MATHEMATICAL MODELING AND COMPUTING, Vol. 10, No. 2, pp. 421–434 (2023)

Nonlinear dynamics of kinetic fluctuations and quasi-linear relaxation in plasma

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(Received 3 January 2023; Revised 12 April 2023; Accepted 20 April 2023)

We propose an approximation of pair correlations for solving the equations of the kinetic theory of long-wave (or large-scale) fluctuations in gaseous media. The basic ones are the general nonlinear equations of the large-scale fluctuations theory at the kinetic stage of system evolution, derived from the first principles of statistical mechanics. We show that based on the equations of the long-wave fluctuations kinetics in the case of weak interaction between particles, in the approximation of pair fluctuations it is possible to reproduce the main results of the quasi-linear theory of plasma. Thus, the well-known quasi-linear theory of plasma is provided with a first-principle justification.

Keywords: *distribution function; correlation functions; long-wave fluctuations; kinetic theory of fluctuations; nonlinear dynamic equations; approximation of pair correlations; quasi-linear relaxation in plasma.*

2010 MSC: 82C40 **DOI:** 10.23939/mmc2023.02.421

1. Introduction

Traditionally it is accepted that in general approaches for describing kinetic processes in gaseous media can be mostly divided into two large subgroups (see, for example, [1]). The presence of one subgroup is associated with the possibility of assuming the interaction between particles of the gaseous medium to be small (while not applying such restrictions on the density of particles in the system). The existence of the second-mentioned subgroup is caused by the possibility of assuming the density of particles to be small and the interaction between them to be arbitrary, if only not so strong, that the bound states of particles could arise.

In the first mentioned case, the basic kinetic equation with one-particle distribution function $f(\mathbf{x},\mathbf{p},t)$ by coordinates $\mathbf{x}$ and impulses $\mathbf{p}$ at time t is so-called Fokker–Planck equation:

$$\frac{\partial f(x,t)}{\partial t}+\frac{\mathbf{p}}{m}\frac{\partial f(x,t)}{\partial \mathbf{x}}-\frac{\partial U(x;f)}{\partial \mathbf{x}}\frac{\partial f(x,t)}{\partial \mathbf{p}}=L(x;f), \tag{1}$$

where m is the mass of a particle and the notation $x\equiv(\mathbf{x},\mathbf{p})$ is introduced. The quantities $U(x;f)$ and $L(x;f)$ in (1), which are called the self-consistent (or mean) field and the collision integral, respectively, are determined by the formulas:

$$U(x;f)\equiv\int \mathrm{d}^3x' V(\mathbf{x}-\mathbf{x}')\int \mathrm{d}^3p' f(\mathbf{x}',\mathbf{p}',t), \quad L(x;f)=-\frac{\partial}{\partial p_i}J_i(x;f). \tag{2}$$

It should be noted that the value $V(\mathbf{x}-\mathbf{x}')$ in expression (2) is the potential energy of the pair interaction between particles located at the points $\mathbf{x}$, $\mathbf{x}'$. Let us also pay attention to the fact that the collision integral $L(x;f)$ in Fokker–Planck kinetic equation (1) has the form of divergence in the momentum space. The explicit form of the functional $J_i(x;f)$ in its simplest form is determined by expression [1]:

$$J_i(x;f) = C \int \mathrm{d}^3p' |\mathbf{p}-\mathbf{p}'|^{-3} \left((\mathbf{p}-\mathbf{p}')^2\delta_{ik} - (\mathbf{p}-\mathbf{p}')_i(\mathbf{p}-\mathbf{p}')_j\right) \times \left(\frac{\partial f(x)}{\partial p_k} f(x') - \frac{\partial f(x')}{\partial p'_k} f(x)\right)\Big|_{\mathbf{x}'=\mathbf{x}}, \quad (3)$$

where

$$C = \frac{m}{8\pi}\int_0^\infty \mathrm{d}q\, q^3 V_q^2, \quad V_q = \int \mathrm{d}^3x\, V(\mathbf{x})e^{-i\mathbf{q}\mathbf{x}}.$$

As for the second subgroup of approaches for describing kinetic processes in gaseous media, it is based on using Boltzmann kinetic equation for the one-particle distribution function $f(\mathbf{x},\mathbf{p},t)$:

$$\frac{\partial f(x_1,t)}{\partial t} + \frac{\mathbf{p}_1}{m}\frac{\partial f(x_1,t)}{\partial \mathbf{x}_1} = \int \mathrm{d}^3p_2 \int \mathrm{d}\Omega |\mathbf{p}_2-\mathbf{p}_1| \sigma(\vartheta, |\mathbf{p}_2-\mathbf{p}_1|) \left\{ f(\mathbf{x}_1,\mathbf{p}'_1,t) f(\mathbf{x}_1,\mathbf{p}'_2,t) - f(\mathbf{x}_1,\mathbf{p}_1,t) f(\mathbf{x}_1,\mathbf{p}_2,t)\right\} \equiv L(x_1;f). \quad (4)$$

As can be seen, there is no mean field in this equation, and the collision integral $L(x_1;f)$ has a more complex structure than (2), (3). The main characteristic of the collision integral is the differential cross section $\sigma(\vartheta, |\mathbf{p}_2-\mathbf{p}_1|)$, where ϑ is the angle between vectors $\mathbf{p}_2-\mathbf{p}_1$ and $\mathbf{p}'_2-\mathbf{p}'_1$. Note that the element of the solid angle $\mathrm{d}\Omega$, along which the integration in (4) is carried out, is also associated with the same angle, see [1–3] for details.

However, we should remind that the kinetic equations (1), (4), that are used to describe the non-equilibrium processes in various gaseous media were obtained, in fact, by neglecting the long-wave fluctuations in the described systems. In other words, when the mentioned equations were derived, we substantially used the assumption of rapid attenuation of correlations while increasing the distance between the i-th and the j-th particles of the system, $|\mathbf{x}_i-\mathbf{x}_j| \gtrsim r_0$, where r_0 is the characteristic radius of interaction between particles. Meanwhile, when studying the dynamics of long-wave fluctuations, it is necessary to deal with systems with a large, rising in time radius of correlations [4–9], so the assumption of rapid decrease in correlations $|\mathbf{x}_i-\mathbf{x}_j| \gtrsim r_0$, which underlies [1–3], does not perform. Since the impact of long-wave fluctuations on relaxation processes in such systems can be very significant, the question arises as to the method of describing the kinetics and hydrodynamics of long-wave fluctuations. In order to develop such a technique, the approaches used in [1–3] should be significantly modified. This modification [8] is related to the following circumstances. Approaches [1–3] are based on the idea of Bogolyubov about the hierarchy of system relaxation times. According to it, at the kinetic stage of system evolution many-particle distribution functions $f_S(x_1,\ldots,x_S;t)$ depend on time t only through the one-particle distribution function $f(x,t)$ dependency on time (the so-called functional hypothesis):

$$f_S(x_1,\ldots,x_S;t) \underset{t\gg\tau_0}{\longrightarrow} f_S(x_1,\ldots,x_S;f(x',t)), \quad (5)$$

where τ_0 is the so-called time of chaotization, with order of magnitude comparable with the time of one collision [1]. However, this assumption (5) becomes unfair in the presence of long-wave fluctuations in the system [8].

2. Smoothed many-particle distribution functions

In order to formulate a functional hypothesis modified for the presence of long fluctuations in the system, we consider the smoothed S-particle distribution functions $\tilde{f}_S(x_1,\ldots,x_S;t)$, that arise from the ordinary many-particle distribution functions $f_S(x_1,\ldots,x_S;t)$ due to the transition to the asymptotic region $|\mathbf{x}_i-\mathbf{x}_j| \gg r_0$,

$$f_S(x_1,\ldots,x_S;t) \underset{|\mathbf{x}_i-\mathbf{x}_j|\gg r_0}{\longrightarrow} \tilde{f}_S(x_1,\ldots,x_S;t) \equiv \mathcal{P} f_S(x_1,\ldots,x_S;t) \quad (6)$$

($\mathcal{P}$ is the symbol for the smoothing operation). It is clear that due to such a limit transition, a significant simplification must occur in the description of the system.

Let us go into more detail on the concept of the smoothing operation. Even if the initial many-particle distribution functions were smooth (on the r_0 scale) functions $\mathbf{x}_i$, $i=1,2,\ldots,S$, as a result of the time evolution, they would acquire a complex irregular character at $|\mathbf{x}_i - \mathbf{x}_j| \lesssim r_0$ $(i, j = 1, 2, \ldots, S)$, which reflects the irregular on the r_0 scale properties of the potential energy of the S particles interaction. The character of this irregularity can be illustrated by the example of the function $\varphi(r)$ (here, r plays the role of $|\mathbf{x}_i - \mathbf{x}_j|$), which has two spatial scales of change, $\varphi(r) = \varphi\left(\frac{r}{r_0}, \frac{r}{L}\right)$ (r_0 is the characteristic microscopic scale of change over small distances, $L \gg r_0$ is the characteristic macroscopic scale of change over long distances). Then the smoothing operation of the function $\varphi(r)$ is defined by the expression:

$$\tilde{\varphi}(r) = \mathcal{P}\varphi(r) \equiv \varphi\left(\infty, \frac{r}{L}\right), \quad \mathcal{P}^2 = \mathcal{P}. \tag{7}$$

It is clear that the smoothing operation satisfies the properties:

$$\mathcal{P}\varphi_1(r)\varphi_2(r) = \tilde{\varphi}_1(r)\tilde{\varphi}_2(r), \quad \mathcal{P}\left\{\varphi_1(r) + \varphi_2(r)\right\} = \tilde{\varphi}_1(r) + \tilde{\varphi}_2(r), \tag{8}$$
$$\mathcal{P}\frac{\partial \varphi(r)}{\partial r} = \frac{\partial \tilde{\varphi}(r)}{\partial r}.$$

If $\varphi(r)$ is a smooth function of r (i.e., a function that does not contain the small scale r_0), then

$$\tilde{\varphi}(r) = \varphi(r), \quad \varphi(r) \equiv \varphi(r/L) \tag{9}$$

whence it follows that $\mathcal{P}^2 = \mathcal{P}$. In particular, if $\varphi(r) = c = \text{const}$, then $\tilde{c} = c$. If the function $\psi(r) \equiv \psi(r/r_0)$ is a function that does not contain the scale L, i.e., $\psi(r) = 0$ for $r > r_0$, then

$$\mathcal{P}\psi(r) = 0. \tag{10}$$

It follows that

$$\mathcal{P}\psi(r)\tilde{\varphi}(r) = 0, \quad \mathcal{P}\int \mathrm{d}r'\psi(r - r')\tilde{\varphi}(r) = \int \mathrm{d}r'\psi(r - r')\tilde{\varphi}(r). \tag{11}$$

In particular, since the many-particle correlation functions of the equilibrium systems $g^0_S(x_1, \ldots, x_S)$ are concentrated at $|\mathbf{x}_i - \mathbf{x}_j| \lesssim r_0$, then $\mathcal{P}g^0_S(x_1, \ldots, x_S) = 0$. Formulas (6)–(11) fully determine the operation of smoothing the small-scale fluctuations and specify a set of microscopic variables (namely, the smoothed many-particle distribution functions $\tilde{f}_S(x_1, \ldots, x_S; t)$) at the fluctuation-kinetic stage of system evolution.

Therefore, according to Bogolyubov–Peletminskii reduced description method [1,2,8], it is assumed that at times $t \gg \tau_0$ ($\tau_0 \sim r_0/\nu$ is the time of chaotization, ν is the average velocity of the particle) the state of the system is completely described by the smoothed many-particle distribution functions $\tilde{f}_S(x_1, \ldots, x_S; t)$ (see (6)). It means that the exact many-particle distribution functions $f_S(x_1, \ldots, x_S; t)$ at $t \gg \tau_0$ will depend on time and initial many-particle distribution functions only through the smoothed many-particle distribution functions (the functional hypothesis, cf. (5)):

$$f_S(x_1, \ldots, x_S; t) \underset{t \gg \tau_0}{\longrightarrow} f_S\big(x_1, \ldots, x_S; \tilde{f}_1(t), \tilde{f}_2(t), \ldots\big). \tag{12}$$

Thus, although the exact distribution functions $f_S(x_1, \ldots, x_S; t)$ at times $t \lesssim \tau_0$ depend, generally speaking, on the initial many-particle distribution functions $f_{S'}(x_1, \ldots, x_{S'}; 0)$, this dependence is simplified for times much greater than τ_0 and is contained only in functions $\tilde{f}_1(t), \tilde{f}_2(t), \ldots$, when the functions f_S become functionals. In this sense, functionals (12) are universal and do not depend on the nature of the initial conditions for many-particle distribution functions (see [8]). We should note that according to (6), functionals (12) must satisfy the following condition:

$$f_S\big(x_1, \ldots, x_S; \tilde{f}_1(t), \tilde{f}_2(t), \ldots\big) \underset{|\mathbf{x}_i - \mathbf{x}_j| \gg r_0}{\longrightarrow} \mathcal{P}f_S\big(x_1, \ldots, x_S; \tilde{f}_1(t), \tilde{f}_2(t), \ldots\big) \equiv \tilde{f}_S(x_1, \ldots, x_S; t), \tag{13}$$

The further task of constructing the kinetic theory of long-wave fluctuations consists in deriving equations for smoothed distribution functions $\tilde{f}_S(x_1, \ldots, x_S; t)$. However, we will not describe here the actual procedure for obtaining such equations, referring the reader for details to works [8, 9]. We will present only the final result, namely the form of equations of fluctuation kinetics in two cases — for systems with a weak interaction between particles and for systems with a low density of particles and an arbitrary interaction between them (however, we assume that the bound states of particles are not formed during such an interaction). Note that the derivation of these equations significantly uses the principle of spatial weakening correlations for smoothed many-particle distribution functions, written as follows:

$$\tilde{f}_S(x_1,\ldots,x_S;t) \underset{|\mathbf{y}-\mathbf{z}|\to\infty}{\longrightarrow} \tilde{f}_{S'}(y_1,\ldots,y_{S'};t)\tilde{f}_{S''}(z_1,\ldots,z_{S''};t), \quad S'+S''=S. \tag{14}$$

Here relationship (14) should be understood so that the distance between the subgroup of S' particles characterized by the set of coordinates and momentum $y_1,\ldots,y_{S'}$ and the subgroup of S'' particles characterized by the set of coordinates and momentum $z_1,\ldots,z_{S''}$ $(x_1,\ldots,x_S \equiv y_1,\ldots,y_{S'},z_1,\ldots,z_{S''})$ goes to infinity (see [1–3, 8]).

3. General kinetic equations of the theory of long-wave fluctuations

So, the kinetic equations in the second order perturbation theory for the weak interaction between particles are written in the form:

$$\begin{aligned}\frac{\partial}{\partial t}\tilde{f}_S(x_1,\ldots,x_S;t) &+ \sum_{n=1}^{S}\frac{\mathbf{p}_n}{m}\frac{\partial}{\partial \mathbf{x}_n}\tilde{f}_S(x_1,\ldots,x_S;t)\\ &- \sum_{n=1}^{S}\frac{\partial}{\partial \mathbf{p}_n}\int \mathrm{d}x_{S+1}\frac{\partial V(\mathbf{x}_n-\mathbf{x}_{S+1})}{\partial \mathbf{x}_n}\tilde{f}_{S+1}(x_1,\ldots,x_{S+1};t)\\ &= \sum_{n=1}^{S}\int \mathrm{d}x'_n\int \mathrm{d}x'_{S+1}K(x_n;x'_n,x'_{S+1})\tilde{f}_{S+1}(x_1,\ldots,x'_n,\ldots,x_S,x'_{S+1};t),\end{aligned} \tag{15}$$

where the kernel $K(x;x',x'')$ is defined by the expression:

$$\begin{aligned}K(x;x',x'') = C\delta(x'-x'')\frac{\partial}{\partial p_i}\left(\frac{\partial}{\partial p'_j}\delta(x-x') - \delta(x-x')\frac{\partial}{\partial p''_j}\right)&\\ \times |\mathbf{p}-\mathbf{p}''|^{-3}\left\{(\mathbf{p}-\mathbf{p}'')^2\delta_{ij} - (\mathbf{p}-\mathbf{p}'')_i(\mathbf{p}-\mathbf{p}'')_j\right\},&\end{aligned} \tag{16}$$

$$\delta(x-x') \equiv \delta(\mathbf{x}-\mathbf{x}')\delta(\mathbf{p}-\mathbf{p}'),$$

and the constant C is given by the formula (3).

For systems with a low density of particles and an arbitrary interaction between them (if only the bound states of particles are not formed during such an interaction), the equations of the long-wave fluctuations kinetics can be given the following form:

$$\begin{aligned}\frac{\partial}{\partial t}\tilde{f}_S(x_1,\ldots,x_S;t) &+ \sum_{n=1}^{S}\frac{\mathbf{p}_n}{m}\frac{\partial}{\partial \mathbf{x}_n}\tilde{f}_S(x_1,\ldots,x_S;t)\\ &= \sum_{n=1}^{S}\int \mathrm{d}x'_n\int \mathrm{d}x'_{S+1}K(x_n;x'_n,x'_{S+1})\tilde{f}_{S+1}(x_1,\ldots,x'_n,\ldots,x_S,x'_{S+1};t),\end{aligned} \tag{17}$$

where the kernel $K(x_1;x',x'')$ in the zeroth approximation by the gradients of $\tilde{f}_S$ values is given by the expression (see also (4)):

$$\begin{aligned}K(x_1;x',x'') = \delta(\mathbf{x}_1-\mathbf{x}')\delta(\mathbf{x}_1-\mathbf{x}'')\int \mathrm{d}^3p_2\int \mathrm{d}\Omega|\mathbf{p}_2-\mathbf{p}_1|\sigma(\vartheta,|\mathbf{p}_2-\mathbf{p}_1|)&\\ \times\left\{\delta(\mathbf{p}''-\mathbf{p}_1)\delta(\mathbf{p}'-\mathbf{p}_2) - \delta(\mathbf{p}'-\mathbf{p}_1)\delta(\mathbf{p}''-\mathbf{p}_2)\right\}.&\end{aligned} \tag{18}$$

We should note that equations (15), (17) are non-closed equations. In fact, as we can see, we have to deal with infinite chains of equations since the right-hand side of the equation for the function $\tilde{f}_S(t)$ includes the function $\tilde{f}_{S+1}(t)$. If we want to write the equation for the function $\tilde{f}_{S+1}(t)$, then the function $\tilde{f}_{S+2}(t)$ will enter its right-hand side, and so further. Thus, we obtain for smoothed many-particle distribution functions an analogue of the famous Bogolyubov–Born–Green–Yvon–Kirkwood (BBGYK) chain of equations, see, for example, [1, 2].

Note that the kinetic equations of the theory of long-wave fluctuations (i.e., equations for smoothed distribution functions) can also be written as a system of equations for the smoothed one-particle distribution function $\tilde{f}(x;t)$ and smoothed correlation functions $\tilde{g}_S(x_1,\ldots,x_S;t)$. To determine the correlation functions $\tilde{g}_S(x_1,\ldots,x_S;t)$ of arbitrary order S, it is convenient to consider some generating functionals [8]. Let us first consider the generating functional $F(u;\tilde{f})$ of smoothed many-particle distribution functions $\tilde{f}_S(x_1,\ldots,x_S;t)$ defined by the formula:

$$F(u;\tilde{f}) = 1 + \sum_{S=1}^{\infty} \frac{1}{S!} \int \mathrm{d}x_1 \ldots \int \mathrm{d}x_S\, u(x_1)\ldots u(x_S)\tilde{f}_S(x_1,\ldots,x_S;t). \tag{19}$$

The many-particle distribution functions $\tilde{f}_S(x_1,\ldots,x_S;t)$ are found by functional differentiation of the functional $F(u;\tilde{f})$ by the functional argument $u(x)$:

$$\tilde{f}_S(x_1,\ldots,x_S;t) = \left.\frac{\delta^S F(u;\tilde{f})}{\delta u(x_1)\ldots\delta u(x_S)}\right|_{u=0}. \tag{20}$$

We can formulate the principle of correlations spatial weakening (14) in the language of the generating functional $F(u;\tilde{f})$. For this purpose, we choose as a functional argument $u(x)$ the sum of two functions $u_{\mathbf{X}}(x)$ and $u_{\mathbf{Y}}(x)$, the first one is different from zero only for values of $\mathbf{x}$ close to $\mathbf{X}$, and the second one — for values of $\mathbf{x}$ close to $\mathbf{Y}$. Then it follows from (14) that the principle of correlations spatial weakening can be represented in the language of the generating functionals in the following form:

$$F(u_{\mathbf{X}}+u_{\mathbf{Y}};\tilde{f}) \underset{|\mathbf{X}-\mathbf{Y}|\to\infty}{\longrightarrow} F(u_{\mathbf{X}};\tilde{f})F(u_{\mathbf{Y}};\tilde{f}). \tag{21}$$

Let us now consider the functional $G(u;\tilde{g})$ related to the generating functional $F(u;\tilde{f})$ by the formula:

$$F(u;\tilde{f}) = \exp G(u;\tilde{g}). \tag{22}$$

It is easy to see that according to (21)

$$G(u_{\mathbf{X}}+u_{\mathbf{Y}};\tilde{g}) \underset{|\mathbf{X}-\mathbf{Y}|\to\infty}{\longrightarrow} G(u_{\mathbf{X}};\tilde{g}) + G(u_{\mathbf{Y}};\tilde{g}). \tag{23}$$

It follows that the functions $\tilde{g}_S(x_1,\ldots,x_S;t)$, which can be found from the functional

$$G(u;\tilde{g}) = \sum_{S=1}^{\infty} \frac{1}{S!} \int \mathrm{d}x_1 \ldots \int \mathrm{d}x_S u(x_1)\ldots u(x_S)\tilde{g}_S(x_1,\ldots,x_S;t) \tag{24}$$

by its S-fold functional differentiation

$$\tilde{g}_S(x_1,\ldots,x_S;t) = \left.\frac{\delta^S G(u;\tilde{g})}{\delta u(x_1)\ldots\delta u(x_S)}\right|_{u=0}, \tag{25}$$

satisfy the relationship

$$\tilde{g}_S(x_1,\ldots,x_{S'},y_{S'+1},\ldots,y_S;t) \underset{|\mathbf{X}-\mathbf{Y}|\to\infty}{\longrightarrow} 0. \tag{26}$$

Therefore, the functional $G(u;\tilde{g})$ is the generating functional of smoothed correlation functions $\tilde{g}_S(x_1,\ldots,x_S;t)$. From (22), (25) it is clear that

$$\begin{aligned}
\tilde{f}_1(x) &= \tilde{g}_1(x),\\
\tilde{f}_2(x_1,x_2) &= \tilde{f}_1(x_1)\tilde{f}_1(x_2) + \tilde{g}_2(x_1,x_2),\\
\tilde{f}_3(x_1,x_2,x_3) &= \tilde{f}_1(x_1)\tilde{f}_1(x_2)\tilde{f}_1(x_3)\\
&\quad + \tilde{f}_1(x_1)\tilde{g}_2(x_2,x_3) + \tilde{f}_1(x_2)\tilde{g}_2(x_1,x_3) + \tilde{f}_1(x_3)\tilde{g}_2(x_1,x_2) + \tilde{g}(x_1,x_2,x_3),\\
&\ldots.
\end{aligned} \tag{27}$$

Note that we can also write the system of equations of the fluctuation kinetics in terms of the above functional (24). For example, in the case of low density and an arbitrary interaction between particles (without the possibility of formation of bound states of particles), the chain of equations (17) can be written in the form:

$$\frac{\partial G(u;\tilde g)}{\partial t}+\int \mathrm{d}x\, u(x)\frac{\mathbf{p}}{m}\frac{\partial}{\partial \mathbf{x}}\frac{\delta G(u;\tilde g)}{\delta u(x)} \\ =\int \mathrm{d}x\int \mathrm{d}x'\int \mathrm{d}x''\, u(x)K(x;x',x'')\left\{\frac{\delta^2 G(u;\tilde g)}{\delta u(x')\delta u(x'')}+\frac{\delta G(u;\tilde g)}{\delta u(x')}\frac{\delta G(u;\tilde g)}{\delta u(x'')}\right\}, \quad (28)$$

where the kernel $K(x;x',x'')$ is defined by the expression (18). The kinetic equations of the long-wave fluctuations in the second order perturbation theory for the weak interaction between particles (15) acquire in terms of the functional (24) a form similar to (28), although somewhat more cumbersome:

$$\frac{\partial G(u;\tilde g)}{\partial t}+\int \mathrm{d}x\, u(x)\frac{\mathbf{p}}{m}\frac{\partial}{\partial \mathbf{x}}\frac{\delta G(u;\tilde g)}{\delta u(x)} \\ -\int \mathrm{d}x\int \mathrm{d}x'\, u(x)\frac{\mathbf{p}}{m}\frac{\partial V(\mathbf{x}-\mathbf{x}')}{\partial \mathbf{x}}\frac{\partial}{\partial \mathbf{p}}\left\{\frac{\delta^2 G(u;\tilde g)}{\delta u(x)\delta u(x')}+\frac{\delta G(u;\tilde g)}{\delta u(x)}\frac{\delta G(u;\tilde g)}{\delta u(x')}\right\} \\ =\int \mathrm{d}x\int \mathrm{d}x'\int \mathrm{d}x''\, u(x)K(x;x',x'')\left\{\frac{\delta^2 G(u;\tilde g)}{\delta u(x')\delta u(x'')}+\frac{\delta G(u;\tilde g)}{\delta u(x')}\frac{\delta G(u;\tilde g)}{\delta u(x'')}\right\}, \quad (29)$$

where the kernel $K(x;x',x'')$ is defined by the expression (16). We emphasize that in both considered cases, the general equations of the long-wave fluctuations kinetics (28), (29) are totally equivalent to the corresponding equations (15) and (17). However, taking into account the explicit form of the functional $G(u;\tilde g)$ (see (24)–(27)), it is easy to conclude that kinetic equations of the long-wave fluctuations (28), (29) are the system of evolution equations for the one-particle distribution function and all correlation functions of any order. And are significantly nonlinear in contrast to infinite chains of equations (15), (17) for many-particle distribution functions. We should also note that in more general cases, the equations of the long-wave fluctuations kinetics have a much more complicated form than (28), (29), due to the specific form of the collision integrals in (15), (17) (see [8,9] for details).

Let us also notice that the obtained equations (28), (29) allow solution $\tilde g_S=0$, $S\geqslant 2$, $\tilde g_1(t)\equiv \tilde f(t)\neq 0$. At the same time, equations (28), (29) turn into (4), (1), respectively. However, such a solution corresponds to the very specific initial conditions $\tilde g_S(t=0)=0$, $S\geqslant 2$.

In addition, the stationary solutions of equations (28), (29) corresponding to the state of statistical equilibrium have the form

$$\tilde g_S=0,\quad S\geqslant 2,\quad f_1=f_0, \quad (30)$$

where f_0 is the equilibrium one-particle distribution function (e.g., the Maxwell function or the Boltzmann function). The situation (30) is related to the fact that long-wave fluctuations are absent in the state of statistical equilibrium. The short-wave fluctuations in the equilibrium state are determined by the functionals $f_S(x_1,\ldots,x_S;f)$, in which the Maxwell distribution f_0 should be substituted instead of the functional argument $f(x)$. At the same time, the many-particle distribution functions obtained in this way coincide with the Gibbs many-particle distribution functions [2].

As for the above-mentioned possibility of existing of more complex cases, here we can point to at least more complex systems. Indeed, the long-wave fluctuations can significantly affect both the kinetic and hydrodynamic stages of evolution in such complex systems as dissipative or active media [10]. They can play an important role, for example, in the processes of propagation and multiplicity of neutrons in environments, at least of the type of nuclear reactors, where neutron multiplicity processes take place [11]. In this work, we will focus on illustrating the role of the long-wave fluctuations kinetics in plasma. We will show that the processes of the so-called quasi-linear relaxation in plasma are directly related to the evolution of long-wave fluctuations at its kinetic stage.

4. Equation of dynamics of pair kinetic fluctuations

To solve the problem announced at the end of the previous section, we will use the equation of the long-wave fluctuations kinetics in the approximation of weak interaction between particles, see (29). To derive the equations of the quasi-linear approximation in plasma, we will be interested in the evolution of kinetic fluctuations in a fully ionized plasma, provided that particle collisions in plasma are neglected [12, 13]. The last circumstance immediately imposes a certain restriction on the time interval of the quasi-linear approximation existence:

$$\tau_0 \ll t \ll \tau_r, \tag{31}$$

where τ_r is the characteristic time of relaxation determined by the collision integral (see above). The time τ_0 that limits this interval from below is found from the requirement of applying the obtained equations (15) or (29) of the long-wave fluctuations kinetics in the approximation of weak interaction to the description of the fluctuation-kinetic stage of evolution in a collisionless plasma. Indeed, the slow change in time of the smoothed many-particle distribution functions $\tilde{f}_S(x_1, \ldots, x_S; t)$, that satisfy equation (15), in relation to the plasma fluctuating media should indicate the smallness of the characteristic periods of plasma oscillations in comparison to the characteristic time t of the function change $\tilde{f}_S(x_1, \ldots, x_S; t)$, $\omega_{ch} t \gg 1$, where ω_{ch} is the characteristic frequency of plasma oscillations. In a one-component (electron) plasma, the fluctuation processes we intend to study in the future, from the entire spectrum of characteristic frequencies of plasma oscillations (see, e.g., [14] for details) only the plasma frequency remains (it is called the Langmuir frequency) ω_{Le}:

$$\omega_{Le} = \left(\frac{2\pi e^2 n}{m} \right)^{1/2}, \tag{32}$$

where e is the elementary charge (the absolute value of the electron charge), m is the electron mass and $n = \int \mathrm{d}^3 p f_1(\mathbf{p})$ is the electron density in an equilibrium plasma. We recall that when studying processes in a one-component fully ionized plasma, one naturally assumes the presence of an ion component that is much more inert compared to the electronic one, its role in this approximation is reduced to ensuring the condition of quasi-neutrality (see, e.g., [12, 13]). According to the above, the time interval in which the collisionless approximation should be valid is given by the relationship:

$$\omega_{Le}^{-1} \ll t \ll \tau_r. \tag{33}$$

Note that conditions under which the quasi-linear approximation may be valid are explained in detail in [13].

As it turns out, to derive the basic equations of the quasi-linear theory of plasma, it is enough to take into account the presence of only pair correlations. Equations of the long-wave fluctuations kinetics taking into account only the pair correlation functions $\tilde{g}(x_1, x_2; t) \equiv g(x_1, x_2; t)$, according to (15) or (29), can be written in the form:

$$\begin{aligned}
\dot{f}(x,t) &= \left(-\frac{\mathbf{p}}{m}\frac{\partial}{\partial \mathbf{x}} + \frac{\partial U(\mathbf{x})}{\partial \mathbf{x}}\frac{\partial}{\partial \mathbf{p}} \right) f(x,t) + \frac{\partial}{\partial \mathbf{p}} \int \mathrm{d}x' \, \frac{\partial V(\mathbf{x}-\mathbf{x}')}{\partial \mathbf{x}} g(x,x';t), \\
\dot{g}(x_1,x_2;t) &= \left(-\frac{\mathbf{p}_1}{m}\frac{\partial}{\partial \mathbf{x}_1} + \frac{\partial U(\mathbf{x}_1)}{\partial \mathbf{x}_1}\frac{\partial}{\partial \mathbf{p}_1} \right) g(x_1,x_2;t) + \left(-\frac{\mathbf{p}_2}{m}\frac{\partial}{\partial \mathbf{x}_2} + \frac{\partial U(\mathbf{x}_2)}{\partial \mathbf{x}_2}\frac{\partial}{\partial \mathbf{p}_2} \right) g(x_1,x_2;t) \\
&\quad + \frac{\partial f(x_1,t)}{\partial \mathbf{p}_1}\frac{\partial}{\partial \mathbf{x}_1} \int \mathrm{d}x'' V(\mathbf{x}_1 - \mathbf{x}'') g(x'',x_2;t) + \frac{\partial f(x_2,t)}{\partial \mathbf{p}_2}\frac{\partial}{\partial \mathbf{x}_2} \int \mathrm{d}x'' g(x_1,x'';t) V(\mathbf{x}'' - \mathbf{x}_2),
\end{aligned} \tag{34}$$

where $V(\mathbf{x}-\mathbf{x}')$ is the potential energy of Coulomb interaction between the plasma electrons located at the points $\mathbf{x}$ and $\mathbf{x}'$:

$$V(\mathbf{x}-\mathbf{x}') = \frac{e^2}{|\mathbf{x}-\mathbf{x}'|}, \tag{35}$$

and the value $U(\mathbf{x})$ is given by the expression:

$$U(\mathbf{x}) = \int \mathrm{d}x' \, V(\mathbf{x}-\mathbf{x}') f(x'). \tag{36}$$

Please note that in equations (34) and further we omit the subscripts "1" in the one-particle distribution function $f(x,t)$ and "2" in the pair correlation function $g(x,x';t)$. In addition, here and hereafter we omit the "tilde" sign above both of them, which was previously used to indicate the smoothness of these functions in the sense explained above, see (6)–(14). Finally, let us emphasize that the equations (34) are nonlinear ones.

The characteristic size r_0 of the spatial localization of small-scale fluctuations, which plays an important role in determining the smoothing operation (see Section 2), is given by the Debye screening radius r_D:

$$r_0 \sim r_D = \left(\frac{\kappa T}{4\pi e^2 n}\right)^{1/2}, \tag{37}$$

where T is the temperature of plasma electronic component and κ is the Boltzmann constant.

Equation (34) taking into account formulas (35)–(37) is a closed system of equations describing the evolution of a one-component collisionless plasma during the times determined by the relation (33). In the spatially homogeneous case, $f(x,t) \equiv f(\mathbf{p},t)$ the first of the equations (34) can be written in the form:

$$\dot{f}(\mathbf{p},t) = -\frac{\partial}{\partial p_i} I_i(\mathbf{p}), \tag{38}$$

where the electron flux density in the momentum space $I_i(\mathbf{p})$ is given by the expression:

$$I_i(\mathbf{p}) = \frac{i}{(2\pi)^3}\int \mathrm{d}^3k\, k_i V_{-\mathbf{k}} \int \mathrm{d}^3p\, g_{\mathbf{k}}(\mathbf{p},\mathbf{p};t), \tag{39}$$

values $V_{\mathbf{k}}$, $g_{\mathbf{k}}(\mathbf{p},\mathbf{p};t)$ are the Fourier transforms of a potential energy $V(\mathbf{x}-\mathbf{x}')$ and pair correlation function $g(x,x';t)$, respectively:

$$V_{\mathbf{k}} = \int \mathrm{d}\mathbf{x}\, \exp(-i\mathbf{k}\mathbf{x})V(\mathbf{x}) = \frac{4\pi e^2}{k^2}, \quad g_{\mathbf{k}}(\mathbf{p},\mathbf{p};t) = \int \mathrm{d}\mathbf{x}\, \exp(-i\mathbf{k}\mathbf{x}) g(\mathbf{x},\mathbf{p},\mathbf{p};t), \tag{40}$$

moreover, due to the fact that $g(x,x';t) = g(x',x;t)$ (see (25)), for the value $g_{\mathbf{k}}$ the ratio is valid:

$$g^*_{\mathbf{k}}(\mathbf{p},\mathbf{p};t) = g_{-\mathbf{k}}(\mathbf{p},\mathbf{p};t). \tag{41}$$

Note that in formulas (39), (41) we took into account that in the spatially homogeneous case the pair correlation function $g(x,x';t)$ depends only on the difference in coordinates $\mathbf{x}$, $\mathbf{x}'$: $g(x,x';t) \equiv g(\mathbf{x}-\mathbf{x}',\mathbf{p},\mathbf{p}';t)$.

The Fourier component $g_{\mathbf{k}}(\mathbf{p},\mathbf{p}';t)$ in accordance with (34) satisfies the equation:

$$\begin{aligned}\dot{g}_{\mathbf{k}}(\mathbf{p}_1,\mathbf{p}_2;t) = &-i\frac{\mathbf{k}}{m}(\mathbf{p}_1-\mathbf{p}_2)g_{\mathbf{k}}(\mathbf{p}_1,\mathbf{p}_2;t)\\ &+ i\,\mathbf{k}\frac{\partial f(\mathbf{p}_1,t)}{\partial \mathbf{p}_1}V_{\mathbf{k}}\int \mathrm{d}\mathbf{p}'\, g_{\mathbf{k}}(\mathbf{p}',\mathbf{p}_2;t) - i\,\mathbf{k}\frac{\partial f(\mathbf{p}_2,t)}{\partial \mathbf{p}_2}V_{-\mathbf{k}}\int \mathrm{d}\mathbf{p}'\, g_{-\mathbf{k}}(\mathbf{p}',\mathbf{p}_1;t).\end{aligned} \tag{42}$$

5. Solving equations of dynamics of pair kinetic fluctuations and equation of the quasi-linear plasma theory

Equations (41), (42) in combination with (38), (39) should be the starting point when deriving the main equations of the quasi-linear theory of plasma. The next task is to find solutions of equation (42). Following the method of variables separation, we will look for a solution of this equation in the form:

$$g_{\mathbf{k}}(\mathbf{p}_1,\mathbf{p}_2;t) = g_{\mathbf{k}}(\mathbf{p}_1,t)\, g_{-\mathbf{k}}(\mathbf{p}_2,t), \tag{43}$$

while, according to (41), the functions $g_{\mathbf{k}}(\mathbf{p},t)$ and $g_{-\mathbf{k}}(\mathbf{p},t)$ are related to each other by the relationship:

$$g^*_{\mathbf{k}}(\mathbf{p},t) = g_{-\mathbf{k}}(\mathbf{p},t). \tag{44}$$

From (42) follows the equation of motion for the value $g_{\mathbf{k}}(\mathbf{p},t)$:

$$\dot{g}_{\mathbf{k}}(\mathbf{p},t) = -i\,\frac{\mathbf{k}\mathbf{p}}{m}g_{\mathbf{k}}(\mathbf{p},t) + i\,\mathbf{k}\frac{\partial f(\mathbf{p},t)}{\partial \mathbf{p}}\Phi_{\mathbf{k}}(t), \tag{45}$$

where the following notation is introduced:

$$\Phi_{\mathbf{k}}(t) \equiv V_{\mathbf{k}}\int \mathrm{d}\mathbf{p}\, g_{\mathbf{k}}(\mathbf{p},t) \tag{46}$$

(value $\frac{1}{e}\Phi_{\mathbf{k}}(t)$ can be given the meaning of the Fourier transform of the fluctuating field potential, see [12]).

Since we assume that in the state of the system under consideration, the distribution function $f(\mathbf{p},t)$ in the time interval (33) changes slowly over time, we will look for the solution of equation (45) in the form of a decomposition of the function $g_{\mathbf{k}}(\mathbf{p},t)$ into the Fourier integral by time

$$g_{\mathbf{k}}(\mathbf{p},t) = \int_{-\infty}^{+\infty} \mathrm{d}\omega\, \exp(-i\omega t)\, g_{\mathbf{k}}(\mathbf{p},\omega), \tag{47}$$

considering in the main approximation that the function $f(\mathbf{p},t)$ in (45) does not depend on time at all, $f(\mathbf{p},t) \approx f(\mathbf{p})$. In this approximation, we obtain the following equation for the Fourier transform $g_{\mathbf{k}}(\mathbf{p},\omega)$ from (45):

$$-i\left(\omega - \mathbf{k}\frac{\mathbf{p}}{m}\right) g_{\mathbf{k}}(\mathbf{p},\omega) = i\,\mathbf{k}\frac{\partial f(\mathbf{p})}{\partial \mathbf{p}}\Phi_{\mathbf{k}}(\omega), \tag{48}$$

where

$$\Phi_{\mathbf{k}}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}t\, \exp(i\omega t)\Phi_{\mathbf{k}}(t). \tag{49}$$

The solution of (48) is:

$$g_{\mathbf{k}}(\mathbf{p},\omega) = A(\mathbf{k},\mathbf{p})\,\delta\left(\omega - \mathbf{k}\frac{\mathbf{p}}{m}\right) - \frac{1}{\omega - \mathbf{k}\frac{\mathbf{p}}{m} + i0}\mathbf{k}\frac{\partial f(\mathbf{p})}{\partial \mathbf{p}}\Phi_{\mathbf{k}}(\omega), \tag{50}$$

where $A(\mathbf{k},\mathbf{p})$ are the arbitrary functions, being subject to a natural restriction related to the fact that the functions $g(\mathbf{x}-\mathbf{x}',\mathbf{p},\mathbf{p}';t)$ found according to formulas (40), (43), (47), (50) must satisfy all the properties of pair correlation functions. It follows from (44) that functions $A(\mathbf{k},\mathbf{p})$ must satisfy the relationships:

$$A^*(\mathbf{k},\mathbf{p}) = A(-\mathbf{k},\mathbf{p}). \tag{51}$$

Further, the entire admissible set of such functions will be denoted by $A_\mu(\mathbf{k},\mathbf{p})$, where μ is some symbolic discrete or continuous parameter and functions $A(\mathbf{k},\mathbf{p}) \equiv A_\mu(\mathbf{k},\mathbf{p})$ can depend on it (see below, (65)).

The obtained formula (50) taking into account (46), (49) allows us to find the value $\Phi_{\mathbf{k}}(\omega)$ in terms of $A_\mu(\mathbf{k},\mathbf{p})$:

$$\Phi_{\mu,\mathbf{k}}(\omega) = A_\mu(\mathbf{k},\omega)\varepsilon^{-1}(\mathbf{k},\omega), \tag{52}$$

where the notation is introduced:

$$A_\mu(\mathbf{k},\omega) \equiv V_{\mathbf{k}}\int \mathrm{d}\mathbf{p}\, A_\mu(\mathbf{k},\mathbf{p})\,\delta\left(\omega - \mathbf{k}\frac{\mathbf{p}}{m}\right), \quad A^*_\mu(\mathbf{k},\omega) = A_\mu(-\mathbf{k},-\omega). \tag{53}$$

Further substituting (52) into (50), we obtain the following expression for the value $g_{\mathbf{k}}(\mathbf{p},\omega)$:

$$g_{\mu,\mathbf{k}}(\mathbf{p},\omega) = A_\mu(\mathbf{k},\mathbf{p})\,\delta\left(\omega - \mathbf{k}\frac{\mathbf{p}}{m}\right) - \frac{A_\mu(\mathbf{k},\omega)}{\varepsilon(\mathbf{k},\omega)\left(\omega - \mathbf{k}\frac{\mathbf{p}}{m} + i0\right)}\mathbf{k}\frac{\partial f(\mathbf{p})}{\partial \mathbf{p}}\Phi_{\mu,\mathbf{k}}(\omega). \tag{54}$$

Value $\varepsilon(\mathbf{k},\omega)$ in (52), (54) is the complex permittivity of a plasma:

$$\varepsilon(\mathbf{k},\omega) = \varepsilon^*(-\mathbf{k},-\omega) = \varepsilon_1(\mathbf{k},\omega) + i\varepsilon_2(\mathbf{k},\omega) = 1 + V_{\mathbf{k}}\mathbf{k}\int \frac{\mathrm{d}\mathbf{p}}{\omega - \mathbf{k}\frac{\mathbf{p}}{m} + i0}\frac{\partial f(\mathbf{p})}{\partial \mathbf{p}}. \tag{55}$$

Since we study the long-wave fluctuations in a fully ionized plasma, it is natural to assume that $k\,r_D \ll 1$ (considerations about the conditions of application of such an approximation in the quasi-linear theory of plasma are set out in detail in [13]). Within the approximation $k\,r_D \ll 1$ the real $\varepsilon_1(\mathbf{k},\omega)$ and imaginary $\varepsilon_2(\mathbf{k},\omega)$ parts of the complex permittivity of the plasma can be represented as (see also [12–14]):

$$\varepsilon_1(\mathbf{k},\omega) \approx 1 - \frac{\omega_{Le}^2}{\omega^2}, \quad \varepsilon_2(\mathbf{k},\omega) \approx -\pi V_{\mathbf{k}}\int \mathrm{d}\mathbf{p}\,\delta\left(\omega_{Le} - \mathbf{k}\frac{\mathbf{p}}{m}\right)\mathbf{k}\frac{\partial f(\mathbf{p})}{\partial \mathbf{p}}. \tag{56}$$

When deriving expressions (56), one should use the representation (see, e.g., [1]):

$$(z + i0)^{-1} = P\frac{1}{z} - i\pi\delta(z),$$

where the symbol P means that during the further integration (see (55)) the principal value of the integral is taken.

The values $\Phi_{\mu,\mathbf{k}}(t)$, $g_{\mu,\mathbf{k}}(\mathbf{p},t)$ can be found from (52), (47), (54) using the theory of residues, so the question arises of finding the zeros of function $\varepsilon(\mathbf{k},\omega)$, see (55), (56). As usual (see, e.g, [12, 14]), we will find the zeros of function $\varepsilon(\mathbf{k},\omega)$ from the equation

$$\varepsilon(\mathbf{k},\omega_0 - i\gamma_{\mathbf{k}}) = 0, \quad |\omega_0| \gg |\gamma_{\mathbf{k}}|, \tag{57}$$

where ω_0 is the frequency of free oscillations in a plasma and $\gamma_{\mathbf{k}}$ is the decrement (increment) of oscillations. For such fluctuations to exist in general, the strict inequality in (57) must be satisfied. And for this, an assumption is necessary about the smallness of the imaginary part of the permittivity $\varepsilon_2(\mathbf{k},\omega)$ compared to its real part $\varepsilon_1(\mathbf{k},\omega)$. If this assumption holds true, the frequency ω_0 and decrement (increment) $\gamma_{\mathbf{k}}$ are determined by the expressions:

$$\varepsilon_1(\mathbf{k},\omega_0) = 0, \quad \gamma_{\mathbf{k}} \approx \varepsilon_2(\mathbf{k},\omega_0)\left\{\frac{\partial\varepsilon_1(\mathbf{k},\omega)}{\partial\omega}\right\}^{-1}_{\omega=\omega_0}, \tag{58}$$

from where, taking into account (56), we obtain:

$$\omega_0^{(1.2)} = \pm\omega_{Le}, \quad \gamma_{\mathbf{k}}^{(1)} = \gamma_{\mathbf{k}}^{(2)} \equiv \gamma_{\mathbf{k}} = \gamma_{-\mathbf{k}}, \tag{59}$$

$$\gamma_{\mathbf{k}} = \frac{1}{2}\omega_{Le}\varepsilon_2 = -\frac{1}{2}\omega_{Le}\pi V_{\mathbf{k}}\int \mathrm{d}\mathbf{p}\,\delta\left(\omega_{Le} - \mathbf{k}\frac{\mathbf{p}}{m}\right)\mathbf{k}\frac{\partial f(\mathbf{p})}{\partial\mathbf{p}}.$$

It follows from the last formula (59) that the value $\gamma_{\mathbf{k}}$ can be called, with equal reason, both an increment and a decrement, depending on the sign of the derivative $\frac{\partial f}{\partial\varepsilon}$.

According to formulas (52), (54), (59) the result of calculating the integrals

$$\Phi_{\mu,\mathbf{k}}(t) = \int_{-\infty}^{\infty} \mathrm{d}\omega\, \exp(-i\omega t)\,\Phi_{\mu,\mathbf{k}}(\omega), \quad g_{\mu,\mathbf{k}}(\mathbf{p},t) = \int_{-\infty}^{\infty} \mathrm{d}\omega\, \exp(-i\omega t)\, g_{\mu,\mathbf{k}}(\mathbf{p},\omega)$$

can be represented as:

$$\Phi_{\mu,\mathbf{k}}(t) \approx \frac{1}{2}\left\{\Phi^-_{\mu,\mathbf{k}}(t) + \Phi^+_{\mu,\mathbf{k}}(t)\right\}, \tag{60}$$

$$g_{\mu,\mathbf{k}}(\mathbf{p},t) \approx \exp\left(-i\mathbf{k}\frac{\mathbf{p}}{m}t\right)\left\{A_\mu(\mathbf{k},\mathbf{p}) + 2\pi i\frac{A_\mu(\mathbf{k},\mathbf{kp}/m)}{\varepsilon(\mathbf{k},\mathbf{kp}/m)}\mathbf{k}\frac{\partial f(\mathbf{p})}{\partial\mathbf{p}}\right\}$$
$$+ \frac{1}{2}\left\{\Phi^-_{\mu,\mathbf{k}}(t)\frac{1}{\omega_{Le} + \mathbf{k}\frac{\mathbf{p}}{m} - i0} + \Phi^+_{\mu,\mathbf{k}}(t)\frac{1}{\omega_{Le} - \mathbf{k}\frac{\mathbf{p}}{m} - i0}\right\}\mathbf{k}\frac{\partial f(\mathbf{p})}{\partial\mathbf{p}},$$

where the functions $\Phi^\pm_{\mu,\mathbf{k}}(t)$ are defined by the expression

$$\Phi^\pm_{\mu,\mathbf{k}}(t) = \mp 2\pi i\omega_{Le}A_\mu(\mathbf{k},\pm\omega_{Le})\exp(-\gamma_{\mathbf{k}}t \mp i\omega_{Le}t); \tag{61}$$

and, as a result of (44), (53) these functions are related to each other by the relationship:

$$\left\{\Phi^+_{\mu,\mathbf{k}}(t)\right\}^* = \Phi^-_{\mu,-\mathbf{k}}(t). \tag{62}$$

When deriving formulas (60) the smallness of the value $\gamma_{\mathbf{k}}$ compared to ω_{Le} is taken into account, therefore, in (60) the sign of approximate equality is used.

We should note that introducing values $\Phi^\pm_{\mu,\mathbf{k}}(t)$ into consideration is not only convenient but also caused by the fact that in the quasi-particle approach of studying oscillations in plasma the operators of plasmons creation and annihilation can be introduced by formulas similar to formulas (61). For this reason, the value

$$J_{\mathbf{k}}(t) = \sum_\mu \Phi^+_{\mu,\mathbf{k}}(t)\,\Phi^-_{\mu,-\mathbf{k}}(t) = \sum_\mu \left|\Phi^+_{\mu,\mathbf{k}}(t)\right|^2 \tag{63}$$

with accuracy up to numerical factors, which are insignificant due to the relative arbitrariness of the functions $A_\mu(\mathbf{k},\mathbf{p})$ (and, therefore, the functions $A_\mu(\mathbf{k},\omega)$, see (53)) coincides with the distribution by wave vectors $\mathbf{k}$ of the intensity of oscillations in plasma (see, e.g., [12, 13]).

Now let us return to the formulas (60), (61). According to (43), the most general form of solution of (42) is given by the expression:

$$g_{\mathbf{k}}(\mathbf{p}_1,\mathbf{p}_2;t) = \sum_\mu g_{\mu,\mathbf{k}}(\mathbf{p}_1,t)\,g_{\mu,-\mathbf{k}}(\mathbf{p}_2,t). \tag{64}$$

This formula explains the very meaning of introducing the index μ in functions $A_\mu(\mathbf{k},\mathbf{p})$. Namely, at $t=0$, there should be enough functions $A_\mu(\mathbf{k},\mathbf{p})$ to construct an arbitrary initial correlation function

$$g_{\mathbf{k}}(\mathbf{p}_1,\mathbf{p}_2;0)=\sum_\mu g_{\mu,\mathbf{k}}(\mathbf{p}_1,0)\,g_{\mu,-\mathbf{k}}(\mathbf{p}_2,0) \tag{65}$$

with their help. Expression (64), taking into account formulas (60), can be given the meaning of the value $g_{\mathbf{k}}(\mathbf{p}_1,\mathbf{p}_2;t)$ expansion by the so-called Van Kampen modes (see [13] for details). Using (64), as well as formulas (39), (46), we can represent the flux density in the momentum space $I_i(\mathbf{p},t)$ in equation (38) as:

$$I_i(\mathbf{p},t)=\frac{i}{(2\pi)^3}\int d\mathbf{k}\,k_i\sum_\mu g_{\mu,\mathbf{k}}(\mathbf{p},t)\,\Phi_{\mu,-\mathbf{k}}(t), \tag{66}$$

where the values $g_{\mu,\mathbf{k}}(\mathbf{p},t)$, $\Phi_{\mu,\mathbf{k}}(t)$ are determined by formulas (60), (61).

Next, let us make an important remark. According to (33), expressions (60), (61) contain rapidly oscillating terms, causing the appearance of the same terms (64), (66). On the other hand, as already mentioned, the possibility of applying the equations of the long-wave fluctuations kinetics in the quasi-linear plasma theory requires a slow change over time of the smoothed distribution functions, see (33) (this, by the way, made it possible to find a solution of equations (42) in the form (47), (60)). It is obvious that appearance of the rapidly oscillating terms is related to the choice of the method of solving equations (42), (45). Therefore, in further calculations, when finding the explicit form of values $g_{\mathbf{k}}(\mathbf{p}_1,\mathbf{p}_2;t)$, $I_i(\mathbf{p},t)$ taking (60), (61) into account, we must keep only those terms that do not contain rapidly oscillating factors of the type $\exp(\pm i\omega_{Le}t)$. This somewhat artificial technique is equivalent to averaging the expression for the "collision integral" over the characteristic periods of plasma oscillations, which are small compared to the characteristic times of change of the one-particle distribution function, see [12]. Excluding rapid oscillations from the consideration allows us to abandon the assumption of invariance in time of the one-particle distribution function (remember that such an approximation was used in our calculations, starting with (48)). And this means that all the physical values that describe the state of the studied system change slowly over time, and the characteristic scales of change of these values over time are of the order of the characteristic time of change of the one-particle distribution function $f(\mathbf{p},t)$.

Taking into account the above remarks, the basic evolution equations of the studied state of the system acquire a rather simple form. For the convenience of presenting the material, let us write the equation of motion for the one-particle distribution function once again (see (38)):

$$\frac{\partial}{\partial t}f(\mathbf{p},t)=-\frac{\partial}{\partial p_i}I_i(\mathbf{p},t), \tag{67}$$

where the electron flux density in the momentum space $I_i(\mathbf{p})$ according to (63), (66) can now be represented in the form:

$$I_i(\mathbf{p},t)=-D_{ij}(\mathbf{p},t)\frac{\partial f(\mathbf{p},t)}{\partial p_j}. \tag{68}$$

The diffusion coefficient in the momentum space $D_{ij}(\mathbf{p})$ is determined by the expression:

$$D_{ij}(\mathbf{p},t)=\frac{1}{16\pi^2}\int d\mathbf{k}\,k_i\,k_j\,\delta\left(\omega_{Le}-\mathbf{k}\frac{\mathbf{p}}{m}\right)\,J_{\mathbf{k}}(t), \tag{69}$$

where the value $J_{\mathbf{k}}(t)$, which is proportional to the distribution by wave number of the intensity of oscillations, according to (63), (61), is given by the formula:

$$J_{\mathbf{k}}(t)=4\pi^2\omega_{Le}^2\exp\big(-2\gamma_{\mathbf{k}}(t)t\big)\sum_\mu\left|A_\mu(\mathbf{k},\omega_{Le})\right|^2.$$

It is easy to see that, taking into account the smallness of the increment (decrement) $\gamma_{\mathbf{k}}$ determined by the formula (59) and the slowness of its change over time, we can assume that the value $J_{\mathbf{k}}(t)$ satisfies the equation:

$$\frac{\partial}{\partial t}J_{\mathbf{k}}(t)=-2\gamma_{\mathbf{k}}(t)J_{\mathbf{k}}(t). \tag{70}$$

Taking into account formulas (59), (68), (69) the closed system of equations (67), (70), describing the coupled relaxation of particles and relaxation of plasma oscillations, is usually called the equations of the quasi-linear plasma theory or the equations of the quasi-linear approximation. They were published for the first time in English-language literature in [15]. Later, a large number of works appeared both with the justification of the quasi-linear approximation and with the analysis of the conditions of its applicability and observation in the experiment, (see [13, 15–18]).

We should note that, despite the rather ancient history of problems related to the quasi-linear theory, it has not lost its relevance until now, see at least [19–22].

6. Conclusions

So, in this paper, we show how the main results of the so-called quasi-linear plasma theory can be reproduced within the framework of the first principles of classical (non-quantum) statistical mechanics. The demonstrated results become possible thanks to the kinetic equations of the long-wave fluctuations, the derivation of which is based on the fundamental first-principle approach of statistical mechanics, namely, the method of reduced description of irreversible processes.

We should note that, despite the rather ancient history of problems related to the quasi-linear theory, it has not lost its relevance until now, see at least [19–22]. The traditional directions for development of the quasi-linear theory remain its generalization to the spatially heterogeneous environments, the description within its framework of the possible instabilities development, etcetera. But it is possible to point to seemingly completely unexpected examples of environments where one should expect the realization of quasi-linear relaxation regimes. Indeed, as can be concluded from the material presented in this paper, there are two necessary conditions for the realization of quasi-linear relaxation. Namely, the existence of the fluctuation-kinetic stage of evolution (and the presence of derived equations of the long-wave fluctuations kinetics) is necessary. In addition, one should be sure of the existence at this stage of the system evolution of time-stable oscillations with high characteristic frequencies (or short periods). As we have seen, it is the largest of the system characteristic periods of oscillation that should determine the time interval for the existence of the collisionless approximation and, therefore, of the quasi-linear regime. For the case considered in this article, the characteristic period was associated with the frequencies of Langmuir plasma oscillations (see (31)).

The specified two main conditions for the quasi-linear regime existence can be implemented, in particular, in systems in which the long-wave fluctuations can significantly affect both the kinetic and hydrodynamic stages of evolution. We mean the previously mentioned complex systems such as dissipative or active media [10], as well as media where neutrons propagate and multiply, such as nuclear reactors [11]. Systems with possible quasi-linear regimes are not limited to the given examples; there may be others. But for each specific system, constructing a quasi-linear approximation is a task of considerable complexity, even in the presence of such a promising approach to the construction of the irreversible processes theory, which is the method of reduced description of the latter.

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Нелінійна динаміка кінетичних флуктуацій та квазілінійна релаксація в плазмі

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Запропоновано наближення парних кореляцій для розв'язку рівнянь кінетичної теорії довгохвильових (або крупномасштабних) флуктуацій у газоподібних середовищах. Базовими є загальні нелінійні рівняння теорії крупномасштабних флуктуацій на кінетичному етапі еволюції системи, виведені із перших принципів статистичної механіки. Показано, що виходячи з рівнянь кінетики довгохвильових флуктуацій у разі слабкої взаємодії між частинками, у наближенні парних флуктуацій можна відтворити основні результати квазілінійної теорії плазми. Тим самим відомій квазілінійній теорії плазми надається першопринципове обґрунтування.

Ключові слова: *функція розподілу; кореляційні функції; довгохвильові флуктуації; кінетична теорія флуктуацій; рівняння нелінійної динаміки; наближення парних кореляцій; квазілінійна релаксація в плазмі.*