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Temperature stresses in a rectangular two-layer plate under the action of a locally distributed temperature field

Musii R. S., Zhydyk U. V., Drohomyretska Kh. T., Svidrak I. H., Shynder V. K.

Lviv Polytechnic National University,
12 S. Bandera Str., 79013, Lviv, Ukraine

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A rectangular isotropic two-layer plate of an irregular structure is considered, the edges of which are freely supported, and a constant temperature is maintained on them. Two-dimensional Kirchhoff-type thermoelasticity equations and two-dimensional heat equations written for an inhomogeneous material were used to study the temperature stresses in the plate. Using the method of double trigonometric series in spatial variables and the Laplace integral transformation over time, the general solutions of boundary value problems of thermoelasticity and heat conductivity for this plate under the action of a locally distributed temperature field specified at the initial moment of time are written down. The normal stresses in the layers of the plate are numerically analyzed depending on the geometric parameters, heat transfer coefficient, and time.

Keywords: *two-layer plate; irregular structure; local heat exchange; temperature stresses.*
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1. Introduction

Rectangular plates of a layered structure are widely used in many branches of modern technology, in particular for protection against unwanted thermal effects, to increase the strength and rigidity of structures, in the construction industry. Therefore, the calculation of temperature stresses in such structures is an important engineering task.

Elements of inhomogeneous structures under the action of thermomechanical loadings were studied [1–5] by many scientists and analytical solutions based on two-dimensional equations of classical and various refined theories were constructed [5–9]. Using Green's function method, three-dimensional thermoelastic fields in plates under the action of heat sources were studied in [10]. Numerical methods were used [11, 12] for the analysis of non-stationary heat processes in inhomogeneous plates. In [12] studies of the influence of thermomechanical connectivity on the stress-deformed state of composite structures were performed. The behaviour of composite plates under the condition of loss of temperature stability is considered in papers [12,13]. The solution to the dynamic problem of thermomechanics for an electroconductive non-ferromagnetic plate under the action of electromagnetic impulses of micro- and nanosecond duration was constructed [14]. In [15], the stress-strain state of a layered cylindrical shell under local convective heating was investigated. A more detailed review of various models and methods of studying the non-homogeneous thin-walled structures is given in works [1, 2, 16, 17].

This article investigates the thermostressed state of layered isotropic rectangular plates under the action of a non-stationary temperature loading, given at the initial moment of time.

2. The basic systems of equations

Consider a rectangular plate with dimensions $a \times b$ and a constant thickness $2h$, which is made of an inhomogeneous isotropic material in the transverse direction. We refer the points of the plate space to the rectangular Cartesian coordinate system x, y, z .

Let the plate be heated by heat sources and the external environment by means of convective heat exchange through the side surfaces $z = \pm h$. To determine the thermoelastic state of such a plate, we will use the two-dimensional mathematical model of Kirchhoff. This model for thermostressed problems consists of two independent systems of equations: a system of thermoelasticity equations and a system of heat conductivity equations.

The system of equations of thermoelasticity. The equations of equilibrium of the plate in the displacements u, v, w of the middle surface have the form:

$$A\left(\frac{\partial^2}{\partial x^2}+\frac{1-\nu}{2}\frac{\partial^2}{\partial y^2}\right)u+\frac{1+\nu}{2}A\frac{\partial^2}{\partial x\partial y}v-B\left(\frac{\partial^3}{\partial x^3}+\frac{\partial^3}{\partial x\partial y^2}\right)w=A^t\frac{\partial T_1}{\partial x}+\frac{B^t}{h}\frac{\partial T_2}{\partial x},$$

$$\frac{1+\nu}{2}A\frac{\partial^2}{\partial x\partial y}u+A\left(\frac{\partial^2}{\partial y^2}+\frac{1-\nu}{2}\frac{\partial^2}{\partial x^2}\right)u-B\left(\frac{\partial^3}{\partial y^3}+\frac{\partial^3}{\partial x^2\partial y}\right)w=A^t\frac{\partial T_1}{\partial y}+\frac{B^t}{h}\frac{\partial T_2}{\partial y},$$

$$-B\left(\frac{\partial^3}{\partial x^3}+\frac{\partial^3}{\partial x\partial y^2}\right)u-B\left(\frac{\partial^3}{\partial y^3}+\frac{\partial^3}{\partial x^2\partial y}\right)v+D\left(\frac{\partial^4}{\partial x^4}+\frac{\partial^4}{\partial y^4}+2\frac{\partial^4}{\partial x^2\partial y^2}\right)w$$
$$-\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\left(B^tT_1+\frac{D^t}{h}T_2\right). \quad (1)$$

Here $\{A,B,D\}=\frac{1}{1-\nu^2}\{E_1,E_2,E_3\}$, $\{A^t,B^t,D^t\}=\frac{1}{1-\nu}\{\beta_1,\beta_2,\beta_3\}$, $t(x,y,z,\tau)$ is temperature field, $T_j=\frac{2j-1}{2h^j}\int_{-h}^{h}t\,z^{j-1}\,dz$ $(j=1,2)$ are integral temperature characteristics;

$$E_i=\int_{-h}^{h}E(z)z^{i-1}dz,\quad \beta_i=\int_{-h}^{h}E(z)\,\alpha_t(z)\,z^{i-1}dz\quad (i=1,2,3), \quad (2)$$

ν is Poisson's ratio, which is considered constant, $E(z)$ and $\alpha_t(z)$ are the modulus of elasticity and the coefficient of linear thermal expansion, which depend on the coordinate z.

Physical equations for stresses and displacements have the form:

$$\sigma_x=\frac{E(z)}{1-\nu^2}\left[\frac{\partial u}{\partial x}+\nu\frac{\partial v}{\partial y}-z\left(\frac{\partial^2 w}{\partial x^2}+\nu\frac{\partial^2 w}{\partial y^2}\right)-(1+\nu)\alpha_t(z)t\right],$$
$$\sigma_y=\frac{E(z)}{1-\nu^2}\left[\nu\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}-z\left(\nu\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)-(1+\nu)\alpha_t(z)t\right],$$
$$\sigma_{xy}=\frac{E(z)}{2(1+\nu)}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}-2z\frac{\partial^2 w}{\partial x\partial y}\right). \quad (3)$$

Physical equations for internal forces N_x, N_y, N_{xy} and moments M_x, M_y, M_{xy} in the middle surface of the plate are obtained from relations (3) by integrating them over the thickness of the plate. We obtain

$$N_x=A\left(\frac{\partial u}{\partial x}+\nu\frac{\partial v}{\partial y}\right)-B\left(\frac{\partial^2}{\partial x^2}+\nu\frac{\partial^2}{\partial y^2}\right)w-A^tT_1-\frac{B^t}{h}T_2,$$
$$N_y=A\left(\nu\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)-B\left(\nu\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)w-A^tT_1-\frac{B^t}{h}T_2,$$
$$M_x=B\left(\frac{\partial u}{\partial x}+\nu\frac{\partial v}{\partial y}\right)-D\left(\frac{\partial^2}{\partial x^2}+\nu\frac{\partial^2}{\partial y^2}\right)w-B^tT_1-\frac{D^t}{h}T_2,$$
$$M_y=B\left(\nu\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)-D\left(\nu\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)w-B^tT_1-\frac{D^t}{h}T_2,$$
$$N_{xy}=\frac{1-\nu}{2}\left[A\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)-2B\frac{\partial^2 w}{\partial x\partial y}\right],$$
$$M_{xy}=\frac{1-\nu}{2}\left[B\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)-2D\frac{\partial^2 w}{\partial x\partial y}\right]. \quad (4)$$

For the uniqueness of the solution of system (1), it is necessary to set the appropriate boundary conditions. In the case of hinged support of the edges of the plate, we have:

$$x=0,a\colon\ w=0,\ v=0,\ M_x=0,\ N_x=0;$$

$$y = 0, b\colon\ w = 0,\ u = 0,\ M_y = 0,\ N_y = 0. \tag{5}$$

Conditions (5) together with the system of equations (1) make up the boundary value problem of the theory of temperature stresses for inhomogeneous isotropic plates in displacements. Based on the known displacements and integral temperature characteristics, from relations (4) we determine the forces and moments of the middle surface of the plate. The temperature stresses at an arbitrary point of the plate can be found using formulas (3). Note that in the partial case of a homogeneous material, system (1) gives two independent systems of equations: to determine the plane stress state and plate bending.

3. System of heat conduction equations

The integral characteristics of temperature T_1 and T_2, which are included in the free members of system (1), must be determined from the corresponding equations of heat conductivity under the boundary conditions specified on the surfaces $z = \pm h$ and at the edges of the plate. For convective heat exchange on the plate $z = \pm h$ surfaces with a linear dependence of the temperature on the transverse coordinate z, the system of heat conduction equations is written in the form

$$\Delta_1 T_1 - \varepsilon_1^t T_1 + \Delta_2 T_2 - \varepsilon_2^t T_2 - C_1 \frac{\partial T_1}{\partial \tau} - C_2 \frac{\partial T_2}{\partial \tau} = -f_1,$$

$$\Delta_2 T_1 - \varepsilon_2^t T_1 + \Delta_3 T_2 - \left(\frac{\Lambda_1}{h^2} + \varepsilon_1^t\right) T_2 - C_2 \frac{\partial T_1}{\partial \tau} - C_3 \frac{\partial T_2}{\partial \tau} = -f_2. \tag{6}$$

Here

$$\Delta_i = \Lambda_i \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right); \quad \{\Lambda_i, C_i\} = \int_{-h}^{h} \{\lambda(z), c_e(z)\} \left(\frac{z}{h}\right)^{i-1} dz, \quad (i = 1, 2, 3). \tag{7}$$

$$f_j = t_1^z \varepsilon_j^t + t_2^z \varepsilon_{3-j}^t + W_j^t, \quad \varepsilon_j^t = \left(\alpha^+ - (-1)^j \alpha^-\right),$$

$$t_j^z = \frac{1}{2}\left(t_z^+ - (-1)^j t_z^-\right), \quad W_j^t = \int_{-h}^{h} w_t \left(\frac{z}{h}\right)^{j-1} dz, \quad (j = 1, 2),;$$

$\lambda(z)$ is heat conductivity coefficient; t_z^+, t_z^- are temperatures of the media on the surfaces $z = h$ and $z = -h$, respectively; α^+, α^- are coefficients of heat transfer from these surfaces; $c_e(z)$ is specific heat capacity; τ is time variable; w_t is density of heat sources.

For the uniqueness of solution (6), it is necessary to add appropriate boundary and initial conditions to system (6). If zero temperature is maintained at the edges of the plate, and the distribution of the temperature field is specified at the initial moment of time, then we have the following conditions:

$$x = 0, a\colon\ T_1 = T_2 = 0;$$

$$y = 0, b\colon\ T_1 = T_2 = 0; \tag{8}$$

$$\tau = 0\colon\ T_1(x, y, 0) = T_1^0(x, y),\ T_2(x, y, 0) = T_2^0(x, y). \tag{9}$$

4. Methods of solving problems of thermoelasticity and heat conduction

Let the plate consist of a package of rigidly interconnected N homogeneous isotropic layers with different properties $\{E^{(k)}, \alpha_t^{(k)}, \lambda^{(k)}, c_e^{(k)}\}$ and different thicknesses h_k. We assume that the hypothesis about the nature of the temperature distribution along the thickness of the plate is fulfilled for the entire package of constituent layers. Then, according to the methodology outlined in [3], the thermophysical characteristics of the layered plate can be represented using asymmetric unit functions $S_\pm(z)$ in the form

$$q(z) = q_1 + \sum_{k=1}^{N-1} (q_{k+1} - q_k)\, S_+(z - z_k). \tag{10}$$

Here $q(z) = \{E(z), \alpha_t(z), \lambda(z), c_e(z)\}$, $q_k = \{E^{(k)}, \alpha_t^{(k)}, \lambda^{(k)}, c_e^{(k)}\}$ are physical and mechanical characteristics of the k-th layer of the plate, z_k is the coordinate of the interface between the k-th and the $k+1$-th layers, $z_k = -h + \sum_{m=1}^{k} h_m$,

$$S_+(x) = \begin{cases} 1, & x > 0 \\ 0, & x \leqslant 0 \end{cases}, \quad S_-(x) = \begin{cases} 1, & x \geqslant 0 \\ 0, & x < 0 \end{cases}.$$

By substituting relation (10) in (2) and (7), we obtain expressions of integral characteristics E_i, β_i, Λ_i, C_i in terms of the physical properties of the layers $E^{(k)}$, $\alpha_t^{(k)}$, $\lambda^{(k)}$, $c_e^{(k)}$. In the case of a two-layer plate, we have expressions for the modulus of elasticity E_i

$$E_1 = h\left[2E^{(1)} + \left(E^{(2)} - E^{(1)}\right)\left(2 - \frac{h_1}{h}\right)\right],$$

$$E_2 = h^2\left[\frac{1}{2}\left(E^{(2)} - E^{(1)}\right)\frac{h_1}{h}\left(2 - \frac{h_1}{h}\right)\right],$$

$$E_3 = \frac{h^3}{3}\left\{2E^{(1)} + \left(E^{(2)} - E^{(1)}\right)\left[1 + \left(1 - \frac{h_1}{h}\right)^3\right]\right\} \tag{11}$$

and for other integral characteristics the expressions will be similar.

To solve the systems of differential equations of heat conductivity (6) and thermoelasticity (1), we use the method of double trigonometric series. To do this, we expand the functions of temperature loads f_j, the unknown functions of displacements u, v, w and integral characteristics of temperature T_j into double Fourier series in such a way that the boundary conditions (5) and (8) are satisfied:

$$f_j = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} f_{jmn} \sin\frac{\pi m}{a}x \, \sin\frac{\pi n}{b}y. \tag{12}$$

$$u = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} U_{mn} \cos\frac{\pi m}{a}x \, \sin\frac{\pi n}{b}y,$$

$$v = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} V_{mn} \sin\frac{\pi m}{a}x \, \cos\frac{\pi n}{b}y,$$

$$w = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} W_{mn} \sin\frac{\pi m}{a}x \, \sin\frac{\pi n}{b}y, \tag{13}$$

$$T_1 = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} T_{1mn} \sin\frac{\pi m}{a}x \, \sin\frac{\pi n}{b}y,$$

$$T_2 = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} T_{2mn} \sin\frac{\pi m}{a}x \, \sin\frac{\pi n}{b}y. \tag{14}$$

Then the heat conductivity equation (6) after substituting (12) and (14) in it will take the form

$$\frac{dT_{1mn}}{d\tau_1} + C'\frac{dT_{2mn}}{d\tau_1} + g_1 T_{1mn} + g_2 T_{2mn} = f_{1mn},$$

$$C''\frac{dT_{1mn}}{d\tau_1} + \frac{dT_{2mn}}{d\tau_1} + g_3 T_{1mn} + g_4 T_{2mn} = f_{2mn}. \tag{15}$$

Here

$$g_1 = \tilde{\Lambda}_1\left(\mu_m^2 + \mu_n^2\right) + \mathrm{Bi}_1; \quad g_2 = \tilde{\Lambda}_2\left(\mu_m^2 + \mu_n^2\right) + \mathrm{Bi}_2;$$

$$g_3 = \left[\tilde{\Lambda}_2\left(\mu_m^2 + \mu_n^2\right) + \mathrm{Bi}_2\right]\tilde{C}; \quad g_4 = \left[\tilde{\Lambda}_3\left(\mu_m^2 + \mu_n^2\right) + \mathrm{Bi}_1 + \tilde{\Lambda}_1\right]\tilde{C};$$

$$\mu_m = \frac{\pi m h}{a}; \quad \mu_n = \frac{\pi n h}{b}; \quad \tau_1 = \frac{2\lambda_0}{hC_1}\tau; \quad C' = \frac{C_2}{C_1}; \; C'' = \frac{C_2}{C_3}; \; \tilde{C} = \frac{C_1}{C_3}; \quad \tilde{\Lambda}_i = \frac{\Lambda_i}{2h\lambda_0}; \quad \mathrm{Bi}_i = \frac{\varepsilon_i^t h}{2\lambda_0};$$

$$f_{1mn} = \mathrm{Bi}_1 t^z_{1mn} + \mathrm{Bi}_2 t^z_{2mn} + W^t_{1mn}\frac{h}{2\lambda_0} = Q_{1mn}(x,y)F_1(\tau);$$
$$f_{2mn} = \left(\mathrm{Bi}_2 t^z_{1mn} + \mathrm{Bi}_1 t^z_{2mn} + W^t_{2mn}\frac{h}{2\lambda_0}\right)\tilde{C} = Q_{2mn}(x,y)F_2(\tau);$$

λ_0 is some characteristic heat conductivity coefficient.

The solution of the system (15) under the initial conditions (9) is found by the method of the integral Laplace transform

$$T_{1mn} = \sum_{\substack{j=1\\k\neq j}}^{2} \frac{1}{C^*(p_j - p_k)}\Big\{(p_j - g_4)Q_{1mn}Z_{1j}(\tau) - (C'p_j - g_2)Q_{2mn}Z_{2j}(\tau) + \big[(p_j - g_4)T'_{0nm} - (C'p_j - g_2)T''_{0nm}\big]\exp(-p_j\tau_1)\Big\},$$

$$T_{2mn} = \sum_{\substack{j=1\\k\neq j}}^{2} \frac{1}{C^*(p_j - p_k)}\Big\{(p_j - g_1)Q_{2mn}Z_{2j}(\tau) - (C''p_j - g_3)Q_{1mn}Z_{1j}(\tau) + \big[(p_j - g_1)T''_{0nm} - (C''p_j - g_3)T'_{0nm}\big]\exp(-p_j\tau_1)\Big\}. \quad (16)$$

Here

$$\left\{Q^z_{jnm}, T'_{0nm}, T''_{0nm}\right\} = \frac{4}{ab}\int_0^a\int_0^b \left\{Q^z_j, T'_0, T''_0\right\}(x,y)\sin\frac{\pi m x}{a}\sin\frac{\pi n y}{b}\,dx\,dy, \quad (j = 1,2)\,; \quad (17)$$

$$Z_{ij} = \int_0^{\tau_1} F_i(u)\exp\big(-p_j(\tau_1 - u)\big)du, \quad (i,j = 1,2); \quad (18)$$

$-p_j$ are the roots of a quadratic equation $C^*p^2 + (g_1 + g_4 - C'g_3 - C''g_2)\,p + g_1g_4 - g_2g_3 = 0$;

$$C^* = 1 - C'C''; \quad T'_0 = T^0_1 + C'T^0_2; \quad T''_0 = T^0_2 + C''T^0_1.$$

The system of differential equations of equilibrium (1), after substituting solutions (13), (14) into them, is transformed into a system of algebraic equations for determining the Fourier coefficients U_{mn}, V_{mn}, W_{mn} of the desired displacements. We write this system in matrix form

$$\begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{12} & m_{22} & m_{23} \\ m_{13} & m_{23} & m_{33}\end{pmatrix}\begin{pmatrix} U_{mn} \\ V_{mn} \\ W_{mn}\end{pmatrix} = \begin{pmatrix} s_1 \\ s_2 \\ s_3\end{pmatrix}T_{1mn} + \begin{pmatrix} r_1 \\ r_2 \\ r_3\end{pmatrix}T_{2mn} \quad (19)$$

Here

$$m_{11} = -A\left(\mu_m^2 + \frac{1-\nu}{2}\mu_n^2\right); \quad m_{12} = -\frac{1+\nu}{2}A\mu_m\mu_n; \quad m_{13} = \frac{B}{h}\left(\mu_m^2 + \mu_n^2\right)\mu_m;$$
$$m_{22} = -A\left(\mu_n^2 + \frac{1-\nu}{2}\mu_m^2\right); \quad m_{23} = \frac{B}{h}\left(\mu_m^2 + \mu_n^2\right)\mu_n; \quad m_{33} = \frac{D}{h^2}\left(\mu_m^2 + \mu_n^2\right)^2;$$

$$s_1 = A^th\mu_m;\ s_2 = A^th\mu_n;\ s_3 = -B^t\left(\mu_m^2 + \mu_n^2\right); \quad r_1 = B^t\mu_m; \quad r_2 = B^t\mu_n; \quad r_3 = -\frac{D^t}{h}\left(\mu_m^2 + \mu_n^2\right).$$

The solution of system (19) can be found in the form

$$\begin{pmatrix} U_{mn} \\ V_{mn} \\ W_{mn}\end{pmatrix} = \frac{1}{|M|}\sum_{i=1}^{3}\left[\begin{pmatrix} s_iM_{i1} \\ s_iM_{i2} \\ s_iM_{i3}\end{pmatrix}T_{1mn} + \begin{pmatrix} r_iM_{i1} \\ r_iM_{i2} \\ r_iM_{i3}\end{pmatrix}T_{2mn},\right] \quad (20)$$

where $|M|$ is the determinant of the matrix, $(m_{ij})_{3\times3}$, M_{ij} are algebraic additions to the elements of this matrix.

Based on known displacements and integral temperature characteristics, all other components of the stress-strain state of the plate can be found using known formulas.

5. Analysis of numerical results

Numerical studies were performed for a two-layer plate, which is heated by the temperature field given at the initial moment of time by the expression

$$T_1^0(x,y) = t^* \cos\frac{\pi(x-x_0)}{2a_0}\cos\frac{\pi(y-y_0)}{2b_0}N(x)N(y), \quad T_2^0(x,y)=0. \tag{21}$$

Here $N(x) = [S_-(x-(x_0-a_0)) - S_+(x-(x_0+a_0))]$, $N(y) = [S_-(y-(y_0-b_0)) - S_+(y-(y_0+b_0))]$, $t^* = \text{const}$; x_0, y_0 are coordinates of the centre of the heating area; $2a_0 \times 2b_0$ is the size of this area.

Then, from equations (17) and (21), we obtain the following expressions for the Fourier coefficients

$$T_{mn}^0 = \frac{4t^*\sin\frac{\pi m x_0}{a}\cos\frac{\pi m a_0}{a}\sin\frac{\pi n y_0}{b}\cos\frac{\pi n b_0}{b}}{\pi^2\frac{a_0 b_0}{ab}\left[\left(\frac{a}{2a_0}\right)^2 - m^2\right]\left[\left(\frac{b}{2b_0}\right)^2 - n^2\right]}, \quad \text{if} \quad m \neq \frac{a}{2a_0},\ n \neq \frac{b}{2b_0};$$

$$T_{mn}^0 = \frac{2t^*\sin\frac{\pi m x_0}{a}\sin\frac{\pi n y_0}{b}\cos\frac{\pi n b_0}{b}}{\pi m\frac{b_0}{b}\left[\left(\frac{b}{2b_0}\right)^2 - n^2\right]}, \quad \text{if} \quad m = \frac{a}{2a_0},\ n \neq \frac{b}{2b_0};$$

$$T_{mn}^0 = \frac{2t^*\sin\frac{\pi m x_0}{a}\cos\frac{\pi m a_0}{a}\sin\frac{\pi n y_0}{b}}{\pi n\frac{a_0}{a}\left[\left(\frac{a}{2a_0}\right)^2 - m^2\right]}, \quad \text{if} \quad m \neq \frac{a}{2a_0},\ n = \frac{b}{2b_0},$$

$$T_{mn}^0 = \frac{t^*\sin\frac{\pi m x_0}{a}\sin\frac{\pi n y_0}{b}}{mn}, \quad \text{if} \quad m = \frac{a}{2a_0},\ n = \frac{b}{2b_0}.$$

During the calculations, it was assumed that the heat transfer coefficients from the plate $z = \pm h$ surfaces are equal $\alpha^+ = \alpha^- = \alpha_z$, the ambient temperature is zero, and there are no heat sources.

For the materials of the plate layers, steel and ceramics were chosen. The bottom layer is made of steel with the following physical and mechanical properties:

$$E^{(1)} = 192\,\text{GPa}; \quad \nu^{(1)} = 0.3; \quad \alpha_t^{(1)} = 17\cdot 10^{-6}\,1/\text{K}; \quad \lambda^{(1)} = 16.7\,\text{W/mK}; \quad c_e^{(1)} = 500\,\text{J/kg K}.$$

The upper layer of the plate is made of ceramics, for which the physical and mechanical properties are as follows:

$$E^{(2)} = 117\,\text{GPa}; \quad \nu^{(2)} = 0.3; \quad \alpha_t^{(2)} = 7.11\cdot 10^{-6}\,1/\text{K}; \quad \lambda^{(2)} = 2.036\,\text{W/mK}; \quad c_e^{(2)} = 615\,\text{J/kg K}.$$

The values of other parameters are equal: $h/a = 0.025$; $x_0 = a/2$; $y_0 = b/2$; $a_0/a = 0.25$; $b_0/b = 0.25$; $\lambda_0 = \lambda^{(1)}$.

For the given parameters, the dimensionless normal stresses $\sigma_1' = \frac{\sigma_x}{E^{(1)}\alpha_t^{(1)}t^*}$ and $\sigma_2' = \frac{\sigma_y}{E^{(1)}\alpha_t^{(1)}t^*}$ were calculated in the centre of the heating area $(a/2, b/2)$. The change of these stresses depending on the dimensionless transverse coordinate $z' = z/h$ for $\text{Bi} = 1$, $a/b = 3$, $h_2/h_1 = 0.3$ at different moments of time τ' is shown in Figures 1 and 2.

The stresses in the layers are linear in nature and have a break at the interface between the layers. In the second layer, both stresses are tensile, and in the first layer, the stress σ_1' is compressive, and the stress σ_2' changes from a tensile value on the surface $z' = -1$ to a compressive value at the interface of the layers. It was found that the maximum stress values are reached at the line of separation of layers. Over time, the stresses decrease and level out across the thickness of the plate.

The change in maximum stresses over time for the values $\text{Bi} = 1$, $a/b = 3$ and different values of the ratio of layer thicknesses $h' = h_2/h_1$ in the first $\sigma_i'(1)$ and second $\sigma_i'(2)$ layers are illustrated in Figures 3–6.

It was obtained that with a decrease in the parameter h', the maximum stresses in the first layer decrease and increase in the second. The maximum stress values are acquired at the initial moment of time, and with the passage of time, as a result of heat transfer, the temperature drops, and the stresses decrease.

The dependence of the maximum stresses in each layer on the ratio of plate lengths a/b for $h' = 0.3$, $\tau' = 0.01$ and different values of the heat transfer coefficient Bi is shown in Figures 7–10.

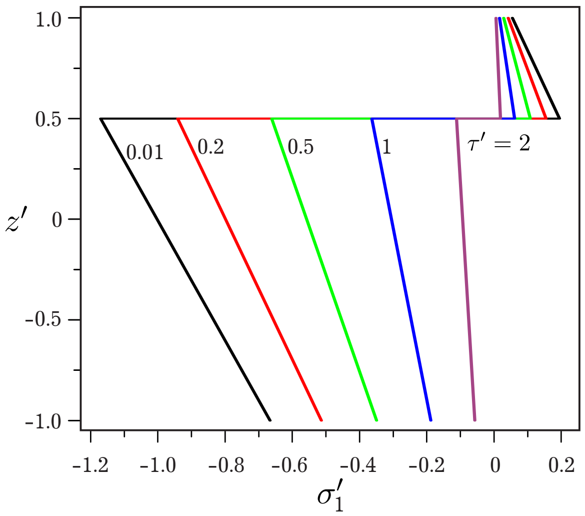

Fig. 1. Change in normal stresses σ'_1 from transverse coordinate z' for Bi $= 1$, $a/b = 3$, $h_2/h_1 = 0.3$, at different time points τ'.

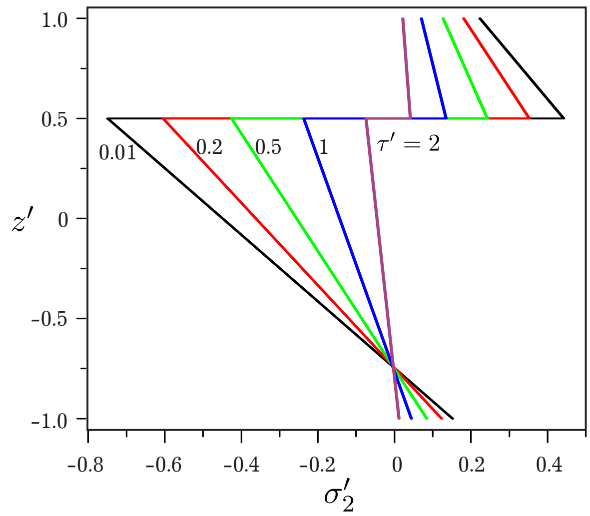

Fig. 2. Change in normal stresses σ'_2 from transverse coordinate z' for Bi $= 1$, $a/b = 3$, $h_2/h_1 = 0.3$, at different time points τ'.

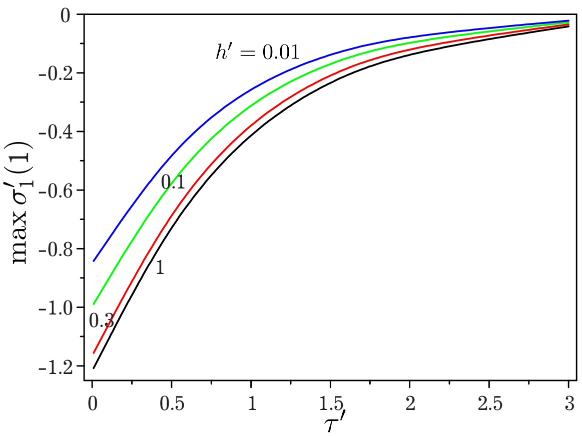

Fig. 3. Change in time τ' of the maximum stresses $\sigma'_1(1)$ in the first layer for Bi $= 1$, $a/b = 3$, and different values of the parameter of the ratio of layer thicknesses $h' = h_2/h_1$.

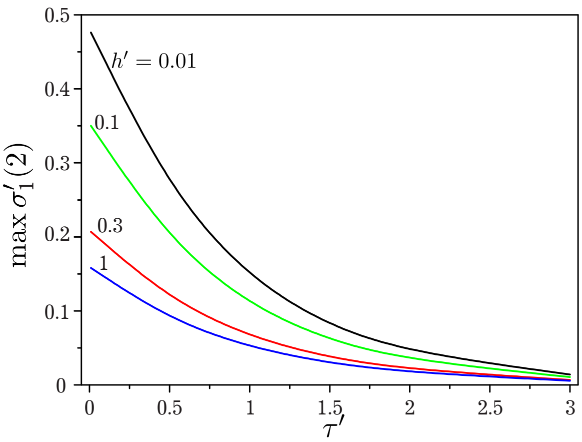

Fig. 4. Change in time τ' of the maximum stresses $\sigma'_1(2)$ in the second layer for Bi $= 1$, $a/b = 3$, and different values of the parameter of the ratio of layer thicknesses $h' = h_2/h_1$.

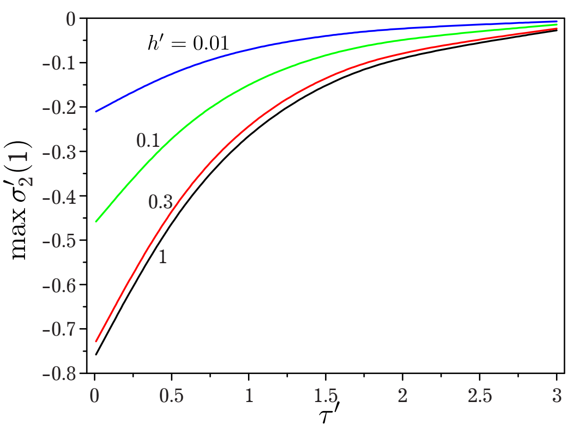

Fig. 5. Change in time τ' of the maximum stresses $\sigma'_2(1)$ in the first layer for Bi $= 1$, $a/b = 3$, and different values of the parameter of the ratio of layer thicknesses $h' = h_2/h_1$.

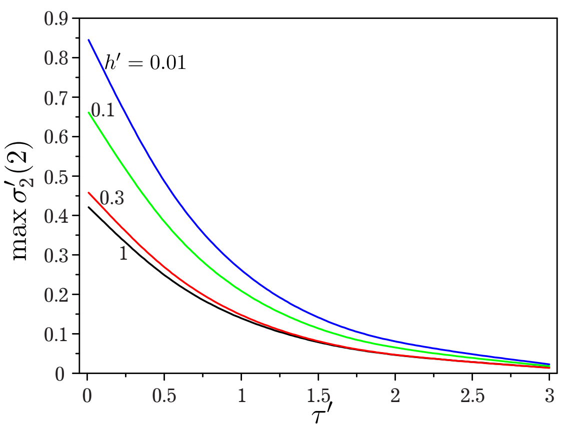

Fig. 6. Change in time τ' of the maximum stresses $\sigma'_2(2)$ in the second layer for Bi $= 1$, $a/b = 3$, and different values of the parameter of the ratio of layer thicknesses $h' = h_2/h_1$.

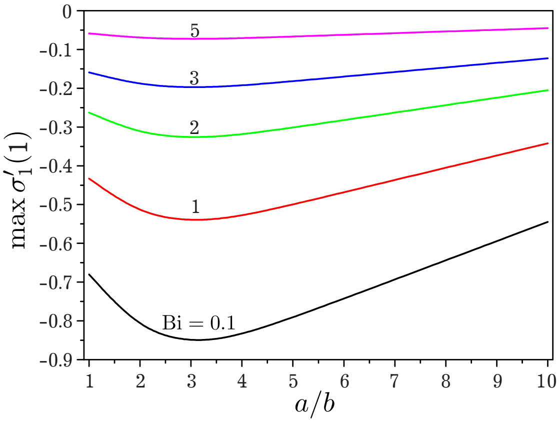

Fig. 7. Dependence of stresses $\max \sigma_1'(1)$ in the first layer for $h' = 0.3$, $\tau' = 0.01$ from relation a/b and different coefficient Bi values.

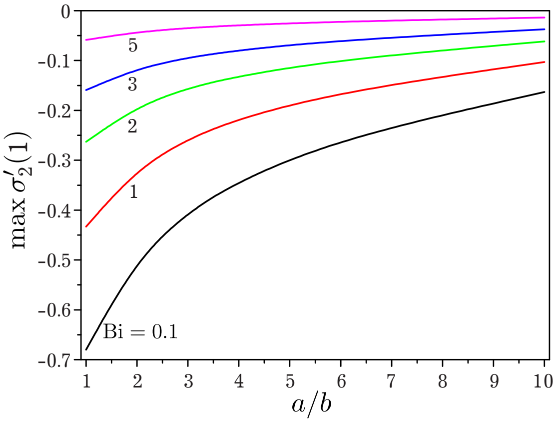

Fig. 8. Dependence of stresses $\max \sigma_2'(1)$ in the first layer for $h' = 0.3$, $\tau' = 0.01$ from relation a/b and different coefficient Bi values.

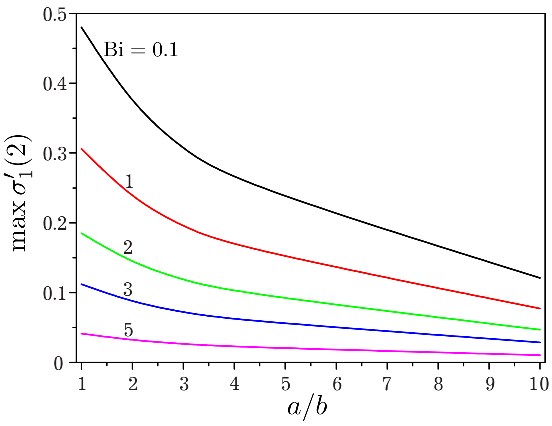

Fig. 9. Dependence of stresses $\max \sigma_1'(2)$ in the second layer for $h' = 0.3$, $\tau' = 0.01$ from relation a/b and different coefficient Bi values.

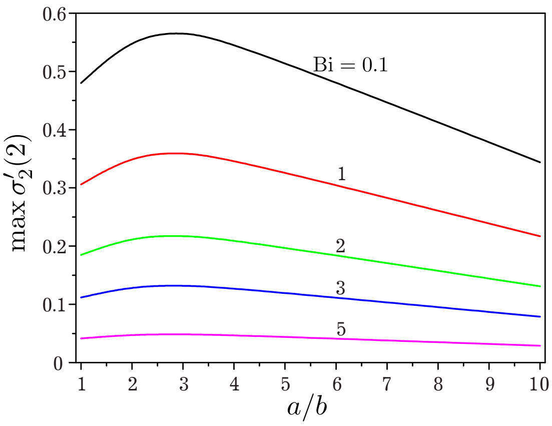

Fig. 10. Dependence of stresses $\max \sigma_2'(2)$ in the second layer for $h' = 0.3$, $\tau' = 0.01$ from relation a/b and different coefficient Bi values.

The stress $\sigma_2'(1)$ and $\sigma_1'(2)$ reach their maximum values for the square plate ($a/b = 1$), and the stress $\sigma_1'(1)$ and $\sigma_2'(2)$ — for the ratio $a/b = 3$. As the stress ratio a/b increases, the stress decreases monotonically. It was obtained that at a given time the stresses are greater when the heat transfer from the plate surfaces $z = \pm h$ is less.

6. Conclusions

Based on the linear equations of the Kirchhoff theory, the normal stresses of a two-layer isotropic rectangular plate, which is heated by the temperature field specified at the initial moment, were investigated. Heat exchange with the environment takes place through the side surfaces of the plate. The closed solution of the considered problem was found by the methods of trigonometric Fourier series in spatial variables and integral Laplace transform in time. The dependence of normal stresses in the considered plate on geometric parameters, heat transfer coefficient and time is graphically illustrated. The obtained results can be used to analyse the stress state of an isotropic rectangular plate with coatings.

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Температурні напруження у прямокутній двошаровій пластині за дії локально розподіленого температурного поля

Мусій Р. С., Жидик У. В., Дрогомирецька Х. Т., Свідрак І. Г., Шиндер В. К.

Національний університет "Львівська політехніка",
вул. С.Бандери 12, Львів, 79013, Україна

Розглянуто прямокутну ізотропну двошарову пластину нерегулярної структури, краї якої вільно оперті і на них підтримується стала температура. Для дослідження температурних напружень в пластині використано двовимірні рівняння термопружності типу Кірхгофа і двовимірні рівняння теплопровідності, записані для неоднорідного матеріалу. З використанням методу подвійних тригонометричних рядів за просторовими змінними та інтегрального перетворення Лапласа за часом записано загальні розв'язки крайових задач термопружності і теплопровідності для даної пластини за дії локально розподіленого температурного поля, заданого в початковий момент часу. Числово проаналізовано нормальні напруження в шарах пластини залежно від геометричних параметрів, коефіцієнта тепловіддачі та часу.

Ключові слова: *двошарова пластина; нерегулярна структура; локальний теплообмін; температурні напруження.*