Global dynamic of spatio-temporal fractional order SEIR model

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The global analysis of a spatio-temporal fractional order SEIR infection epidemic model is studied and analyzed in this paper. The dynamics of the infection is described by four partial differential equations with a fractional derivative order and with diffusion. The equations of our model describe the evolution of the susceptible, the exposed, the infected and the recovered individuals with taking into account the spatial diffusion for each compartment. At first, we will prove the existence and uniqueness of the solution using the results of the fixed point theorem, and the equilibrium points are established and presented according to $\mathcal{R}_0$. Next, the bornitude and the positivity of the solutions of the proposed model are established. Using the Lyapunov direct method it has been proved that the global stability of the each equilibrium depends mainly on the basic reproduction number $\mathcal{R}_0$. Finally, numerical simulations are performed to validate the theoretical results.

1. Introduction

Humain population have known several dangerous epidemics and pandemics. According to the world health organization, the infectious diseases have caused more than 17 million deaths each year [1–3]. Mathematical modeling has produced good results to apprehend those infectious disease. The first mathematical model of the infectious diseases spread is formulated and analyzed by Daniel Bernoulli in 1760 [4]. This model has a principal objective to evaluate the effectiveness of the variolation of the susceptible against smallpox. In 1911, Roland Ross was preoccupied with studying the dynamics of malaria, he managed to develop differential equations that model the spread of malaria. Drawing on the ideas of Roland Ross, Kermack and McKrndrick in 1927 proposed the classic model SIR [5] which represents three classes (compartments), class S for healthy individuals, class I for infected individuals and class R for recovered ones. An extension of SIR models was set up based on the remark that for certain diseases, a certain part of the infected population does not present any symptoms, which gave rise to the SEIR model, with E represents the exposed individuals [6–8].

Most of the classical models take into account only the temporal variable t [9–12]. However, the time t is not the only variable that influences the infection propagation. For this reason, many authors consider also the propagation according to the space x [7,13]. The optimal control of a spatio-temp SIR model with diffusion is studied in [14]. The numerical study of an SIR epidemic model with diffusion is studied in [15]. An asymptotic study of SIR reaction–diffusion model with a linear source is fulfilled in [16].

For several problems, the state of the system always depends on its history. Since the derivative of fractional order has an interesting property called memory effect, modeling with fractional derivative

becomes essential. The fractional derivative enlarged the region of stability of dynamical systems, it turned out that many phenomena in different fields can be described very successfully by models using differential equations of fractional order [17–19].

Dynamical analysis of a delayed fractional-order SIR model with saturated incidence and treatment functions is studied in [20]. Later, the fractional order SIR model with generalized incidence rate is studied in [21]. The global analysis of a time fractional order spatio-temporal SIR model is studied in a recent work [22]. The authors of the latter work have proposed the following model:

$$
\begin{cases}
{}_0^C D_t^\alpha S(t,x) = \lambda_S \Delta S(t,x) + \Lambda - \beta S(t,x)I(t,x) - (\mu + \gamma_1)\,S(t,x), \\
{}_0^C D_t^\alpha I(t,x) = \lambda_I \Delta I(t,x) + \beta S(t,x)I(t,x) - (\mu + \gamma_3)I(t,x), \\
{}_0^C D_t^\alpha R(t,x) = \lambda_R \Delta R(t,x) + \gamma_1 S(t,x) + \gamma_3 I(t,x) - \mu R(t,x),
\end{cases}
\tag{1}
$$

with ${}_0^C D_t^\alpha$ is the time fractional derivative of order α in the sense of Caputo with $0 < \alpha \leqslant 1$.

The birth rate of the susceptible is denoted Λ, β is the infection rate, γ_1 is the recovery rate of the susceptible, γ_3 is the recovery rate of the infected individuals and μ is the natural mortality. The positive constants λ_S, λ_I and λ_R are the diffusion coefficients of the susceptible, the infected and the recovered, respectively.

In most diseases, after the initial incubation, the host remains in a latent phase before becoming infectious, and this period may not be negligible compared to the infectious period. Therefore, it makes sense to add a new compartment of the exposed to the epidemiological model. Hence, we will add this new compartment to the previous work. In this work, we propose an SEIR diffusion model with the fractional derivative:

$$
\begin{cases}
{}_0^C D_t^\alpha S(x,t) = \lambda_S \Delta S(x,t) + \Lambda - \beta S(x,t)I(x,t) - (\mu + \gamma_1)S(x,t), \\
{}_0^C D_t^\alpha E(x,t) = \lambda_E \Delta I(x,t) + \beta S(x,t)I(x,t) - (\mu + k + \gamma_2)E(x,t), \\
{}_0^C D_t^\alpha I(x,t) = \lambda_I \Delta I(x,t) + kE(x,t) - (\mu + \gamma_3)I(x,t), \\
{}_0^C D_t^\alpha R(x,t) = \lambda_R \Delta R(x,t) + \gamma_1 S(x,t) + \gamma_2 E(x,t) + \gamma_3 I(x,t) - \mu R(x,t).
\end{cases}
\tag{2}
$$

The new parameters of our suggested model are as follows. The diffusion coefficient of the new exposed class E is given by λ_E, k is the rate that exposed become infectious and γ_2 is the recovery rate for the exposed individuals. The transfer diagram of the infection dynamics is given in Fig. 1.
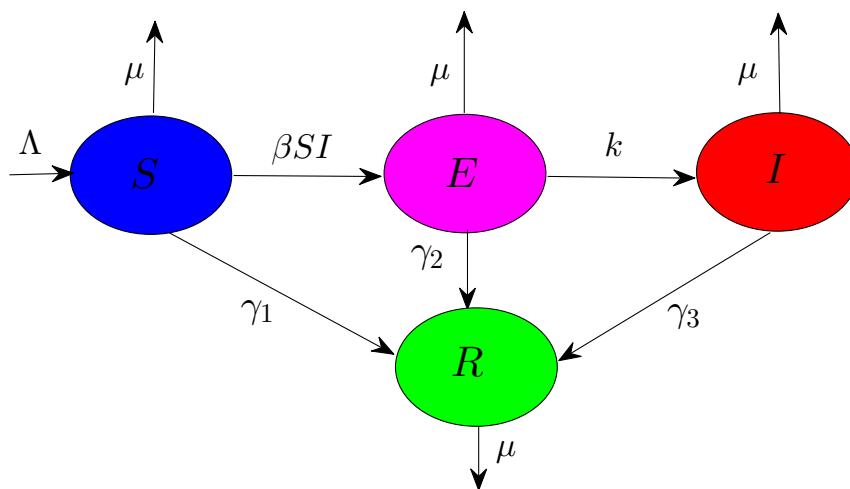

Fig. 1. The transfer diagram of the spatio-temporal model SEIR.

For biological reasons, we must choose the initial conditions as positive functions

$$S(x,0) = S_0, \quad E(x,0) = E_0, \quad I(x,0) = I_0 \quad \text{and} \quad R(x,0) = R_0, \quad x \in \Omega. \tag{3}$$

Where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. The normal derivatives of the class S, E, I and R at the boundary of Ω are zero, which means biologically that the population remain inside the boundary.

$$\frac{\partial S(x,t)}{\partial v} = \frac{\partial E(x,t)}{\partial v} = \frac{\partial I(x,t)}{\partial v} = \frac{\partial R(x,t)}{\partial v} = 0, \quad (x,t) \in \partial\Omega \times [0,T], \tag{4}$$

with $\frac{\partial}{\partial v}$ denotes the outward normal derivative on $\partial\Omega$.

This document is organized as follows. First, we present the necessary Definitions and Lemmas in the next section. In Section 3, we will give the result of existence and state the equilibria. The global stability is given Section 4. Section 5 is devoted to different numerical simulations. We conclude in the last section.

2. Preliminaries

We first present Green's formula [23]:

Theorem 1. *Let Ω be a domain of $\mathbb{R}^2$ or $\mathbb{R}^3$, and $n(x)$ its exterior normal. Let u and v be two regular functions, w a field of vectors defined on Ω. So,*

$$\int_\Omega \operatorname{div} w \, dx = \int_{\partial\Omega} w \cdot n \, d\sigma \quad \text{(divergence formula)},$$

$$\int_\Omega (\Delta u) v \, dx = -\int_\Omega \nabla u \cdot \nabla v \, dx + \int_{\partial\Omega} \frac{\partial u}{\partial n} v \, d\sigma \quad \text{(Green formula)}.$$

Definition 1. *The Mittag–Leffler function, $E_\alpha(z)$, is given as*

$$E_\alpha(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(k\alpha + 1)}, \quad \alpha > 0, \quad z \in \mathbb{C}$$

with

$$\Gamma(x) = \int_0^{+\infty} e^{-t} t^{x-1} dt$$

is the Gamma function

Definition 2 (Riemann–Liouville fractional integral [24, 25]). *Let f be a function such that $f \in L^1(\mathbb{R}^+)$, the fractional Riemann–Liouville integral, with $\alpha > 0$ of f is*

$$I^\alpha f(t) = \int_0^t \frac{1}{\Gamma(\alpha)} (t-s)^{\alpha-1} f(s) \, ds.$$

Definition 3 (The derivative of fractional order in the sense of Caputo [25]). *Let $\alpha > 0$, and let $n \in \mathbb{N}$ check $n-1 < \alpha \leqslant n$. The Caputo fractional derivative of order α applied to the function $f \in C^n([0, +\infty), \mathbb{R})$ is given by*

$${}_0^C D_t^\alpha f(t) = I^{n-\alpha} D^n f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{\alpha+1-n}} \, ds$$

with $D = \frac{d}{dt}$.

For $n = 1$, we have $0 < \alpha < 1$, so, we have

$${}_0^C D_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f'(s)}{(t-s)^\alpha} \, ds.$$

LaSalle's principle of invariance is a widely used tool to study the asymptotic behavior of solutions of differential equations, see [26, 27].

Theorem 2 (LaSalle's principle of invariance). *Let x^* be an equilibrium point of a Cauchy problem. The equilibrium x^* is asymptotically stable if there exists a continuous function $\mathcal{V}$ defined on a neighborhood $U \subset \mathbb{R}^n$ of x^* with values in $\mathbb{R}$, differentiable on $U \setminus \{x^*\}$ such that:*

   i) $\mathcal{V}(x^*) = 0$ and $\mathcal{V}(x) > 0$ if $x \neq x^*$.

  ii) $\dot{\mathcal{V}} \leqslant 0$ on $U \setminus \{x^*\}$.

 iii) The set $S = x \in U / \dot{\mathcal{V}}(x) = 0$ does not contain any trajectory of the system other than $x(t) = x^*$.

3. Existence result and the equilibria

3.1. Existence result

Let $\mathbb{X} = C(\bar{\Omega}, \mathbb{R})$ where $\mathbb{X}^4$ is a Banach space with the usual norms. We set $J = (S, E, I, R)$, $J_0 = (S_0, E_0, I_0, R_0)$, $\lambda = (\lambda_S, \lambda_E, \lambda_I, \lambda_I)$ and let A be the linear diffusion operator

$$A \colon \mathcal{D}(A) \subset \mathbb{X}^4 \to \mathbb{X}^4, \quad AJ = \lambda \Delta J = (\lambda_S \Delta S, \lambda_E \Delta E, \lambda_I \Delta I, \lambda_R \Delta R), \quad \forall J \in \mathcal{D}(A),$$

where

$$\mathcal{D}(A) = \left\{ J \in \mathbb{X}^4 \colon \Delta J \in \mathbb{X}^4, \frac{\partial J}{\partial v} = 0_{\mathbb{R}^4} \text{ for all } x \in \partial \Omega \right\}.$$

Consider the function f defined by $f \colon [0, T] \times \mathbb{X}^4 \mapsto \mathbb{X}^4$, with

$$f(t, J(t)) = f(J(t)) = \big(f_1(J(t)), f_2(J(t)), f_3(J(t)), f_4(J(t)) \big),$$

$$\begin{cases} f_1(J(t)) = \Lambda - \beta S(t, x) I(t, x) - (\mu + \gamma_1) S(t, x), \\ f_2(J(t)) = \beta S(t, x) I(t, x) - (\mu + k + \gamma_2) E(t, x), \\ f_3(J(t)) = k E(t, x) - (\mu + \gamma_3) I(t, x), \\ f_4(J(t)) = \gamma_1 S(t, x) + \gamma_2 E(t, x) + \gamma_3 I(t, x) - \mu R(t, x), \end{cases} \tag{5}$$

we can rewrite the model (2)–(3) in the following expression:

$$\begin{cases} {}_0^C D_t^\alpha J = AJ + f(J(t)), \\ J(0) = J_0, \end{cases} \tag{6}$$

with $J = (S, E, I, R)$ and $J_0 = (S_0, E_0, I_0, R_0)$.

The proposition (3.3) in [28] allows to show the existence, uniqueness and non-negativity of the solutions of the problem (6).

Proposition 11. Let $0 < \alpha \leqslant 1$, for $J^0 \in \mathcal{D}(A)$, the problem (6) has a unique positive solution $J \in C([0, T]; X^4)$ with

$$J(t) = \int_0^\infty \Phi_\alpha(\theta) Q(t^\alpha \theta) J^0 d\theta + F(t),$$

and

$$F(t) = \alpha \int_0^t \int_0^\infty \theta (t - \tau)^{\alpha - 1} \Phi_\alpha(\theta) Q\left((t - \tau)^\alpha \theta\right) f(\tau) \, d\theta \, d\tau,$$

$\Phi_\alpha(\theta)$ is a probability density function defined on $(0, \infty)$.

Proof. It is clear that f is continuous and Lipschitzian, and A a linear operator defined on set $\mathcal{D}(A) \subset X^4$ dances without itself so according to [28] there is a unique positive solution to the problem (6). ∎

It remains to show that the solution is bounded. Biologically we have S, E, I and R are positive functions on $T = \Omega \times [0, +\infty]$ we also have

$$N(x, t) = \int_\Omega [S(x, t) + E(x, t) + I(x, t) + R(x, t)] \, dx.$$

The fractional derivative of order α in the sense of Caputo:

$${}_0^C D_t^\alpha N(x, t) = \int_\Omega \left[{}_0^C D_t^\alpha S(x, t) + {}_0^C D_t^\alpha E(x, t) + {}_0^C D_t^\alpha I(x, t) + {}_0^C D_t^\alpha R(x, t) \right] dx,$$

$$= \int_\Omega \big(\lambda_S \Delta S(x,t) + \lambda_E \Delta E(x,t) + \lambda_I \Delta I(x,t) + \lambda_R \Delta R(x,t)\big)\, dx$$

$$+ \int_\Omega \big[\Lambda - \beta S(x,t)I(t,x) - (\mu + \gamma_1)S(x,t) + \beta S(x,t)I(x,t)$$

$$- (\mu + k + \gamma_2)E(x,t) + kE(x,t) - (\mu + \gamma_3)I(x,t) + \gamma_1 S(x,t)$$

$$+ \gamma_2 E(x,t) + \gamma_3 I(x,t) - \mu R(x,t)\big]\, dx.$$

According to Green's formula and the initial conditions, we have

$$\int_\Omega \big(\lambda_S \Delta S(x,t) + \lambda_E \Delta E(x,t) + \lambda_I \Delta I(x,t) + \lambda_R \Delta R(x,t)\big)\, dx = 0,$$

then

$$_0^C D_t^\alpha N(x,t) = \int_\Omega \big[\Lambda - \mu(S(x,t) + E(x,t) + I(x,t) + R(x,t))\big]\, dx$$

$$= \Lambda \|\Omega\| - \mu N(x,t).$$

Applying the Laplace transformation one can get:

$$N(t) = N(0)\mathbb{E}_\alpha(-\mu t^\alpha) + \frac{\Lambda}{\mu}\big(1 - \mathbb{E}_\alpha(-\mu t^\alpha)\big),$$

since $0 \leqslant \mathbb{E}_\alpha \leqslant 1$, so

$$N(t) \leqslant N(0) + \frac{\Lambda}{\mu},$$

then, the functions S, E, I and R are bounded.

3.2. The equilbria

Before giving the different equilibria of our problem, we will calculate the basic reproduction number. Indeed, this number is known as the expected average number of new cases of infection, generated by an average infectious individual (during its period of infectivity), in a population consisting entirely of susceptible. To calculate the basic reproduction rate, we will apply the method given by Van Den Driessche and Watmough in [29]:

To determine $\mathcal{R}_0$, we are just need two classes E and I:

$$\begin{cases} _0^C D_t^\alpha E(x,t) = \lambda_E \Delta E(x,t) + \beta S(x,t)I(x,t) - (\mu + k + \gamma_2)E(x,t), \\ _0^C D_t^\alpha I(x,t) = \lambda_I \Delta I(x,t) + kE(x,t) - (\mu + \gamma_3)I(x,t). \end{cases}$$

We have $\mathcal{R}_0 = \rho(\mathbb{F}\mathbb{V}^{-1})$ (ρ is the spectral radius) with

$$\mathbb{F} = \begin{pmatrix} 0 & \frac{\beta\Lambda}{\mu+\gamma_1} \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \mathbb{V} = \begin{pmatrix} \mu + k + \gamma_2 & 0 \\ -k & \mu + \gamma_3 \end{pmatrix},$$

thus $\quad \mathbb{V}^{-1} = \begin{pmatrix} \frac{1}{\mu+k+\gamma_2} & 0 \\ \frac{k}{(\mu+k+\gamma_2)(\mu+\gamma_3)} & \frac{1}{\mu+\gamma_3} \end{pmatrix} \quad$ and $\quad \mathbb{F}\mathbb{V}^{-1} = \begin{pmatrix} \frac{\beta\Lambda k}{(\mu+\gamma_1)(\mu+k+\gamma_2)(\mu+\gamma_3)} & \frac{\beta\Lambda}{(\mu+\gamma_1)(\mu+\gamma_3)} \\ 0 & 0 \end{pmatrix}.$

The characteristic polynomial of the matrix $\mathbb{F}\mathbb{V}^{-1}$ is given by

$$\mathcal{P}(X) = -X\left(\frac{\beta\Lambda k}{(\mu+\gamma_1)(\mu+k+\gamma_2)(\mu+\gamma_3)} - X\right),$$

then

$$\mathcal{R}_0 = \rho(\mathbb{F}\mathbb{V}^{-1}) = \frac{\beta\Lambda k}{(\mu+\gamma_1)(\mu+k+\gamma_2)(\mu+\gamma_3)},$$

hence finally,

$$\mathcal{R}_0 = \frac{\beta\Lambda k}{(\mu+\gamma_1)(\mu+k+\gamma_2)(\mu+\gamma_3)}.$$

Now, we calculate the equilibria. As it is well known X^* is an equilibrium point of (6) if $f(X^*) = 0$,

$$\begin{cases} f_1(J(t)) = 0, \\ f_2(J(t)) = 0, \\ f_3(J(t)) = 0, \\ f_4(J(t)) = 0, \end{cases} \tag{7}$$

then

$$\begin{cases} \Lambda - \beta S(x,t)I(x,t) - (\mu + \gamma_1)S(x,t) = 0, \\ \beta S(x,t)I(x,t) - (\mu + k + \gamma_2)E(x,t) = 0, \\ kE(x,t) - (\mu + \gamma_3)I(x,t) = 0, \\ \gamma_1 S(x,t) + \gamma_2 E(x,t) + \gamma_3 I(x,t) - \mu R(x,t). \end{cases} \tag{8}$$

So,

$$\begin{cases} S = \dfrac{\Lambda}{\beta I + \mu + \gamma_1}, \\ \beta SI - \dfrac{(\mu + k + \gamma_2)(\mu + \gamma_3)}{k}I = 0, \\ E = \dfrac{\mu + \gamma_3}{k}I, \\ R = \dfrac{\gamma_1 S + \gamma_2 E + \gamma_3 I}{\mu}. \end{cases} \tag{9}$$

We have in the second equation of the system (9): $I\left(\beta S - \frac{(\mu+k+\gamma_2)(\mu+\gamma_3)}{k}\right) = 0$.

i) If $I = 0$ then according to the third equation of (9) we have $E = 0$, and the first equation of (9) gives $S = \frac{\Lambda}{\mu+\gamma_1}$, then the last equation gives: $R = \frac{u}{\mu}S = \frac{\Lambda\gamma_1}{\mu(\mu+\gamma_1)}$, then we obtain the infection-free equilibrium given by

$$\mathcal{E}_f = (S_f, 0, 0, R_f) \quad \text{with} \quad S = \frac{\Lambda}{\mu + \gamma_1} \quad \text{and} \quad R = \frac{\Lambda\gamma_1}{\mu(\mu + \gamma_1)}.$$

ii) Now if $I \neq 0$ then $\left(\beta S - \frac{(\mu+k+\gamma_2)(\mu+\gamma_3)}{k}\right) = 0$, then

$$S^* = \frac{(\mu + k + \gamma_2)(\mu + \gamma_3)}{\beta k} = \frac{\Lambda}{\mathcal{R}_0(\mu + \gamma_1)}, \quad \text{with} \quad \mathcal{R}_0 = \frac{\Lambda\beta k}{(\mu + \gamma_1)(\mu + k + \gamma_2)(\mu + \gamma_3)},$$

according to the first equation of (8):

$$I^* = \frac{\Lambda - (\mu + \gamma_1)S^*}{\beta S^*} = \frac{\Lambda - (\mu + \gamma_1)\frac{\Lambda}{\mathcal{R}_0(\mu+\gamma_1)}}{\beta\frac{\Lambda}{\mathcal{R}_0(\mu+\gamma_1)}} = \frac{1 - \frac{1}{\mathcal{R}_0}}{\frac{\beta}{\mathcal{R}_0(\mu+\gamma_1)}},$$

therefore

$$I^* = \frac{(\mathcal{R}_0 - 1)(\mu + \gamma_1)}{\beta}.$$

From equation 3 of (9):

$$E^* = \frac{(\mu\gamma_3)}{k}I^* = \frac{(\mu\gamma_3)}{k}\frac{(\mathcal{R}_0 - 1)(\mu + \gamma_1)}{\beta} = \frac{(\mathcal{R}_0 - 1)\Lambda}{\mathcal{R}_0(\mu + k + \gamma_3)},$$

and finally,

$$R^* = \frac{\gamma_1 S* + \gamma_2 E* + \gamma_3 I*}{\mu},$$

then the equilibrium point with infection is given by

$$\mathcal{E}^* = (S^*, E^*, I^*, R^*) \quad \text{with} \quad S^* = \frac{\Lambda}{\mathcal{R}_0(\mu + \gamma_1)}, \quad I^* = \frac{(\mathcal{R}_0 - 1)(\mu + \gamma_1)}{\beta},$$

$$E^* = \frac{(\mathcal{R}_0 - 1)\Lambda}{\mathcal{R}_0(\mu + k + \gamma_3)} \quad \text{and} \quad R^* = \frac{\gamma_1 S* + \gamma_2 E* + \gamma_3 I*}{\mu}.$$

4. Global stability analysis

Noting that the first three equations do not depend on the bucket $R(x,t)$ and are then decoupled with the last system equation (2). Our attention is therefore focused on the analysis of the following reduced system: for all $(x,t) \in \Omega \times [0,+\infty[$,

$$\begin{cases} {}^C_0 D^\alpha_t S(x,t) = \lambda_S \Delta S(x,t) + \Lambda - \beta S(x,t)I(x,t) - (\mu + \gamma_1)S(x,t), \\ {}^C_0 D^\alpha_t E(x,t) = \lambda_E \Delta I(x,t) + \beta S(x,t)I(x,t) - (\mu + k + \gamma_2)E(x,t), \\ {}^C_0 D^\alpha_t I(x,t) = \lambda_I \Delta I(x,t) + kE(x,t) - (\mu + \gamma_3)I(x,t). \end{cases} \tag{10}$$

First of all, we have the following lemma.

Lemma 1 (Lemma 3.1 in [30]). *Let Ψ be a positive function defined by $\Psi(y) = y - \ln(y) - 1$, $y > 0$ and $y(t) \in \mathbb{R}+^*$ is a continuous differentiable function for all $\alpha \in [0,1]$ and $t \geqslant t_0$,*

$$ {}^C_{t_0} D^\alpha_t \left[y^* \Psi\left(\frac{y(t)}{y^*} \right) \right] \leqslant \left(1 - \frac{y^*}{y(t)} \right) {}^C_{t_0} D^\alpha_t y(t), \quad y \in \mathbb{R}^*_+. $$

The demonstration is well detailed in [30].

First, we discuss the global stability of the infection-free equilibrium $\mathcal{E}_f$ and the immune-free infection equilibrium $\mathcal{E}^*$.

Theorem 3. *If $\mathcal{R}_0 \leqslant 1$ then the infection-free equilibrium $\mathcal{E}_f$ is globally asymptotically stable.*

Proof. Let $\mathcal{V}$ be the positive function defined by

$$\mathcal{V}(x,t) = \int_\Omega \left[\frac{S_f}{\varepsilon} \Psi\left(\frac{S(x,t)}{Sf} \right) + \frac{1}{\varepsilon} \Psi E(x,t) + \frac{1}{k} I(x,t) \right] dx, \quad \text{with} \quad \varepsilon = \mu + k + \gamma_2.$$

It is clear that

$$\mathcal{V}(\mathcal{E}_f) = 0.$$

The fractional derivative of order α in the sense of Caputo of $\mathcal{V}$ is given by

$$ {}^C_{t_0} D^\alpha_t \mathcal{V}(x,t) = {}^C_{t_0} D^\alpha_t \int_\Omega \left[\frac{S_f}{\varepsilon} \Psi\left(\frac{S(x,t)}{Sf} \right) + \frac{1}{\varepsilon} \Psi E(x,t) + \frac{1}{k} I(x,t) \right] dx.$$

We apply the results of Lemma 1 and we obtain:

$$\begin{aligned} {}^C_{t_0} D^\alpha_t \mathcal{V}(x,t) &\leqslant \int_\Omega \left(\left(1 - \frac{S_f}{S(x,t)} \right) {}^C_0 D^\alpha_t S(x,t) + {}^C_0 D^\alpha_t E(x,t) + {}^C_0 D^\alpha_t I(t,x) \right) dx \\ &\leqslant \int_\Omega \left[\frac{1}{\varepsilon} \left(1 - \frac{S_f}{S(x,t)} \right) (\Lambda - \beta S(x,t)I(x,t) - (\mu + \gamma_1)S(x,t)) \right] dx \\ &+ \int_\Omega \left[\frac{1}{\varepsilon} \beta S(x,t)I(x,t) - E(x,t) + E(x,t) - \frac{(\mu + \gamma_3)}{k} I(x,t) \right] dx \\ &+ \int_\Omega \left(\frac{1}{\varepsilon} \lambda_S \Delta S(x,t) + \frac{1}{\varepsilon} \lambda_E \Delta E(x,t) + \frac{1}{k} \lambda_I \Delta I(x,t) \right) dx \\ &- \int_\Omega \frac{\lambda_S}{\varepsilon} \frac{S_f \Delta S(x,t)}{S(x,t)} dx. \end{aligned}$$

According to Green's formula and the boundary conditions, we have

$$\int_\Omega (\lambda_S \Delta S(x,t) + \lambda_E \Delta E(x,t) + \lambda_I \Delta I(x,t)) \, dx = 0,$$

and

$$-\int_\Omega \lambda_S \frac{S_f}{S(x,t)} \Delta S(x,t) \, dx = -\lambda_S S_f \int_\Omega \frac{\|\nabla S(x,t)\|^2}{S(x,t)^2} dx.$$

Hence,

$$
{}^{C}_{t_0}D_t^{\alpha}\mathcal{V}(x,t) \leqslant -\frac{1}{\varepsilon}\lambda_S S_f \int_{\Omega} \frac{\|\nabla S(x,t)\|^2}{S(x,t)^2}dx + \int_{\Omega}\left[\frac{1}{\varepsilon}\beta S_f - \frac{(\mu+\gamma_3)}{k}\right]I(x,t)dx
$$
$$
+ \int_{\Omega}\frac{1}{\varepsilon}\left[\Lambda - (\mu+\gamma_1)S(x,t) - \frac{\Lambda S_f}{S(x,t)} + S_f(\mu+\gamma_1)\right]dx,
$$

since $\Lambda = (\mu+\gamma_1)S_f$,

$$
\Lambda - (\mu+\gamma_1)S(x,t) - \frac{\Lambda S_f}{S(x,t)} + (\mu+\gamma_1)S_f = (\mu+\gamma_1)S_f - (\mu+\gamma_1)S(x,t) - \frac{(\mu+\gamma_1)S_f^2}{S(x,t)} + (\mu+\gamma_1)S_f
$$
$$
= \left(2S_f S(x,t) - S(x,t)^2 - S_f^2\right)\frac{\mu+\gamma_1}{S(x,t)}
$$
$$
= -(\mu+\gamma_1)\frac{(S(x,t)-S_f)^2}{S(x,t)} \leqslant 0.
$$

We have $\beta S_f = \frac{\mathcal{R}_0(\mu+k+\gamma_2)(\mu+\gamma_3)}{k}$ and $\mathcal{R}_0 = \frac{\beta k\Lambda}{(\mu+\gamma_1)(\mu+k+\gamma_2)(\mu+\gamma_3)}$, so,

$$
\frac{1}{\varepsilon}\beta S_f I(x,t) - \frac{(\mu+\gamma_3)}{k}I(x,t) = \left[\frac{\beta\Lambda}{(\mu+\gamma_1)(\mu+k+\gamma_2)} - \frac{\mu+\gamma_3}{k}\right]I(x,t)
$$
$$
= \frac{k}{\mu+\gamma_3}\left(\mathcal{R}_0 - 1\right)I(x,t),
$$

so, finally

$$
{}^{C}_{t_0}D_t^{\alpha}\mathcal{V}(x,t) \leqslant -\frac{1}{\varepsilon}\lambda_S S_f \int_{\Omega}\frac{\|\nabla S(x,t)\|^2}{S(x,t)^2}dx - \frac{\mu+\gamma_1}{\varepsilon}\int_{\Omega}\frac{(S(x,t)-S_f)^2}{S(x,t)}dx + \frac{k}{\mu+\gamma_3}\left(\mathcal{R}_0-1\right)\int_{\Omega}I(x,t)dx.
$$

If $\mathcal{R}_0 \leqslant 1$ then ${}^{C}_{t_0}D_t^{\alpha}\mathcal{V}(x,t) \leqslant 0$

$$
\text{If} \quad {}^{C}_{t_0}D_t^{\alpha}V(x,t) = 0 \quad \text{so:} \quad S = S_f, \quad E = 0 \quad \text{and} \quad I = 0,
$$
$$
\left\{(S,E,I) \in \mathbb{R}_+^3 : {}^{C}_{0}D_t^{\alpha}V(x,t) = 0\right\} = \{\mathcal{E}_f\}.
$$

Then according to the principle of LaSalle invariance if $\mathcal{R}_0 \leqslant 1$ then $\mathcal{E}_f$ is globally asymptotically stable. ∎

Theorem 4. *The equilibrium point $\mathcal{E}^*$ is globally asymptotically stable if $\mathcal{R}_0 > 1$.*

Proof. Consider the positive function defined by

$$
\mathcal{V}_1(x,t) = \int_{\Omega}\left[S^*\Psi\left(\frac{S(x,t)}{S^*}\right) + E^*\Psi\left(\frac{E(x,t)}{E^*}\right) + \frac{\mu+k+\gamma_2}{k}I^*\Psi\left(\frac{I(x,t)}{I^*}\right)\right]dx.
$$

According to Lemma 1 the fractional derivative of $\mathcal{V}_1(t)$ is verified:

$$
{}^{C}_{0}D_t^{\alpha}\mathcal{V}_1(x,t) \leqslant \int_{\Omega}\left[\left(1-\frac{S^*}{S}\right){}^{C}_{0}D_t^{\alpha}S(x,t) + \left(1-\frac{E^*}{E}\right){}^{C}_{0}D_t^{\alpha}E(x,t)\right.
$$
$$
\left. + \frac{\mu+k+\gamma_2}{k}\left(1-\frac{I^*}{I}\right){}^{C}_{0}D_t^{\alpha}I(x,t)\right]dx
$$
$$
\leqslant \int_{\Omega}\left(1-\frac{S^*}{S}\right)(\lambda_S\Delta S + \Lambda - \beta SI - (\mu+\gamma_1)S)\,dx
$$
$$
+ \int_{\Omega}\left(1-\frac{E^*}{E}\right)(\lambda_E\Delta E + \beta SI - (\mu+k+\gamma_2)E)\,dx
$$
$$
+ \frac{\mu+k+\gamma_2}{k}\int_{\Omega}\left(1-\frac{I^*}{I}\right)(\lambda_I\Delta I + kE - (\mu+\gamma_3)I)\,dx
$$
$$
\leqslant \int_{\Omega}\left[\lambda_S\Delta S + \lambda_E\Delta E + \lambda_I\Delta I - \lambda_S S^*\frac{\Delta S}{S} - \lambda_E E^*\frac{\Delta E}{E} - \lambda_I I^*\frac{\Delta I}{I}\right]dx
$$

$$+ \int_\Omega \left[\Lambda - \beta SI - (\mu + \gamma_1)S - \Lambda \frac{S^*}{S} + \beta S^* I + (\mu + \gamma_1)S^* \right.$$

$$+ \beta SI - (\mu + k + \gamma_2)E - \beta SI \frac{E^*}{E} + (\mu + k + \gamma_2)E^*$$

$$+ (\mu + k + \gamma_2)E - \frac{(\mu + k + \gamma_2)(\mu + \gamma_3)}{k} I - (\mu + k + \gamma_2)E \frac{I^*}{I} dx$$

$$+ \left. \frac{(\mu + k + \gamma_2)(\mu + \gamma_3)}{k} I^* \right] dx.$$

According to Green's formula and the boundary conditions,

$$\int_\Omega \left(\lambda_S \Delta S(x,t) + \lambda_E \Delta E(x,t) + \lambda_I \Delta I(x,t) \right) dx = 0,$$

and

$$- \int_\Omega \lambda_S S^* \frac{\Delta S(x,t)}{S(x,t)} = - \int_\Omega \lambda_S S^* \frac{\|\nabla S(x,t)\|^2}{S(x,t)^2} dx \leqslant 0,$$

$$- \int_\Omega \lambda_E E^* \frac{\Delta E(x,t)}{E(x,t)} = - \int_\Omega \lambda_E E^* \frac{\|\nabla E(x,t)\|^2}{E(x,t)^2} \leqslant 0,$$

$$- \int_\Omega \lambda_I I^* \frac{\Delta I(x,t)}{I(x,t)} dx = - \int_\Omega \lambda_I I^* \frac{\|\nabla I(x,t)\|^2}{I(x,t)^2} dx \leqslant 0.$$

Since $\Lambda = (\mu + \gamma_1)S^* + (\mu + k + \gamma_2)E^*$ and $\frac{E^*}{I^*} = \frac{\mu + \gamma_3}{k}$, thus

$$\begin{aligned} {}_0^C D_t^\alpha \mathcal{V}_1(x,t) \leqslant \int_\Omega &\left[(\mu + \gamma_1)S^* + (\mu + k + \gamma_2)E^* - (\mu + \gamma_1)S - (\mu + \gamma_1)\frac{(S^*)^2}{S} \right. \\ &+ \mu + \gamma_1)S^* - (\mu + k + \gamma_2)E^* \frac{S^*}{S} + \beta S^* I - \beta SI \frac{E^*}{E} + (\mu + k + \gamma_2)E^* \\ &- (\mu + k + \gamma_2)E^* \frac{I}{I^*} - (\mu + k + \gamma_2)E^* \frac{E}{E^*} \frac{I^*}{I} + (\mu + k + \gamma_2)E^* \\ &\left. - \lambda_S S^* \frac{\|\nabla S\|^2}{S^2} - \lambda_E E^* \frac{\|\nabla E\|^2}{E^2} - \lambda_I I^* \frac{\|\nabla I\|^2}{I^2} \right] dx. \end{aligned}$$

Since, $\beta S^* I = (\mu + k + \gamma_2)E^* \frac{I}{I^*}$, then

$$\begin{aligned} {}_0^C D_t^\alpha \mathcal{V}_1(x,t) \leqslant & - \int_\Omega \left(\lambda_S S^* \frac{\|\nabla S\|^2}{S^2} + \lambda_E E^* \frac{\|\nabla E\|^2}{E^2} + \lambda_I I^* \frac{\|\nabla I\|^2}{I^2} \right) dx \\ & + (\mu + \gamma_1)S^* \int_\Omega \left(2 - \frac{S}{S^*} - \frac{S^*}{S} \right) dx + (\mu + k + \gamma_2)E^* \int_\Omega \left(3 - \frac{S^*}{S} - \frac{SIE^*}{EI^*S^*} - \frac{EI^*}{IE^*} \right) dx. \end{aligned}$$

Since the arithmetic mean is always less than the geometric mean then:

$$2 - \frac{S}{S^*} - \frac{S^*}{S} \leqslant 0,$$

and

$$3 - \frac{S^*}{S} - \frac{SIE^*}{EI^*S^*} - \frac{EI^*}{IE^*} \leqslant 0.$$

Therefore

$${}_0^C D_t^\alpha \mathcal{V}_1(x,t) \leqslant 0,$$

after a simple calculation we can conclude that if ${}_0^C D_t^\alpha \mathcal{V}_1(t) = 0$, so, $S = S^*$, $E = E^*$ and $I = I^*$, thus

$$\left\{ (S, E, I) \in \mathbb{R}_+^3 : {}_0^C D_t^\alpha \mathcal{V}_1(x,t) = 0 \right\} = \{ \mathcal{E}^* \} .$$

So, according to the principle of LaSalle invariance if $\mathcal{R}_0 > 1$ then $\mathcal{E}^*$ is globally asymptotically stable.   ∎

5. Numerical simulations

In this section, we present the results of numerical simulations to validate the theoretical results of the previous section. We used the finite difference numerical method with the Euler scheme for the approximation of the diffusion expression. And Euler's fractional method as mentioned in explains [31, 32] for the fractional derivative of order α in Caputo's sense. We have used the one-dimensional interval $0 \leqslant x \leqslant L$ and the time $0 \leqslant t \leqslant T$ with $L = 4$, and $T = 200$. The initial conditions were chosen as constants. We used an explicit numerical method with the space step $h_x = 0.01$ and the time step $h_t = 0.1$. The program was implemented with Matlab.
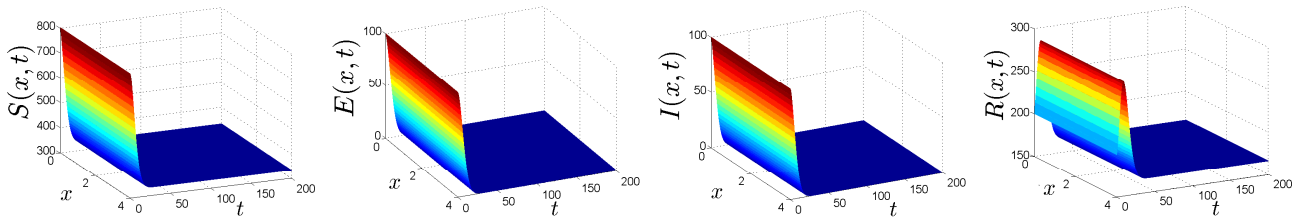
Fig. 2. The dynamics of classes S, I and R showing the stability of the free equilibrium $\mathcal{E}_f$.

Figure 2 shows the dynamics of classes S, E, I and R for the following parameters $\alpha = 0.6$, $\Lambda = 10$, $\mu = 0.02$, $\beta = 0.00001$, $k = 0.00053$, $\lambda_S = 0.01$, $\lambda_E = 0.01$, $\lambda_I = 0.01$, $\lambda_R = 0.01$, $\gamma_1 = 0.01$, $\gamma_2 = 0.01$ and $\gamma_3 = 0.01$. With these parameters, we have the basic reproduction rate is $\mathcal{R}_0 = 0.00192888 \leqslant 1$ and we observe the convergence toward the free equilibrium $\mathcal{E}_f = (333.33, 0, 0, 166.66)$. So, according to Theorem 3 we have $\mathcal{E}_f$ is globally asymptotically stable, which means that the spatio-temporal dynamics converges towards $\mathcal{E}_f$.
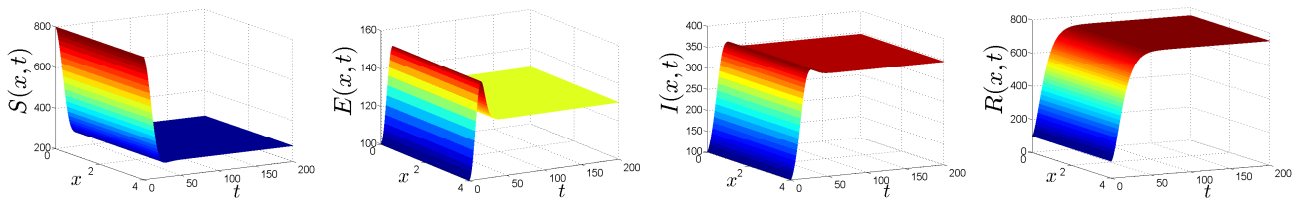
Fig. 3. The dynamics of classes S, E, I and R showing the stability of the endemic equilibrium $\mathcal{E}^*$.

Figure 3 shows the dynamics of classes S, E, I and R for the following parameters $\alpha = 0.6$, $\Lambda = 15$, $\mu = 0.01$, $\beta = 0.0001$, $k = 0.053$, $\lambda_S = 0.01$, $\lambda_E = 0.01$, $\lambda_I = 0.01$, $\lambda_R = 0.01$, $\gamma_1 = 0.01$, $\gamma_2 = 0.01$, and $\gamma_3 = 0.01$. With these parameters, we have $\mathcal{R}_0 = 2.7226 > 1$ and we observe the convergence toward the endemic equilibrium $\mathcal{E}^* = (275.471, 130, 344.520, 750)$. So, according to Theorem 4, we therefore have $\mathcal{E}^*$ is globally asymptotically stable, which means biologically that the infection persists.

6. Conclusion

In this work, we have presented and studied an epidemic model SEIR described by differential equations with diffusion and a fractional derivative of order α. We have taken into account the presence of the exposed class E and the propagation according to the space x. First, we have demonstrated the existence, uniqueness, positivity and boundness of the solution of the proposed dynamic model. Then, we have determined the basic reproduction number $\mathcal{R}_0$. After presenting two equilibrium points of the model as a function of basic reproduction number $\mathcal{R}_0$, we have proved the global stability of the disease-free equilibrium $\mathcal{E}_f$ when $\mathcal{R}_0 \leqslant 1$, which means the extinction of the disease. However, when $\mathcal{R}_0 > 1$, the endemic equilibrium $\mathcal{E}^*$ is globally asymptotically stable, which means biologically that the disease persists. The paper ends with some numerical simulations to validate the founded theoretical results.

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Глобальна динаміка просторово-часової моделі SEIR дробового порядку

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У цій статті досліджено та проаналізовано глобальний аналіз просторово-часової моделі епідемії інфекції SEIR дробового порядку. Динаміка інфекції описується чотирма диференціальними рівняннями в частинних похідних з дробовим порядком похідної та з дифузією. Рівняння нашої моделі описують еволюцію сприйнятливих, виявлених, інфікованих і одужавших осіб з урахуванням просторової дифузії для кожного компартменту. Спочатку доведено існування та єдиність розв'язку, використовуючи результати теореми про нерухому точку, а точки рівноваги встановлені та представлені відповідно до $\mathcal{R}_0$. Далі встановлено обмеженість і позитивність розв'язків запропонованої моделі. За допомогою прямого методу Ляпунова було доведено, що глобальна стійкість кожної рівноваги залежить головним чином від основного числа відтворення $\mathcal{R}_0$. Нарешті, чисельне моделювання виконується для підтвердження теоретичних результатів.

Ключові слова: *глобальна стійкість; часова дробовість; реакційно–дифузійні системи; просторово-часова епідемічна модель SEIR; основний репродукційний параметр.*