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MODELING THE INFLUENCE OF THE SHAPE OF THE LOCAL HEAT FLOW INTENSITY DISTRIBUTION ON THE SURFACE OF A SEMI-INFINITE BODY ON THE STRESS STATE IN THE VICINITY OF A SUBSURFACE CRACK

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Abstract. *Purpose.* A mathematical model to determine the two-dimensional thermoelastic state in a semi-infinite solid weakened by an internal crack under conditions of local heating is examined. Heat flux due to frictional heating on the local area of the body causes changes in temperature and stresses in the body, which significantly affects its strength, as it can lead to crack growth and local destruction. Therefore, the study of the problem of frictional heat is of practical interest. This paper proposes to investigate the stress-deformed state in the vicinity of the crack tip, depending on the crack placement.

Methodology. The methods for studying the two-dimensional thermoelastic state of a body with cracks as stress concentrators are based on the method of complex variable function. Reducing the problem of stationary heat conduction and thermoelasticity to singular integral equations (SIE) of the first kind, the numerical solution by the method of mechanical quadrature was obtained.

Findings. In this paper, we present graphical dependencies of stress intensity factors (SIF) at the crack tip on the angle orientation of the crack as well as forms of the intensity distribution of the local heat flux. The obtained results will be used later to determine the critical value of the intensity of the local heat flux from equations of limit equilibrium at which crack growth and the local destruction of the body occur.

Originality. The scientific novelty lies in the fact that the solutions to two-dimensional problems of heat conduction and thermoelasticity for a half-plane containing a crack due to local heating by a heat flux were obtained. This would make it possible to obtain a comparative analysis of the intensity of thermal stresses around the top of the crack, depending on the form of distribution of the intensity of the heat flow on the surface of the body.

Practical value. The practical value is the ability to extend our knowledge of the real situation in the thermoelastic elements of engineering structures with the crack that operate under conditions of heat stress (frictional heat) in various industries, particularly in mechanical engineering. The results of specific values of SIF at the crack tip in graphs may be useful in the development of sustainable modes of structural elements in terms of preventing the growth of cracks.

Keywords: crack, heat flux, heat condition, thermoelasticity, stress intensity factor, singular integral equation.

Introduction

The contact of sliding or rolling is accompanied by the destruction of bodies due to the initiation and subsequent propagation of cracks [1]. The load due to such an interaction consists of two parts – mechanical, which is caused by contact pressure, and thermal, caused by heat flow due to frictional heat generation. The solution of plane isothermal contact problems for a half-space with a crack was obtained in works [2-6]. The solution of the corresponding temperature problem is constructed here.

According to studies [6], the temperature problem is reduced to the determination of the temperature field and the stress state caused by it in the body during the heating of part of its outer surface (contact area) by a distributed heat flow with an intensity proportional to the specific work of friction.

Problem Statement

Let a semi-infinite body (an elastic heat-conducting half-space) with an internal arbitrarily placed crack of length $2l$ is heated on a section with a surface width of 2a by a heat flux of intensity $q(x)$. We believe that outside the heating area, the surface of the half-space is thermally insulated and free from external forces, and the edges of the crack are thermally insulated, not loaded, and not in contact with each other.

We refer to the half-space to the rectangular coordinate system Oxy so that the axis Ox is located along the edge of the half-plane (axial section of the half-space), and the center of the crack is at a point $O_1(0,-h)$ on the axis Oy. The positive direction of the axis Oy coincides with the external normal to the free surface of the half-space at the point O. In addition, let us introduce a local coordinate system $O_1x_1y_1$, the axis of which is directed along the section line and forms an angle α with the axis Ox (Fig. 1).

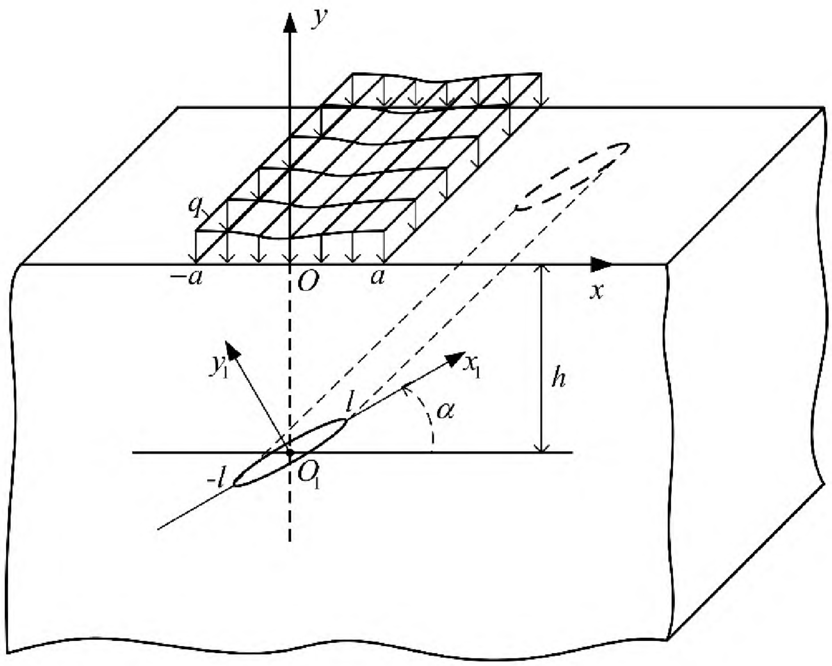

Fig. 1. Scheme of local heating by non-uniform heat flux
of a half-space with an internal near-surface crack

The relationship between these coordinate systems is as follows: $z_1 = (z + ih)\exp(-i\alpha)$, $z = x + iy$, $z_1 = x_1 + iy_1$.

Review of Modern Information Sources on the Subject of the Paper

Previously, cases of a half-space with one internal [7, 8] or edge cut [9], as well as corresponding problems for a periodic system of such cracks [10, 11] were considered. It was assumed that the heating of the half-space by the heat flow takes place on a section of its surface that is finite in width. The intensity of heat flow distribution was constant, and uniform over the entire heating area. This case of thermal load distribution corresponds to the complex modes of operation of friction nodes when the pressure on the contact area is equalized. However, most often in engineering calculations, an elliptical (Hertzian) pressure distribution, and the corresponding heat flow intensity are used [1]. Additional consideration of the microgeometry of bodies corresponds to the quasi-Hertzian (for example, parabolic) contact pressure distribution or a combination of Hertzian and oscillating distributions [12]. Therefore, it is important to analyze and compare the results of the calculations of the stress intensity coefficients near the tips of the internal crack, which correspond to the local heating of the surface of a semi-infinite body by a heat flow that has a different form of the intensity distribution.

The problem of thermal conductivity

The temperature $T(x,y)$ of the half-space with a crack is written as the sum of the temperature $T_0(x,y)$ of the half-space without a crack and the temperature disturbed due to the presence of a crack $T^*(x,y)$: $T(x,y)=T_0(x,y)+T^*(x,y)$. The stationary temperature of the half-space without a crack $T_0(x,y)$ from the heating of the area $|x'|\le a$ of the surface $y'=0$ by the heat flux q is determined from the relation [7]

$$T_0(x,y)=-\frac{1}{\pi\lambda}\int_{-c}^{c} q(\zeta)\ln\sqrt{(x-\zeta)^2+y^2}d\zeta+C, \quad (1)$$

$|x|<\infty, |y|<0$, C is an arbitrary constant, λ is the thermal conductivity coefficient.

The perturbed temperature $T^*(x,y)$ can be found from the solution of the singular integral equation (SIE) [7] concerning the unknown function $\gamma'(t_1)$

$$\frac{1}{\pi}\int_{-l}^{l}\left[\frac{1}{t_1-x_1}+L(t_1,x_1)\right]\gamma'(t_1)dt_1=F(x_1),\ |x_1|<l, \quad (2)$$

where

$$L(t_1,x_1)=\mathrm{Re}\left[\exp(i\alpha)/\left(\zeta_1-\overline{\eta_1}\right)\right],$$

$$\zeta_1=t_1\exp(i\alpha)-ih,\quad \eta_1=x_1\exp(i\alpha)-ih;$$

$$F(x_1)=-\left.\frac{\partial T_0(x,y)}{\partial y_1}\right|_{y_1=0}=\sin\alpha\left.\frac{\partial T_0(x,y)}{\partial x}\right|_{y_1=0}-\cos\alpha\left.\frac{\partial T_0(x,y)}{\partial y}\right|_{y_1=0}. \quad (3)$$

Integral equation (2) for the arbitrary right-hand side has a unique solution under the additional condition of temperature $T^*(x,y)$ continuity while by passing the crack contour in the form

$$\int_{-l}^{l}\gamma'(t_1)dt_1=0. \quad (4)$$

The problem of thermoelasticity

A stationary temperature field $T_0(x,y)$, in the absence of internal heat sources, does not cause stress in a single-connected half-space without a crack. Then we will find the stresses initiated by the perturbed temperature $T^*(x,y)$. The singular integral equation of the corresponding problem of thermoelasticity with respect to the unknown function $Q(t_1)$ has the form [7]

$$\int_{-l}^{l}\left[R(t_1,x_1)Q(t_1)+S(t_1,x_1)\overline{Q(t_1)}\right]dt_1=0,\ |x_1|<l. \tag{5}$$

where

$$S(t_1,x_1)=\frac{\exp(i\alpha)}{2}\left[\frac{\eta_1-\overline{\eta}_1}{(\zeta_1-\overline{\eta}_1)^2}+\frac{1}{\overline{\zeta}_1-\eta_1}-\frac{\exp(-2i\alpha)(\zeta_1-\eta_1)}{(\overline{\zeta}_1-\eta_1)^2}\right],$$

$$R(t_1,x_1)=\frac{1}{t_1-x_1}+\frac{\exp(i\alpha)}{2}\times$$

$$\times\left\{\begin{array}{l}\dfrac{1}{\zeta_1-\overline{\eta}_1}+\dfrac{\exp(-2i\alpha)}{\overline{\zeta}_1-\eta_1}+\\ +(\overline{\eta}_1-\eta_1)\left[\dfrac{1+\exp(-2i\alpha)}{(\overline{\zeta}_1-\eta_1)^2}-\dfrac{2\exp(-2i\alpha)(\zeta_1-\eta_1)}{(\overline{\zeta}_1-\eta_1)^3}\right]\end{array}\right\},$$

$$Q(t_1)=g'(t_1)+2i\beta_t\gamma(t)/(1+\chi),\quad \beta_t=\alpha_t E,\ \chi=3-4\mu. \tag{6}$$

where $g'(t_1)$ is the derivative of the unknown jump of displacements when crossing the crack line; α_t, E, μ are coefficient of linear thermal expansion, Young's modulus, and Poisson's ratio, respectively.

The unambiguity of movements while bypassing the contour of the crack is ensured by the fulfillment of the condition $\int_{L_1} g'(t_1)dt_1=0$ which, taking into account equality (6), will take the form

$$\int_{-l}^{l}Q(t_1)dt_1=-\frac{2i\beta^t}{1+\chi}\int_{-l}^{l}t_1\gamma'(t_1)dt_1. \tag{7}$$

Equation (5) has a unique solution under condition (7).

Algorithm for solving the thermoelasticity problem: first, we find a function $\gamma'(t_1)$ from the system of singular integral equations of the thermal conductivity problem (2), (4) and substitute it into the system of equations of the thermoelasticity problem (5), (7) to determine the function $Q(t_1)$. Then we calculate the stress intensity factors K_I, K_{II}, which are real numbers characterizing the stress-strain state around the crack tips, according to the formula [7]

$$K_I^{\pm}-iK_{II}^{\pm}=\pm\lim_{t_1\to\pm l}\sqrt{2\pi|t_1\mp l|}Q(t_1). \tag{8}$$

Here, the indices «–» refer to the beginning of the crack ($t_k=l_k^-$), and «+» to its end ($t_k=l_k^+$).

Different types of heat flow intensity distribution

The temperature $T_0(x,y)$ in the half-space without a crack, which is heated by the local heat flux of intensity $q(x)$, is written in the form (1). In the general case of heat flow intensity $q(x)$, for integration in relation (1), a technique based on the use of piecewise-constant approximation can be used [13]

$$q(t)=\sum_{i=1}^{n} q(t_i^*)\phi_i(t), \tag{9}$$

where

$$t_i^*=\frac{t_{i-1}+t_i}{2},\quad \phi_i(t)=\begin{cases}1, t\in[t_{i-1},t_i],\\ 0, t\in[t_{i-1},t_i],\end{cases}\quad t_i=\left(i-\frac{n}{2}\right)\delta_t,\ i=0,1,...,n;\qquad \delta_t=\frac{2a}{n}.$$

Taking into account the approximation (9), equality (1) can be written in the form

$$T_0(x,y)=-\frac{1}{\pi\lambda}\sum_{i=1}^{n} q(t_i^*)\int_{t_{i-1}}^{t_i}\ln\sqrt{(x-t)^2+y^2}dt+C,\ |x|<\infty,\ y<0. \tag{10}$$

After integrating in relation (10) and determining the partial derivatives of the temperature $T_0(x,y)$, we obtain expressions for substituting them into the right-hand side of the thermal conductivity equation (2)

$$\left.\frac{\partial T_o(x,y)}{\partial x}\right|_{y_1=0}=\frac{1}{\pi\lambda}\sum_{i=1}^{n} q(t_i^*)\ln\sqrt{\frac{(x-t_i)^2+y^2}{(x-t_{i-1})^2+y^2}}, \tag{11}$$

$$\left.\frac{\partial T_o(x,y)}{\partial y}\right|_{y_1=0}=\frac{1}{\pi\lambda}\sum_{i=1}^{n} q(t_i^*)\left(\operatorname{arctg}\frac{x-t_i}{y}-\operatorname{arctg}\frac{x-t_{i-1}}{y}\right),$$

where $x=x_1\cos\alpha$, $y=x_1\sin\alpha-h$.

Taking into account the linear dependence of the intensity of the frictional heat flow $q(x)$ on the contact pressure, we have

$$q(x)=q_0\varepsilon\left(1-\frac{x^2}{a^2}\right)^{\kappa},\ -a\le x\le a, \tag{12}$$

where q_0 is the maximum value of the heat flow intensity. Here parameter $0\le\kappa\le 2{,}5$ depends on the cleanliness of the surface treatment of the body, the multiplier ε is determined from the condition of equality in the considered distributions of the total amount of heat spent on heating the half-space. Then from equality (12) for $\kappa=0$, $\varepsilon=1$ we will get a uniform distribution of heat flow intensity, for $\kappa=0{,}5$; $\varepsilon=4/\pi$ we will get an elliptical (Hertzian) distribution, for $\kappa=1$; $\varepsilon=1{,}5$ parabolic heat flow distribution. In the latter case, exact expressions for partial derivatives of temperature $T_0(x,y)$ are also obtained

$$\left.\frac{\partial T_o(x,y)}{\partial x}\right|_{y_1=0}=-\frac{1{,}5q_0}{\pi\lambda}\left[\frac{x^2-y^2-a^2}{2}\ln\frac{(x-a)^2+y^2}{(x+a)^2+y^2}+\right.$$
$$\left.+2xy\left(\operatorname{arctg}\frac{x-a}{y}-\operatorname{arctg}\frac{x+a}{y}\right)+2ax\right]; \tag{13}$$

$$\left.\frac{\partial T_o(x,y)}{\partial y}\right|_{y_1=0}=-\frac{1{,}5q_0}{\pi\lambda}\left[\left(x^2-y^2-a^2\right)\left(\operatorname{arctg}\frac{x-a}{y}-\operatorname{arctg}\frac{x+a}{y}\right)-\right.$$
$$\left.-xy\ln\frac{(x-a)^2+y^2}{(x+a)^2+y^2}-2ay\right].$$

In the paper [12] it was established that during the friction of a railway wheel and a rail, the intensity of the heat flow can be modeled by the sum of elliptical and oscillating distributions

$$q(x)=q_0\left(\frac{4}{\pi}\sqrt{1-\frac{x^2}{a^2}}-\frac{1}{4}\cos\frac{5\pi x}{a}\right),\ |x|\le a\,. \tag{14}$$

Distributions of the intensity of the dimensionless frictional heat flow q/q_0 obtained according to (12) and (14) are shown in Fig. 2.

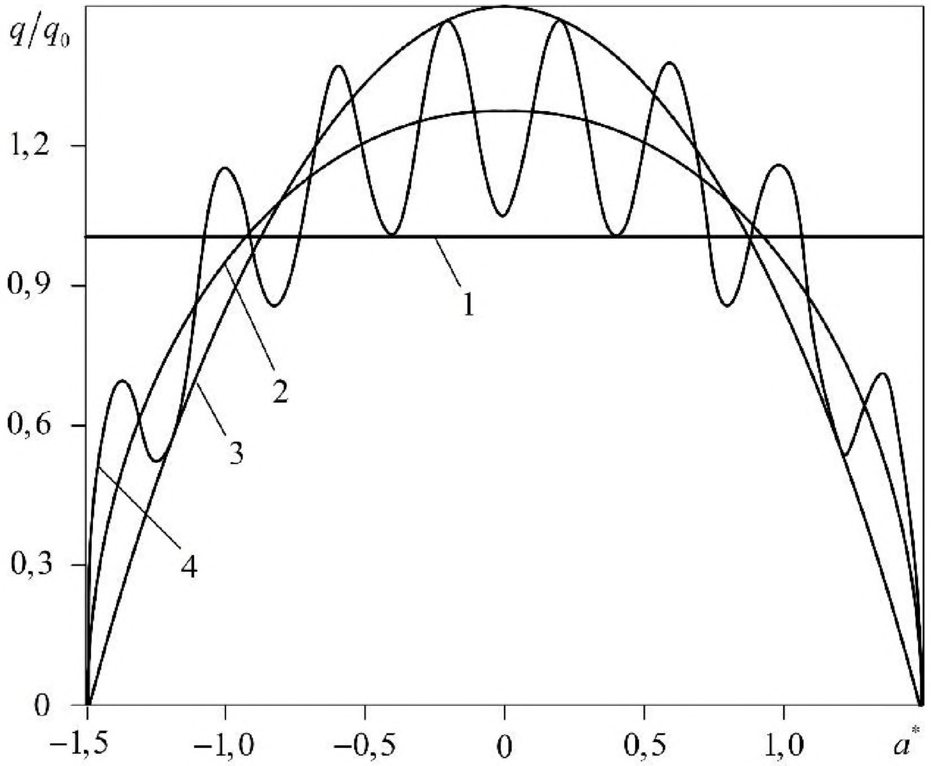

Fig. 2. Dependence of the distribution of the dimensionless heat flow intensity q/q_0 on the parameter $a^*=a/h$: curve 1 corresponds to a uniform distribution; 2 – elliptical; 3 – parabolic; 4 – oscillating

Calculation results

The numerical solution of the SIE systems of the first kind (2), (4) and (5), (7) with a special Cauchy kernel in the class of functions having a root singularity at the ends $x_1=\pm l$ of the integration interval was obtained by the numerical method of mechanical quadrature [14]. The independent input parameters of the problem are the dimensionless half-width of the heating area $a^*=a/h$ and the half-length of the crack $l^*=l/h$. To achieve sufficient accuracy in calculations, it is necessary to take at least 20 collocation points.

Graphs for dimensionless coefficients of stress intensity $k_j^{\pm *}=k_j^{\pm}2\pi\lambda/q_0\beta' l\sqrt{\pi l}$, $j=1,2$ ($k_1^{\pm}=K_I^{\pm}$, $k_2^{\pm}=K_{II}^{\pm}$) using (8), depending on the angle of inclination of the crack for different types of heat flow intensity distribution, shown in Figs. 3–5.

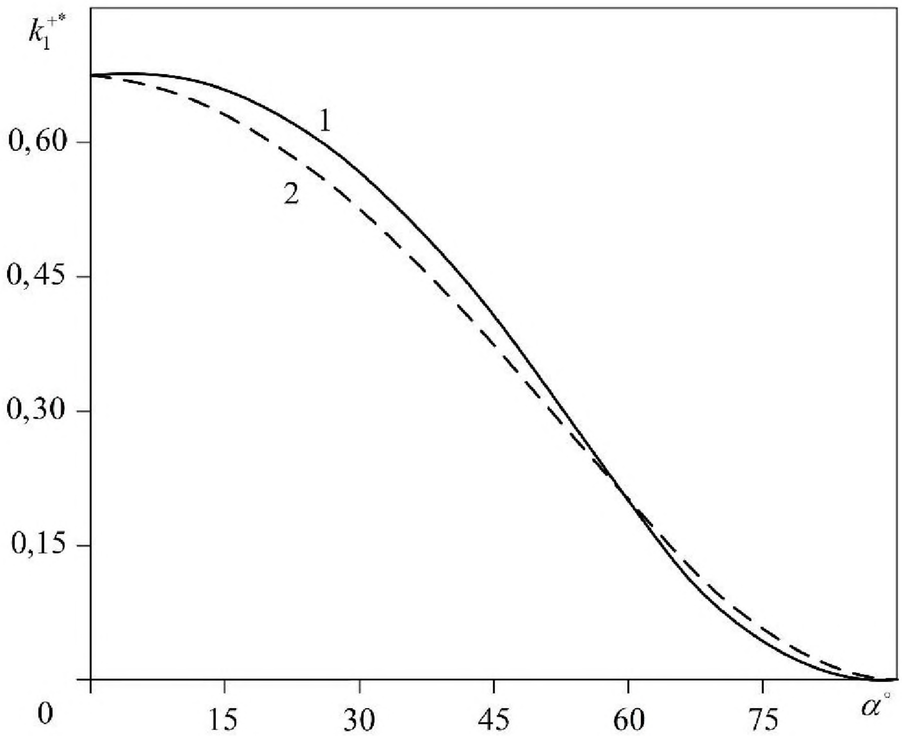

Fig. 3. Dependence of SIF k_1^{+*} on the angle of inclination of the crack α in the case of parabolic heat flow distribution for $a^* = 1.5$; $l^* = l / h = 0.9$: curve 1 – exact solution; 2 – approximate solution

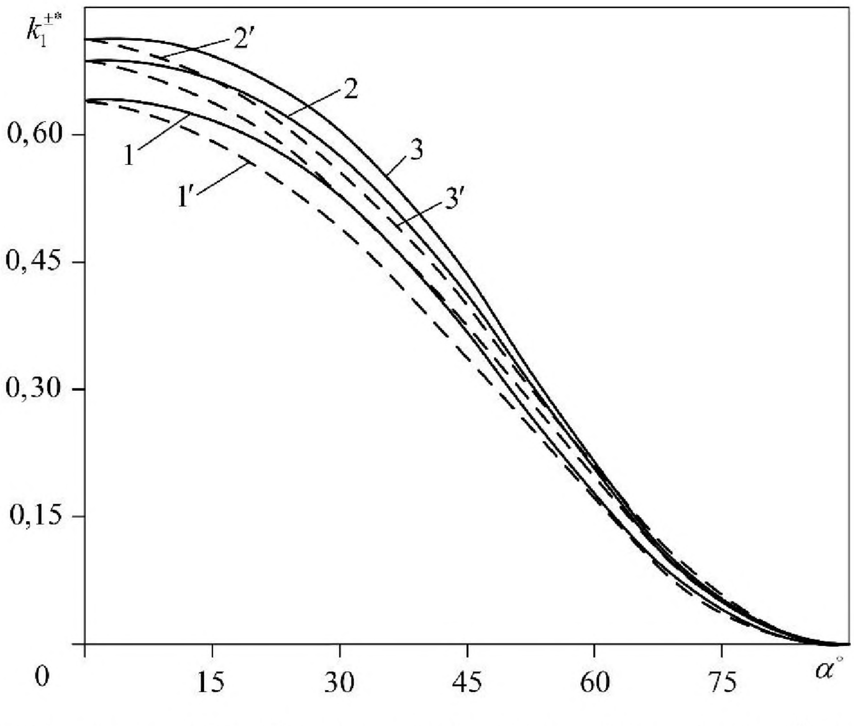

Fig. 4. Dependences of SIF $k_1^{\pm *}$ on the angle of inclination of the crack α for $a^* = 1.5$; $l^* = 0.9$ and different distributions of heat flow intensity (k_1^{+*} – solid curves, k_1^{-*} – dashed curves). Curves 1, 1' – uniform distribution; 2, 2' – elliptical; 3, 3' – parabolic

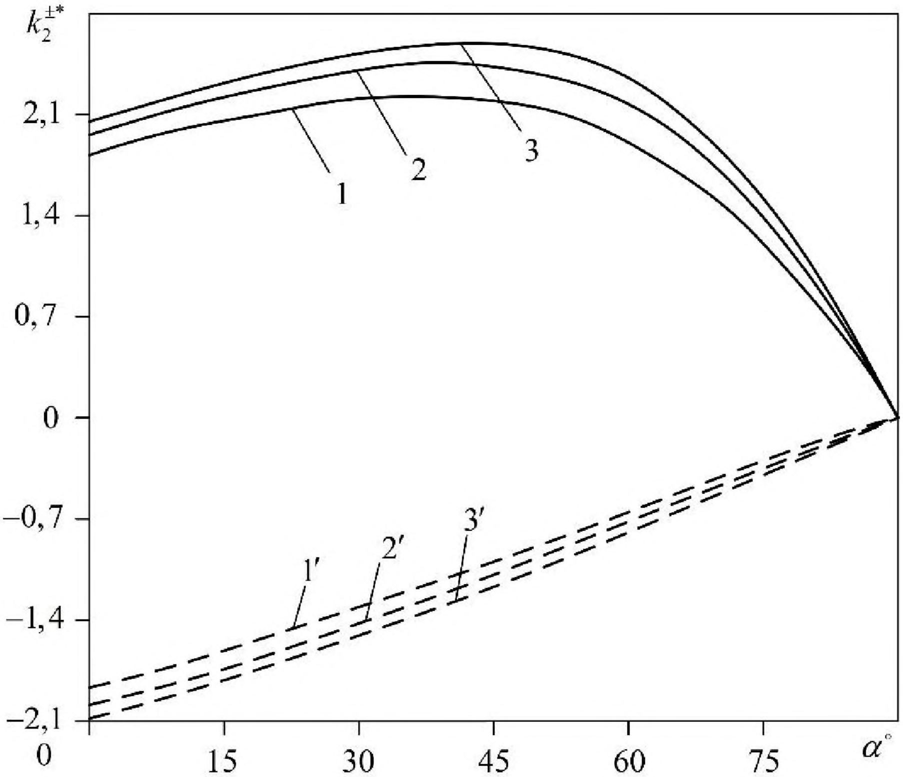

Fig. 5. Dependences of SIF $k_2^{\pm *}$ on the angle of inclination of the crack α for $a^* = 1.5$; $l^* = 0.9$ and different distributions of heat flow intensity (k_1^{+*} – solid curves, k_1^{-*} – dashed curves). Curves 1, 1' – uniform distribution; 2, 2' – elliptical; 3, 3' – parabolic

Solid curves correspond to SIF k_1^{+*}, k_2^{+*} at the top of the crack $x = l$, and the dashed lines k_1^{-*}, k_2^{-*} correspond to the top of the crack $x = -l$. The analysis of the numerical results for the stress intensity coefficients allows us to draw the following conclusions: when the crack rotates from a position perpendicular to the surface of the body to a position of the crack parallel to the surface of the body, the stress intensity coefficient $k_1^{\pm *}$ increases from zero to a maximum value at both tips of the crack. With the above-mentioned rotation of the crack, the intensity coefficient stress $k_2^{\pm *}$ takes a minimum zero value when the crack is located perpendicular to the surface and a maximum value when it is at an angle of 45 degrees to the surface of the body for different forms of heat flow intensity.

Conclusions

1. It was established that, despite the 15 % difference in the maximum values $q(x)$ for the Hertzian and oscillating heat flow intensity distributions (Fig. 2), the corresponding results for the stress intensity coefficients differ by only 1 %.

2. A comparison of the SIF k_1^{+*} values obtained for the parabolic distribution of heat flow intensity using the exact expressions for the derivatives (11) and their approximate values (9) shows a fairly good coincidence of the results even with a small number $(n = 4)$ of points of division of the integration interval $[-a, a]$ (Fig. 3).

3. An increase in the unevenness of the heat flow intensity distribution (increasing parameter k in formula (12) causes a slight increase in both normal (Fig. 4) and tangential (Fig. 5) stresses at the crack tips.

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