomes

$$f(xy)=f(x)\chi_1(y)+\chi_2(x)f(y),\quad x,y\in G. \tag{12}$$

I) If $\chi_1=\chi_2=\chi$ then the equation (12) becomes

$$f(xy)=f(x)\chi(y)+\chi(x)f(y),\quad x,y\in G.$$

Using [9, Proposition 4], we get that $f=\chi a$ where $a$ is a continuous additive function on $G$. On putting $f=\chi a$ in the equation (8) with $\chi_1=\chi_2=\chi$, we find that

$$\int_G \chi(xyt)a(xyt)\,d\mu(t)=\chi(x)a(x)\chi(y)+\chi(x)\chi(y)a(y),\quad x,y\in G,$$

which implies that

$$\chi(x)\chi(y)\int_G(a(x)+a(y)+a(t))\chi(t)\,d\mu(t)=\chi(x)\chi(y)(a(x)+a(y)),$$

for all $x,y\in G$. Then

$$a(x)\int_G\chi(t)\,d\mu(t)+a(y)\int_G\chi(t)\,d\mu(t)+\int_G a(t)\chi(t)\,d\mu(t)=a(x)+a(y),$$

for all $x,y\in G$. Since $\mu(\chi)=1$, we conclude that $\mu(a\chi)=0$. So, we are in the case i) a) of our statement.

II) If $\chi_1(y_0) \neq \chi_2(y_0)$ for a fixed $y_0 \in G$ then using [9, Theorem 11], we obtain that

$$f(x) = \alpha(\chi_1(x) - \chi_2(x)) + A([y_0, x])\chi_1(x), \quad x \in G, \tag{13}$$

where $\alpha$ ranges over $\mathbb{C}$ and $A\colon [G,G] \to \mathbb{C}$ over the continuous additive functions with the transformation property (9). Using (13) in (8) and the fact that $\mu(\chi_1) = \mu(\chi_2) = 1$, we find that

$$\begin{aligned} &\alpha\chi_1(x)\chi_1(y) - \alpha\chi_2(x)\chi_2(y) + \chi_1(x)\chi_1(y)\int_G A([y_0, xyt])\chi_1(t)\,d\mu(t) = \alpha\chi_1(x)\chi_1(y) \\ &\quad - \alpha\chi_1(y)\chi_2(x) + \chi_1(y)\chi_1(x)A([y_0,x]) + \alpha\chi_2(x)\chi_1(y) - \alpha\chi_2(x)\chi_2(y) + \chi_2(x)\chi_1(y)A([y_0,y]), \end{aligned} \tag{E}$$

for all $x, y \in G$.

Since the function $A$ satisfies the transformation property (9), then using [9, Lemma 10], we obtain that

$$A([y_0, xy]) = A([y_0, x]) + \frac{\chi_2(x)}{\chi_1(x)}A([y_0, y]) \quad \text{for all} \quad x, y \in G.$$

So, the equation (E) becomes

$$\chi_2(x)\chi_2(y)\mu(A([y_0,\cdot])\chi_1) = \alpha\chi_1(x)\chi_1(y) \quad \text{for all} \quad x, y \in G.$$

Finally, taking $x = e$ and using the linear independence of different characters, we conclude that $\mu(A([y_0,\cdot])\chi_1) = 0$. So, we are in the case ii) of our statement.

**Case 2.** Suppose that $\gamma \neq 0$. Putting $x = e$ in (12), we find that

$$f(y) = \gamma\chi_1(y) + f(y) - \gamma\chi_2(y) \quad \text{for all} \quad y \in G,$$

which implies that $\chi_1 = \chi_2 = \chi$. So, equation (12) becomes

$$f(xy) = f(x)\chi(y) + \chi(x)f(y) - \gamma\chi(x)\chi(y), \quad x, y \in G. \tag{14}$$

Dividing the right and the left hand sides of (14) by $(-\gamma)$, we get that

$$\frac{-1}{\gamma}f(xy) = \frac{-1}{\gamma}f(x)\chi(y) + \frac{-1}{\gamma}\chi(x)f(y) + \chi(x)\chi(y), \quad x, y \in G. \tag{15}$$

Putting $F = \frac{-1}{\gamma}f$ in (15) we find that

$$F(xy) = F(x)\chi(y) + \chi(x)F(y) + \chi(x)\chi(y), \quad x, y \in G. \tag{16}$$

From Proposition (5), we obtain that $F = \chi(a-1)$, where $a$ is a continuous additive function on $G$ and so

$$f = \gamma\chi(1-a). \tag{17}$$

Replacing the expression of $f$ from (17) into equation (8) with the condition $\chi_1 = \chi_2 = \chi$, we get that

$$\int_G \gamma\chi(xyt)(1 - a(xyt))\,d\mu(t) = \gamma\chi(x)(1-a(x))\chi(y) + \chi(x)\gamma\chi(y)(1-a(y)),$$

for all $x, y \in G$. This implies that

$$\int_G \chi(t)(1 - a(x) - a(y) - a(t))\,d\mu(t) = (1-a(x)) + (1-a(y)), \quad x, y \in G.$$

Since $\mu(\chi) = 1$, we obtain $1 - a(x) - a(y) - \mu(a\chi) = 2 - a(x) - a(y)$, $x, y \in G$, which yields that $\mu(a\chi) = -1$. So, we are in the case i) b) of our statement.

Conversely, simple computations prove that the formulas above for $f$ define solutions of (8). ∎

In the following Proposition, we exhibit the continuous solutions of the equation (8) in the case where $(\mu(\chi_1), \mu(\chi_2)) \neq (1,1)$.

**Proposition 6.** Let $\chi_1$, $\chi_2$ be two continuous characters on $G$ such that $(\mu(\chi_1), \mu(\chi_2)) \neq (1,1)$. Depending on $\chi_1$ and $\chi_2$, the solutions $f \in C(G)$ of the equation (8) are:

i) If $\chi_1 = \chi_2 = \chi$ and $\mu(\chi) = 2$ then $f = \gamma\chi$, $\gamma \in \mathbb{C}\backslash\{0\}$;

ii) If $\mu(\chi_1) = 1$ and $\mu(\chi_2) \neq 1$ then $f(x) = \alpha(\chi_1(x) - \chi_2(x)) + A([y_0, x])\chi_1(x)$, $x \in G$, where $A\colon [G,G] \to \mathbb{C}$ over the continuous additive functions with the transformation property (9) such that $\alpha = \frac{\mu(A([y_0,\cdot])\chi_1)}{(\mu(\chi_2)-1)}$;

iii) otherwise $f = 0$.

Conversely, the formulas above for $f$ define solutions of (8).

**Proof.** Let $\chi_1$, $\chi_2$ be two continuous characters on $G$ such that $(\mu(\chi_1), \mu(\chi_2)) \neq (1,1)$ and let $f$ be a continuous solution of (8). We proceed as in the proof of Theorem 1.

**Case 1.** Suppose that $f(e) = \gamma = 0$.

I) If $\chi_1 = \chi_2 = \chi$ then we find that $f = \chi a$, where $a$ is a continuous additive function on $G$. On putting $f = \chi a$ in the equation (8) with $\chi_1 = \chi_2 = \chi$, we find that

$$\int_G \chi(xyt)a(xyt)\, d\mu(t) = \chi(x)a(x)\chi(y) + \chi(x)\chi(y)a(y), \quad x, y \in G,$$

which means that

$$\chi(x)\chi(y) \int_G (a(x) + a(y) + a(t))\chi(t)\, d\mu(t) = \chi(x)\chi(y)(a(x) + a(y)),$$

for all $x, y \in G$. This yields that

$$(a(x) + a(y))\mu(\chi) + \mu(a\chi) = a(x) + a(y),$$

for all $x, y \in G$. Then

$$(\mu(\chi) - 1)a(xy) = -\mu(a\chi) \quad \text{for all} \quad x, y \in G.$$

Since $\mu(\chi) \neq 1$, the additive function $a$ is constant. We conclude that $a = 0$ and then $f = 0$. Thus, we are in the case iii) of our statement.

II) If $\chi_1(y_0) \neq \chi_2(y_0)$ for a fixed $y_0 \in G$, we obtain that

$$f(x) = \alpha(\chi_1(x) - \chi_2(x)) + A([y_0, x])\chi_1(x), \quad x \in G, \tag{18}$$

where $\alpha$ ranges over $\mathbb{C}$ and $A\colon [G,G] \to \mathbb{C}$ over the continuous additive functions with the transformation property (9). On putting (18) in (8), we find that

$$\begin{aligned}(\alpha + A([y_0, x]))(\mu(\chi_1) - 1)\chi_1(xy) + (\mu(A([y_0, \cdot])\chi_1) - \alpha(\mu(\chi_2) - 1))\chi_2(xy) \\ + \chi_2(x)\chi_1(y)A([y_0, y])(\mu(\chi_1) - 1) = 0 \quad \text{for all} \quad x, y \in G.\end{aligned} \tag{19}$$

Here we discuss three cases:

a) If $\mu(\chi_1) = 1$ and $\mu(\chi_2) \neq 1$, then (19) becomes

$$(\mu(A([y_0, \cdot])\chi_1) - \alpha(\mu(\chi_2) - 1))\chi_2(xy) = 0, \quad x, y \in G,$$

then $\mu(A([y_0, \cdot])\chi_1) = \alpha(\mu(\chi_2) - 1)$. So, we are in the case ii) of our statement.

b) If $\mu(\chi_2) = 1$ and $\mu(\chi_1) \neq 1$, then (19) becomes

$$(\mu(\chi_1) - 1)[(\alpha + A([y_0, x]))\chi_1(xy) + \chi_2(x)\chi_1(y)A([y_0, y])] = 0, \quad x, y \in G.$$

Putting $x = e$ in the last equation and using the fact that $A([y_0, e]) = 0$, we find that $A([y_0, y]) = -\alpha$ for all $y \in G$. Since $A$ is additive, we deduce that $A = 0$, so $\alpha = 0$, which implies that $f = 0$. So, we are in the case iii) of our statement.

c) If $\mu(\chi_2) \neq 1$ and $\mu(\chi_1) \neq 1$, putting $x = e$ in (19), we find that, $A([y_0, y]) = -\alpha$ for all $y \in G$, then $\alpha = 0$, which gives that $f = 0$. So, we are in the case iii) of our statement.

**Case 2.** Suppose that $\gamma = f(e) \neq 0$, then we have necessarily $\chi_1 = \chi_2 = \chi$ and so, we find that

$$f = \gamma\chi(1 - a), \tag{20}$$

where $a$ is a continuous additive function on $G$. Replacing the expression of $f$ from (20) into equation (8), we get that

$$\int_G \gamma\chi(xyt)(1 - a(xyt))\, d\mu(t) = \gamma\chi(x)(1 - a(x))\chi(y) + \chi(x)\gamma\chi(y)(1 - a(y)),$$

for all $x, y \in G$, which implies that

$$\int_G \chi(t)(1 - a(x) - a(y) - a(t))\, d\mu(t) = (1 - a(x)) + (1 - a(y)), \quad x, y \in G,$$

then

$$\mu(\chi)(1 - a(x) - a(y)) - \mu(a\chi) = 2 - a(x) - a(y), \quad x, y \in G.$$

This yields that

$$a(xy)(1-\mu(\chi)) = 2-\mu(\chi)+\mu(a\chi), \quad x,y\in G.$$

Since $\mu(\chi)\neq 1$, we conclude that $a=0$. Then

$$f=\gamma\chi.$$

Replacing this formula into equation (8), we find that $\mu(\chi)=2$, so we are in the case i) of our statement.

Conversely, simple computations prove that the formulas above for $f$ define solutions of (8). ■

In the following corollary we solve the functional equation

$$f(xyz_0) = f(x)\chi_1(y)+\chi_2(x)f(y), \quad x,y\in G, \tag{21}$$

where $\chi_1$ and $\chi_2$ are two continuous characters on $G$ and $z_0$ is a fixed element in $G$ such that $\chi_1(z_0) = \chi_2(z_0) = 1$.

**Corollary 1.** *Let $G$ be a topological group, $z_0$ a fixed constant in $G$ and $\chi_1$, $\chi_2$ are two continuous characters on $G$ such that $\chi_1(z_0)=\chi_2(z_0)=1$. Assume that the function $f\in C(G)$ is a solution of the equation (21). Then we have the following cases:*
*i) If $\chi_1=\chi_2=\chi$ then $f$ has one of the forms:*
*a) $f=\chi a$, where $a\colon G\to\mathbb{C}$ is a continuous additive function such that $a(z_0)=0$.*
*b) $f=\gamma\chi(1-a)$, where $\gamma$ is a constant in $\mathbb{C}$ and $a\colon G\to\mathbb{C}$ is a continuous additive function such that $a(z_0)=-1$.*
*ii) If $\chi_1(y_0)\neq\chi_2(y_0)$ for a fixed $y_0\in G$ then*

$$f(x)=\alpha(\chi_1(x)-\chi_2(x))+A([y_0,x])\chi_1(x), \quad x\in G,$$

*where $\alpha$ ranges over $\mathbb{C}$ and $A\colon [G,G]\to\mathbb{C}$ over the continuous additive functions with the transformation property (9) such that $A([y_0,z_0])=0$. Furthermore if $G$ is Abelian, then the continuous solutions of the equation (21) are the functions of the forms:*

$$f=\alpha(\chi_1-\chi_2), \quad \alpha\in\mathbb{C}.$$

*Conversely, the formulas above for $f$ define solutions of (21).*

**Proof.** As the proof of Theorem 1 with $\mu=\delta_{z_0}$. ■

## 4. Examples

**Example 1.** In view of Corollary 1, we characterize the corresponding continuous solutions of equation (8) which is

$$f(x+y+z_0) = f(x)\chi_1(y)+\chi_2(x)f(y), \quad x,y\in\mathbb{R}. \tag{22}$$

Here $G=(\mathbb{R},+)$, $z_0$ is a fixed element in $\mathbb{R}\backslash\{0\}$ and $\chi_1,\chi_2\colon\mathbb{R}\to\mathbb{C}$ are two continuous characters such that $\chi_1(z_0)=\chi_2(z_0)=1$. Let $f$ be a continuous solution of the equation (22).

The continuous characters on $\mathbb{R}$ are known to be $\chi(x)=e^{\lambda x}$, $x\in\mathbb{R}$, where $\lambda$ ranges over $\mathbb{C}$.

**Case 1**. Assume $\chi_1=\chi_2=\chi$. The condition $\chi(z_0)=1$ implies that $\lambda=\frac{i2k\pi}{z_0}$, where $k\in\mathbb{Z}$, so the relevant characters are of the form $\chi_k(x)=\exp\left(\frac{i2k\pi}{z_0}x\right)$, $x\in\mathbb{R}$ and $k\in\mathbb{Z}$.

The continuous additive functions on $\mathbb{R}$ are the functions of the form $a(x)=\beta x$, $x\in\mathbb{R}$, where the constant $\beta$ ranges over $\mathbb{C}$ (see for instance [9, Corollary 2.4]). In the point i) a) of Corollary 1 we have $a(z_0)=0$ which implies that $\beta=0$ i.e. $a=0$. So, $f=0$ in this case.

The condition $a(z_0)=-1$ in the point i) b) of Corollary 1 implies that $\beta=\frac{-1}{z_0}$, so, $a(x)=\frac{-1}{z_0}x$ for all $x\in\mathbb{R}$. In this case

$$f(x)=\gamma e^{\frac{i2k\pi}{z_0}x}\left(1+\frac{1}{z_0}x\right), \quad x\in\mathbb{R}, \quad \gamma\in\mathbb{C}.$$

**Case 2**. Assume now that $\chi_1\neq\chi_2$. The group $(\mathbb{R},+)$ is Abelian, so, according to Corollary 1, we get that

$$f(x)=\alpha(\chi_1(x)-\chi_2(x))=\alpha\left(e^{\frac{i2k_1\pi}{z_0}x}-e^{\frac{i2k_2\pi}{z_0}x}\right) \quad x\in\mathbb{R},$$

where $\alpha\in\mathbb{C}$ and $k_1,k_2\in\mathbb{Z}$.

In conclusion, the continuous solutions $f\colon \mathbb{R}\to\mathbb{C}$ of the functional equation (22) which is here

$$f(x+y+z_0)=f(x)e^{\frac{i2k_1\pi}{z_0}y}+e^{\frac{i2k_2\pi}{z_0}x}f(y),\quad x,y\in\mathbb{R},$$

are the functions of the forms:
i) If $k_1=k_2=k$, then

$$f(x)=\gamma e^{\frac{i2k\pi}{z_0}x}\left(1+\frac{1}{z_0}x\right),\quad x\in\mathbb{R},$$

where $\gamma\in\mathbb{C}$;
ii) If $k_1\neq k_2$, then

$$f(x)=\alpha(\chi_1(x)-\chi_2(x))=\alpha\left(e^{\frac{i2k_1\pi}{z_0}x}-e^{\frac{i2k_2\pi}{z_0}x}\right),\quad x\in\mathbb{R},$$

where $\alpha\in\mathbb{C}$.

**Example 2.** For an application of our results on a non-Abelian group, we consider the $(ax+b)$-group

$$G:=\left\{\begin{pmatrix} a & b\\ 0 & 1\end{pmatrix}\middle|\, a>0, b\in\mathbb{R}\right\},$$

$Z_0=\left(\begin{smallmatrix} a_0 & 0\\ 0 & 1\end{smallmatrix}\right)$ be a fixed element on $G$ such that $a_0\neq 1$ and let $\mu=\delta_{Z_0}$. We indicate the continuous solutions of the functional equation

$$f(XYZ_0)=f(X)\chi_1(Y)+\chi_2(X)f(Y),\quad X,Y\in G. \tag{23}$$

The continuous characters on $G$ are parameterized by $\lambda\in\mathbb{C}$ as follows (see, e.g., [1, Example 3.13]),

$$\chi_\lambda\begin{pmatrix} a & b\\ 0 & 1\end{pmatrix}=a^\lambda \text{ for } a>0 \text{ and } b\in\mathbb{R}.$$

The continuous additive functions on $G$ are parameterized by $\alpha\in\mathbb{C}$ as follows

$$a_\alpha\begin{pmatrix} a & b\\ 0 & 1\end{pmatrix}=\alpha\ln a.$$

**Case 1**. Suppose $\chi_1=\chi_2=\chi$. The condition $\chi_\lambda(Z_0)=1$ implies that $a_0^\lambda=e^{\lambda\ln a_0}=1$, then $\lambda=\frac{i2k\pi}{\ln a_0}$ and so, $\chi_1\left(\left(\begin{smallmatrix} a & b\\ 0 & 1\end{smallmatrix}\right)\right)=\chi_2\left(\left(\begin{smallmatrix} a & b\\ 0 & 1\end{smallmatrix}\right)\right)=\exp\left(\frac{i2k\pi\ln a}{\ln a_0}\right)$ for a fixed $k\in\mathbb{Z}$. According to Corollary 1, the solutions of the equation (23) are of the forms:
a) $f=\chi_\lambda a_\alpha$ such that $a_\alpha(Z_0)=0$, which implies that $\alpha\ln(a_0)=0$, then $a_\alpha=0$ (because $a_0\neq 1$). So, $f=0$.
b) $f=\gamma\chi_\lambda(1-a_\alpha)$, $\gamma\in\mathbb{C}$ such that $a_\alpha(Z_0)=-1$. This gives that $\alpha=\frac{-1}{\ln a_0}$ and, so,

$$f\begin{pmatrix} a & b\\ 0 & 1\end{pmatrix}=\gamma a^\lambda\left(1+\frac{1}{\ln a_0}\ln a\right) \text{ for } a>0,\ b\in\mathbb{R} \text{ and } \gamma\in\mathbb{C}.$$

**Case 2**. Suppose now $\chi_1\neq\chi_2$. Let $Y_1=\left(\begin{smallmatrix} a_1 & 0\\ 0 & 1\end{smallmatrix}\right)\in G$ such that $\chi_1(Y_1)\neq\chi_2(Y_1)$. Since $\chi_1(Z_0)=\chi_2(Z_0)=1$, then $\chi_1(\left(\begin{smallmatrix} a & b\\ 0 & 1\end{smallmatrix}\right))=a^{\lambda_1}=\exp\left(\frac{i2k_1\pi\ln a}{\ln a_0}\right)$ and $\chi_2(\left(\begin{smallmatrix} a & b\\ 0 & 1\end{smallmatrix}\right))=a^{\lambda_2}=\exp\left(\frac{i2k_2\pi\ln a}{\ln a_0}\right)$ for different fixed $k_1,k_2\in\mathbb{Z}$. The continuous, additive functions on

$$[G,G]=\left\{\begin{pmatrix} 1 & x\\ 0 & 1\end{pmatrix}\middle|\, x\in\mathbb{R}\right\},$$

are given by $A_\alpha\left(\begin{smallmatrix} 1 & b\\ 0 & 1\end{smallmatrix}\right)=\alpha b$, where $\alpha\in\mathbb{C}$ (see [9, Example 19]). By Corollary 1, we get that

$$f\begin{pmatrix} a & b\\ 0 & 1\end{pmatrix}=\gamma(a^{\lambda_1}-a^{\lambda_2})+A_\alpha\left(\left[Y_1,\begin{pmatrix} a & b\\ 0 & 1\end{pmatrix}\right]\right)a^{\lambda_1},\quad \gamma\in\mathbb{C}.$$

In this case, $Y_1\neq Z_0$ because $\chi_1(Z_0)=\chi_2(Z_0)=1$. By simples computations, the condition $A_\alpha[Y_1,Z_0]=0$ is always verified. The transformation property (9) for $A_\alpha$ is

$$A_\alpha\left(\begin{pmatrix} a & b\\ 0 & 1\end{pmatrix}\begin{pmatrix} 1 & x\\ 0 & 1\end{pmatrix}\begin{pmatrix} a & b\\ 0 & 1\end{pmatrix}^{-1}\right)=A_\alpha\left(\begin{pmatrix} 1 & ax\\ 0 & 1\end{pmatrix}\right)=\frac{a^{\lambda_2}}{a^{\lambda_1}}A_\alpha\left(\begin{pmatrix} 1 & x\\ 0 & 1\end{pmatrix}\right),$$

for all $\left(\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}\right) \in G$ and $\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right) \in [G,G]$, which reduces to $\alpha = \alpha a^{\lambda_2-\lambda_1-1}$. Then there are two cases:
1) If $\lambda_2 - \lambda_1 \neq 1$ then $\alpha = 0$, so that $A_\alpha = 0$ and we deduce that

$$f\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} = \gamma(a^{\lambda_1} - a^{\lambda_2}), \quad \gamma \in \mathbb{C}.$$

2) If $\lambda_2 - \lambda_1 = 1$, here any $A_\alpha$ has the transformation property (9). According to Corollary 1, we get that

$$f\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} = \gamma(a^{\lambda_1} - a^{\lambda_2}) + \alpha a_1 b a^{\lambda_1}, \quad \gamma \in \mathbb{C}.$$


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## Інтеграл від розширення формули додавання синуса


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У цій роботі визначено неперервні розв'язки інтегрального функціонального рівняння розширення Стеткара закону додавання синусів. $\int_G f(xyt)d\mu(t) = f(x)\chi_1(y) + \chi_2(x)f(y)$, $x, y \in G$, де $f\colon G \to \mathbb{C}$, $G$ — локально компактна Хаусдорфова група, $\mu$ — регулярна комплекснозначна борелівська міра на $G$ з компактним носієм та $\chi_1$, $\chi_2$ — фіксовані характери на $G$.



**Ключові слова:** *функціональне рівняння; закон додавання синусів; характер; адитивна функція; міра Бореля.*