MATHEMATICAL MODELING AND COMPUTING, Vol. 10, No. 2, pp. 445–457 (2023)

Fractional Brownian motion in financial engineering models

Yanishevskyi V. S., Nodzhak L. S.

Lviv Polytechnic National University,
12 S. Bandera Str., 79013, Lviv, Ukraine

(Received 10 January 2023; Revised 12 May 2023; Accepted 18 May 2023)

An application of fractional Brownian motion (fBm) is considered in stochastic financial engineering models. For the known Fokker–Planck equation for the fBm case, a solution for transition probability density for the path integral method was built. It is shown that the mentioned solution does not result from the Gaussian unit of fBm with precise covariance. An expression for approximation of fBm covariance was found for which solutions are found based on the Gaussian measure of fBm and those found based on the known Fokker–Planck equation match.

Keywords: *fractional Brownian motion; stochastic equation; Fokker–Planck equation; transition probability density; path integrals.*

2010 MSC: 91B70, 91G30, 81S40 **DOI:** 10.23939/mmc2023.02.445

1. Introduction

During the building of many stochastic models of financial engineering, Brownian motion [1–4] plays a fundamental role. The Brownian motion term was first introduced by L. Bachelier for the modelling of option price dynamics. Later Einstein applied it to the modelling of physical processes of diffusion. A mathematical theory of Brownian motion was built by Wiener, which is why it is sometimes called a Wiener process. The Wiener measure, which is a functional measure, is related to the Brownian motion. Brownian motion is a part of stochastic differential equations that allow for the modelling of different Markov processes of diffusion that have a wide variety of applications. In particular, the important role in financial mathematics plays a geometric Brownian motion, which is used to model the price evolution of financial assets and derivatives. Geometric Brownian motion was used by Black and Scholes for the modelling of the option price, which is now known as the Black–Scholes formula. The theory of stochastic differential equations is developed enough at the moment and is applied to various problems of financial engineering. Apart from mentioned models of asset pricing, options [5], and other derivatives are the stochastic models of interest rate [6, 7], models of stochastic volatility, and others [1, 8].

We also consider stochastic models based on fBm as a base [9, 10]. In comparison to Brownian motion, for which increments on time intervals that do not intersect do not correlate, whereas, for fBm, a correlation takes place, which is also called a strong after action. Properties of fBm and its application to a number of applied problems were researched in many works [11–16]. In particular, in works [11, 12], a generalized Ito formula for fBm is given. Based on it, a Fokker–Planck equation for the transition probability density of stochastic process based on fBm was received [13–15, 17]. In the given works, solutions were found for the Fokker–Planck equation for transition probability for which option price was obtained, as well as equations were built for option price dynamics. As a result, generalized Black–Scholes formulas for option price were received. Stochastic processes based on so-called sub-fBm and their application to the modelling of asset price dynamics and based derivatives were considered. Aside from fBm, a generalization of the CEV model for fBm where considered [14, 18].

As it is known, the transition probability density for mentioned stochastic differential equation can be defined based on the probabilistic measure of the specified process. It is understood that both approaches must give the same result as it takes place in the case of Brownian motion [19]. In this paper, a solution of the Fokker–Planck equation for the transition probability density of fBm in the form of path integral was obtained. The same solution was received using the Gaussian measure of fBm, and also the condition where they both match was found. As a result, it was shown that the Fokker–Planck equation given in [9,15,18,20] corresponds to Gaussian measure with a covariance which in a certain way approximates a covariance of fBm.

2. Fractional Brownian motion

FBm $B(\tau)$ in time interval $\tau \in [0,t]$ defines Gaussian process with a zero average and covariance [9, 10, 12]:

$$\begin{aligned} &\langle B(\tau)\rangle = 0, \quad \langle B(\tau)B(s)\rangle = R_H(\tau,s), \\ &R_H(\tau,s) = \frac{1}{2}\big(\tau^{2H} + s^{2H} - |\tau - s|^{2H}\big), \quad s,\tau \in [0,t], \quad 0 < H < 1. \end{aligned} \tag{1}$$

From (1) we obtain for process variation that

$$\langle B(\tau)^2\rangle = \tau^{2H}. \tag{2}$$

For $H = \frac{1}{2}$ from (1) the covariance of Brownian motion results [2, 4, 8]

$$R(\tau,s) = \frac{1}{2}\big(\tau + s - |\tau - s|\big) = \min(\tau,s). \tag{3}$$

A distinctive characteristic of fBm for $H > \frac{1}{2}$ is a long-term time dependency between increases. This can be visually shown by splitting the time interval $[0,t]$ into n equal intervals. Then using (1), it is possible to show that the following applies:

$$r(m) = \langle B(1)(B(1+m) - B(1))\rangle, \quad m \in \{1,\dots,n-1\}, \quad \lim_{n\to\infty} \sum_{m=1}^{n-1} r(m) \to \infty. \tag{4}$$

In case of Brownian motion $H = \frac{1}{2}$, we have that $r(m) = 0$ for $\forall m > 1$.

FBm is also determined based on Brownian motion [9, 10] with the help of stochastic integral:

$$B(\tau) = \int_0^\tau K_H(\tau,s)\,dW(s), \quad 0 < s < \tau < t. \tag{5}$$

Based on (2) and (5), we obtain a connection of covariance with the kernel of integral transformation

$$R_H(\tau,s) = \int_0^{\min(\tau,s)} K_H(\tau,u)K_H(s,u)\,du. \tag{6}$$

For the kernel $K_H(\tau,s)$ a number of equivalent representations exist [9, 10], particularly the following

$$K_H(\tau,s) = c_H s^{\frac{1}{2}-H} \int_s^\tau (u-s)^{H-\frac{3}{2}} u^{H-\frac{1}{2}}\,du, \quad 0 < s < \tau < t, \tag{7}$$

where $c_H = \sqrt{\frac{H(2H-1)}{B(2-2H, H-\frac{1}{2})}}$, $\frac{1}{2} < H < 1$, and $B(x,y)$ denotes a beta function. In the limit $H \downarrow \frac{1}{2}$ with (7) we obtain that $K_H(\tau,s) \to 1$ $(\tau > s)$ and according to (5) fBm matches with Brownian motion. And based on (6), we obtain a covariance (3) for the Brownian motion.

Since fBm is a stochastic Gaussian process with covariance (1), it has a respective Gaussian measure [4]. Let us consider a discrete realization of this process on time interval $[0,t]$. Let us set the following time interval breakdown $(0 < t_1 < t_2 < \ldots < t_n = t)$. We compare a random vector of fBm values $\mathbf{B} = (B_1, B_2, \ldots, B_n)$ in breakdown points for this breakdown. Then probability density of distribution n of dimensional random vector $\mathbf{B}$ (Gaussian measure) is given by the following expression

$$\mu(\mathbf{B}) = (2\pi)^{-\frac{n}{2}} \sqrt{\det \hat{R}^{-1}} \exp\Big(-\frac{1}{2}\sum_{i,j} B_i \hat{R}^{-1}_{ij} B_j\Big). \tag{8}$$

Here R^{-1}_{ij} denotes elements of matrix which is inverted covariance matrix $\hat{R}$

$$\hat{R} = \|R_{ij}\|, \quad R_{ij} = R_H(t_i,t_j), \quad i,j = 1\ldots n.$$

It is understood that the Gaussian measure (8) can be also obtained based on Wiener measure and determining fBm (5). Corresponding calculations are given in Appendix A.

The transition probability density for fBm we will define based on (8) by the following integral

$$K(B,t)=\int_{-\infty}^{\infty}\mu(\mathbf{B})\delta(B-B_n)\prod_{i=1}^{n}dB_i. \tag{9}$$

Let us perform integral transform for δ of function

$$\delta(B-B_n)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-\mathrm{i}x(B-B_n)}dx=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-\mathrm{i}xB}e^{\mathrm{i}\mathbf{X}_0\mathbf{B}}dx, \tag{10}$$

where n denotes a dimensional vector $\mathbf{X}_0=(0,0,\dots,x)$. After substituting (10) into (9) and solving integrals for dB_i $(i=1\dots n)$

$$\int_{-\infty}^{\infty}\mu(\mathbf{B})e^{\mathrm{i}\mathbf{X}_0\mathbf{B}}\prod_{i=1}^{n}dB_i=e^{-1/2\mathbf{X}_0\hat{R}\mathbf{X}_0}=e^{-1/2R_H(t)x^2}$$

we obtain

$$K(B,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-\mathrm{i}xB}e^{-1/2R_H(t)x^2}dx=\frac{1}{\sqrt{2\pi R_H(t)}}e^{-\frac{1}{2}\frac{B^2}{R_H(t)}}. \tag{11}$$

Here we introduce notation $R_H(t)\equiv R_H(t,t)=t^{2H}$ for fBm variation. Transition probability density (11) satisfies Fokker–Planck equation

$$\frac{\partial K(B,t)}{\partial t}=\frac{1}{2}\dot{R}_H(t)\frac{\partial^2K(B,t)}{\partial B^2}. \tag{12}$$

As we can see, the transition probability density for fBm and the respective Fokker–Planck equation is determined using fBm variation (equal to $R_H(t)$). We shall point out that mentioned characteristic is known in many multidimensional independent Gaussian processes with diagonal covariance matrix [9]. Fokker–Planck equation of type (12) and its solutions were obtained for fractional Levy motion with a bit different approach [21].

Among other properties of fBm, we shall point out that it is not a martingale and Markov process [9, 10, 12, 22]. Let us illustrate this an example of dual point transition probability density $K_2(B_2,t_2,B_1,t_1)$, which is defined by the following expression

$$K_2(B_2,t_2,B_1,t_1)=\frac{P(B_2,t_2,B_1,t_1)}{K(B_1,t_1)},\quad 0<t_1<t_2.$$

Here $P(B_2,t_2,B_1,t_1)$ is a dual point probability density of fBm found using (8). For $K_2(B_2,t_2,B_1,t_1)$ none of the martingale condition satisfies

$$\int_{-\infty}^{\infty}K_2(B_2,t_2,B_1,t_1)B_2\,dB_2\neq B_1. \tag{13}$$

Also, the Chapman–Kolmogorov equation is not satisfied

$$K_2(B_3,t_3,B_1,t_1)\neq\int_{-\infty}^{\infty}K_2(B_3,t_3,B_2,t_2)K_2(B_2,t_2,B_1,t_1)\,dB_2,\quad 0<t_1<t_2<t_3, \tag{14}$$

meaning that fBm is not a Markov process.

3. Stochastic differential equation based on fBm

For practical application, the Ito formula plays an important role, which in the case of fBm takes the form of [9,11,13,14,20,23]

$$\begin{aligned}df(\tau,B(\tau))&\approx\frac{\partial f(\tau,B(\tau))}{\partial\tau}d\tau+\frac{\partial f(\tau,B(\tau))}{\partial B(\tau)}dB(\tau)+\frac{1}{2}\frac{\partial^2f(\tau,B(\tau))}{\partial B(\tau)^2}(dB(\tau))^2\\&=\Big(\frac{\partial f(\tau,B(\tau))}{\partial\tau}+H\tau^{2H-1}\frac{\partial^2f(\tau,B(\tau))}{\partial B(\tau)^2}\Big)d\tau+\frac{\partial f(\tau,B(\tau))}{\partial B(\tau)}dB(\tau),\end{aligned} \tag{15}$$

where $f(\tau, B(\tau))$ $(\tau \in [0,t])$ is a stochastic function which has second order derivatives. In formula (15) the $(dB(\tau))^2 \approx H\tau^{2H-1}d\tau$ was used. For the function of stochastic value $r(\tau)$, which is given by the following stochastic equation

$$dr(\tau) = A(r(\tau))\,d\tau + \Sigma(r(\tau))\,dB(\tau), \tag{16}$$

the Ito formula has the following form

$$\begin{aligned} df(\tau, r(\tau)) &\approx \frac{\partial f(\tau, r(\tau))}{\partial \tau}d\tau + \frac{\partial f(\tau, r(\tau))}{\partial r(\tau)}dr(\tau) + \frac{1}{2}\frac{\partial^2 f(\tau, r(\tau))}{\partial r(\tau)^2}(dr(\tau))^2 \\ &= \left(\frac{\partial f(\tau, r(\tau))}{\partial \tau} + H\tau^{2H-1}\Sigma(r(\tau))^2\frac{\partial^2 f(\tau, r(\tau))}{\partial r(\tau)^2}\right)d\tau + \frac{\partial f(\tau, r(\tau))}{\partial r(\tau)}\,dr(\tau). \end{aligned} \tag{17}$$

For transition probability density for stochastic process (16), taking into account the Ito formula (15), (17) we obtain the Fokker–Planck equation [9, 14, 15, 18]

$$\frac{\partial K(r,t)}{\partial t} = \frac{1}{2}\dot{R}_H(t)\frac{\partial^2 \Sigma(r)^2 K(r,t)}{\partial r^2} - \frac{\partial A(r)K(r,t)}{\partial r}. \tag{18}$$

In case of $A(r(\tau)) = 0$, $\Sigma(r(\tau)) = 1$ the given equation matches (12). Also, for $H = 1/2$ we obtain that $\dot{R}_H(t) = 1$ and Fokker–Planck equation (18) are transformed into the equation of Brownian motion [2,4]. The general solution of equation (18) in the form of path integral is given in Appendix C.

Works [9, 13, 20] give a Black-Scholes differential equation for determining option price in case of "clean" fBm and some generalization based on it. As it is known, the Black–Scholes equation is an inverse Kolmogorov equation relative to the Fokker–Planck equation (18). Solutions to the Fokker–Planck equation (18) for transition probability density were researched in [14, 15, 18] for a stochastic differential equation of geometric Brownian motion with "clean" fBm and it is generalizations and also for CEV model with fBm. Based on found probability densities, generalized Black–Scholes formulas for option pricing were obtained. Specifics of Fokker–Planck equations (18) as well as of inverse Kolmogorov equation in mentioned works is a presence of a multiplier with derivative for fBm variation, or it is a generalization in a term with second derivative.

As we already noted, the transition probability density of a stochastic process one can obtain directly by using the measure of the process. It is obvious that the results obtained by the two approaches must be equal. However, as it will be shown, the transition probability density for stochastic value $r(\tau)$ (16) that is built based on Gaussian measure does not match the solution to the Fokker–Planck equation (18). Let us illustrate this for the case of stochastic equation (16) with constant $\Sigma(r(\tau)) = \sigma = \text{const}$.

Let us consider breakdown of time interval $(0 < t_1 < t_2 \ldots t_{n-1} < t_n = t)$ and write a discrete realization of stochastic process (16):

$$dr(t_i) = A(r(t_{i-1}))\,dt_i + \sigma\,dB_i, \quad i \in \{1, \ldots, n\}. \tag{19}$$

The following notations were used:

$$dt_i = t_i - t_{i-1}, \quad dr(t_i) = r(t_i) - r(t_{i-1}), \quad dB_i = B(t_i) - B(t_{i-1}), \quad i \in \{1, \ldots, n\}.$$

We shall find the Gaussian measure for the stochastic process $r(\tau)$ based on fBm measure (Appendix B, formula (55)) by means of variable substitution given by equation (19)

$$\mu(d\mathbf{r}) = (2\pi)^{-\frac{n}{2}} J(\{r_i\})\sqrt{\det(\delta^2 R)^{-1}} \exp\left(-\frac{1}{2\sigma^2}\sum_{i,j}(dr_i - A(r_{i-1})dt_i)(\delta^2\hat{R})^{-1}_{ij}(dr_j - A(r_{j-1})dt_j)\right). \tag{20}$$

Here we used the following notations: $dr_i = dr(t_i)$, $r_i = r(t_i)$, $i \in \{1, \ldots, n\}$, and $J(\{r_i\})$ denotes a Jacobian of variable substitution according to equation (19). The approach of calculating $J(\{r_i\})$ is given in works [19, 24, 25]. Transition probability density we obtain according to formula (9)

$$K(r,t) = \int_{-\infty}^{\infty} \mu(\mathbf{dr})\,\delta(r - r_n)\prod_{i=1}^{n} dr_i. \tag{21}$$

It is easy to see that multiple integral in (21) in limit $n \to \infty$, $\max(dt_i) \to 0$, $i \in \{0, \ldots, n\}$, is not possible to bring to the form (58), (59) (in mentioned formulas one should use $\Sigma(r) = \sigma$). Indeed, in exponent (21), we have a double integral sum; as a result, path integral will contain a double integral over time variable while in formulas (58), (59) in exponent we have only single integrals. It emerges that Fokker–Planck equation (18) corresponds to some approximation of fBm covariance (1), the form of which we shall further find out.

4. Approximation of fBm covariance

The approximation of fBm covariance that we look we shall find from the equality condition of found solutions based on the Fokker–Planck equation with the use of Gaussian measure (55) of fBm with precise covariance. It is easy to see that Fokker–Planck equation (18) corresponds to Gaussian measure with diagonal matrix while matrix $\delta^2\hat{R}$ (54) is not diagonal.

Let us write covariance (1) in an identical form

$$R_H(t,s) = \min(R_H(t), R_H(s)) + \frac{1}{2}\big(|R_H(t) - R_H(s)| - R_H(|t-s|)\big), \tag{22}$$

where $R_H(t) = t^{2H}$ is a fBm variation (see (11)). Let us consider the first term in (22) for covariance

$$\tilde{R}_H(t,s) = \min(R_H(t), R_H(s)). \tag{23}$$

Let us show that the Fokker–Planck equation (18) corresponds to a stochastic process with covariance (23). Let us point out that the transition probability densities for fBm with covariance (22) and (23) match (formula (11)).

Let us consider breakdown of time interval $(0 < t_1 < t_2, \ldots, t_{n-1} < t_n = t)$, then Gaussian measure of stochastic process is given by formula (8) with covariance matrix

$$\tilde{R}_{ij} = \tilde{R}_H(t_i, t_j), \quad (i,j) \in \{1,\ldots,n\}.$$

Let us use the Gaussian measure given by fBm increments (55). Based on formula (54) for matrix $\delta^2\tilde{R}$ we obtain that:

$$(\delta^2\tilde{R})_{ij} = \dot{R}_H(t_{i-1})dt_i\delta_{ij}, \quad (i,j) \in \{1,\ldots,n\}.$$

As a result, the Gaussian measure for stochastic process with covariance (23) we write in the form

$$\tilde{\mu}(d\mathbf{B}) = \prod_{i=1}^{n} \frac{dB_i}{\sqrt{2\pi\dot{R}_H(t_{i-1})dt_i}} \exp\left(-\frac{1}{2}\sum_{i=1}^{n}\frac{dB_i^2}{\dot{R}_H(t_{i-1})dt_i}\right). \tag{24}$$

In continuous case, in the limit $n \to \infty$, $\max(dt_i) \to 0$, $i \in \{0,\ldots,n\}$ for measure (24) we obtain that:

$$d\tilde{\mu}(B) = \mathcal{D}_H B(\tau)\exp\left(-\frac{1}{2}\int_0^t \frac{\dot{B}(\tau)^2}{\dot{R}_H(\tau)}d\tau\right), \quad \mathcal{D}_H B(\tau) = \prod_{\tau}\frac{dB(\tau)}{\sqrt{2\pi\dot{R}_H(\tau)d\tau}}. \tag{25}$$

It is obvious that in the case of $H = \frac{1}{2}$, the measure (25) matches with the Wiener measure for Brownian motion.

Hence, covariance (23) determines a stochastic process with the following properties:

$$\big\langle B(\tau)\big\rangle = 0, \quad \big\langle B(\tau)B(s)\big\rangle = \min(R_H(\tau), R_H(s)), \quad \big\langle B(\tau)^2\big\rangle = R_H(\tau).$$

Increments of stochastic on-time intervals that do not intersect are independent:

$$\big\langle (B(s_2) - B(s_1))(B(t_2) - B(t_1))\big\rangle = 0, \quad s_1 < s_2 < t_1 < t_2,$$

we also obtain that

$$\big\langle dB(\tau)^2\big\rangle = \dot{R}_H(\tau)\,d\tau. \tag{26}$$

The stochastic process with covariance (23) is also a Markov and martingale (following conditions are satisfied (13), (14)).

Based on measure (25), let us build a measure for stochastic process $r(\tau)$ that is given by a stochastic equation (16). For this, we shall use the approach for Brownian motion given in [19,25]. In particular, equation (16) we shall write in an integral form

$$B(\tau) = \int_0^\tau \frac{dr(\tau')}{\Sigma(r(\tau'))} - \int_0^\tau \frac{A(r(\tau'))}{\Sigma(r(\tau'))}\,d\tau'. \tag{27}$$

The first term in the right part (27) determines a stochastic integral, which we consider in the Ito sense. For this we introduce the following stochastic variable

$$x(\tau) - x(0) = \int_0^\tau \frac{dr(\tau')}{\Sigma(r(\tau'))} = \varphi(r(\tau)) - \varphi(r(0)). \tag{28}$$

Let us use the Ito formula (17) for increment $x(\tau)$. We obtain the following stochastic differential equation

$$dx(\tau) = \varphi'(r(\tau))dr(\tau) + \frac{1}{2}\varphi''(r(\tau))dr(\tau)^2 = \left(\frac{A(r(\tau))}{\Sigma(r(\tau))} - \frac{1}{2}\Sigma'(r(\tau))\dot{R}_H(\tau)\right)d\tau + dB(\tau). \tag{29}$$

In formula (29) it is taken into account that $dr(\tau)^2 \approx \Sigma(r(\tau))^2\dot{R}_H(\tau)\,d\tau$. As a result, the stochastic variable $x(\tau)$ satisfies the stochastic differential equation with constant volatility. This approach of transforming to stochastic value with constant volatility was proposed in [25]. In this case, we have a known solution for transition probability density in the form of path integral [19,24,25]. Let us rewrite equation (29) by defining stochastic variable $r(\tau)$ through $x(\tau)$

$$dx(\tau) = A_{eff}(x(\tau),\tau)\,d\tau + dB(\tau), \tag{30}$$

where $A_{eff}(x(\tau),\tau) = \frac{A(\varphi^{-1}(x(\tau)))}{\Sigma(\varphi^{-1}(x(\tau)))} - \frac{1}{2}\Sigma'\big(\varphi^{-1}(x(\tau))\big)\dot{R}_H(\tau)$.

As a result of stochastic equation (30) based on measure (25), we obtain a path integral for transition probability density

$$\tilde{K}(x,x_0,t) = \int_{x_0}^{x} \mathcal{D}_H x(\tau) \exp\left\{-\frac{1}{2}\int_0^t \frac{(\dot{x}(\tau) - A_{eff}(x(\tau)))^2}{\dot{R}_H(\tau)}\,d\tau - \frac{1}{2}\int_0^t A'_{eff}(x(\tau),\tau)\,d\tau\right\}. \tag{31}$$

Here $A'_{eff}(x(\tau),\tau)$ denotes a derivative over argument $x(\tau)$, and also

$$\mathcal{D}_H x(\tau) = \prod_\tau \frac{dx(\tau)}{\sqrt{2\pi\dot{R}_H(\tau)d\tau}}. \tag{32}$$

A term from $A'_{eff}(x(\tau),\tau)$ in exponent (31) is caused by Jacobian during variable substitution, after a transition from $B(\tau)$ to $x(\tau)$ [19,24,25].

Expression (31) gives transition probability density for the supplementary stochastic process $dx(\tau)$ (30). In order to obtain transition probability density $K(r,r_0,t)$ of stochastic process $dr(\tau)$ (16) let us use connections between them

$$K(r,r_0,t) = \frac{1}{\Sigma(r)}\tilde{K}(\varphi(r),\varphi(r_0),t).$$

Next, let us perform variable substitution according to (28) ($x(\tau) = \phi(r(\tau))$) in path integral (31). In particular, for expressions in (31) we obtain:

$$\dot{x}(\tau) = \frac{1}{\Sigma(r(\tau))}\dot{r}(\tau), \quad A_{eff}(x(\tau),\tau) = \frac{A(r(\tau))}{\Sigma(r(\tau))} - \frac{1}{2}\Sigma'(r(\tau))\dot{R}_H(\tau),$$

$$A'_{eff}(x(\tau)) = A'(r(\tau)) - A(r(\tau))\frac{\Sigma'(r(\tau))}{\Sigma(r(\tau))} - \frac{1}{2}\Sigma(r(\tau))\Sigma''(r(\tau))\dot{R}_H(\tau)\,.$$

The measure term (32) is replaced by

$$\mathcal{D}_H r(\tau) = \prod_\tau \frac{dr(\tau)}{\sqrt{2\pi\Sigma^2(r(\tau))\dot{R}_H(\tau)d\tau}}.$$

Performing required calculations (see for details in [19]) for transition probability density, we obtain a path integral form given in formulas (58), (59), (60). This way, the Fokker–Planck equation (18) corresponds to approximation (23) for covariance matrix fo fBm.

As it can be seen, for approximation of covariance (23), the problem of finding the measure of stochastic process solves analytically. In the case of fBm covariance (1), (22), with non-diagonal matrix, the difficulties of finding Gaussian measure are related to obtaining an inverse matrix (8), (20), (55). Let us note that the stochastic process with covariance (23) sets independent increases unlike the fBm (4) which has a long-term time dependency between increases (4). The main difference of stochastic process with covariance (23) from Brownian motions is that the average value of (26) contains an additional multiplier that depends on time. It is easy to show that the stochastic equation (16) is effectively equivalent to the Brownian motion equation with the substitution $dB(\tau) \to \sqrt{\dot{R}_H(\tau)dW(\tau)}$ (we consider the case $H > \frac{1}{2}$).

From formula (22) results that covariance are slightly different $R_H(\tau, s) \approx \tilde{R}_H(\tau, s)$ for close times $\tau \approx s$. Meaning that the second term in (22) one can consider as a perturbation and build an approximation for inverse matrix (54) and for the measure of the process (55).

5. Examples of solving some of the fBm stochastic models

In many financial engineering problems, finding a transition probability density given by a stochastic differential equation is important. Let us bring up some of the solutions to known models, generalized for the case of fBm in the path integral method.

Geometric fractional Brownian motion. This model is a generalization for the known model of geometric Brownian motion and was considered in [9, 13, 18, 20]

$$dS(\tau) = rS(\tau) + \sigma S(\tau)\, dB(\tau). \tag{33}$$

Here $S(\tau)$ is value of option price, r is interest rate, σ is price volatility, $dB(\tau)$ is fBm variable. The transition probability density for stochastic equation (33) is given by path integral (58). Let us substitute respective values from (33). We obtain the following

$$K(S, S_0, t) = \sqrt{\frac{S_0}{\sigma^2 S^3}} \int_{S_0}^{S} \mathcal{D}S(\tau) \exp\left(-\frac{1}{2} \int_{t_0}^{t} \frac{\left(\dot{S}(\tau) - rS(\tau)\right)^2}{\sigma^2 S(\tau)^2 \dot{R}_H(\tau)} d\tau + \frac{1}{2} \int_{t_0}^{t} \left(r - \frac{1}{4}\sigma^2 \dot{R}_H(\tau) \right) d\tau \right),$$

where the measuring element of integral is denoted as

$$\mathcal{D}S(\tau) = \prod_{\tau} \frac{dS(\tau)}{\sqrt{2\pi\sigma^2 S(\tau)^2 \dot{R}_H(\tau) d\tau}}. \tag{34}$$

By variable substitution $S(\tau) \to \exp(x(\tau))$ and $x(\tau) \to x(\tau) + r\tau$ the path integral is transformed in the following way

$$\int_{S_0}^{S} \mathcal{D}S(\tau) \exp\left(-\frac{1}{2} \int_{t_0}^{t} \frac{\left(\dot{S}(\tau) - rS(\tau)\right)^2}{\sigma^2 S(\tau)^2 \dot{R}_H(\tau)} d\tau \right) = \int_{x_0}^{x} \mathcal{D}x(\tau) \exp\left(-\frac{1}{2} \int_{t_0}^{t} \frac{\dot{x}(\tau)^2}{\sigma^2 \dot{R}_H(\tau)} d\tau \right). \tag{35}$$

In formula (35) the following measure element is denoted

$$\mathcal{D}x(\tau) = \prod_{\tau} \frac{dx(\tau)}{\sqrt{2\pi\sigma^2 \dot{R}_H(\tau)\, d\tau}}. \tag{36}$$

For the path integral in the right part of (35), we shall use the known solution [26]; as a result, we obtain the following for transition probability density

$$K(S, S_0, t) = \frac{\exp\left(\frac{r}{2}(t - t_0) - \frac{\sigma^2}{8}\left(R_H(t) - R_H(t_0)\right) \right)}{\sqrt{2\pi\sigma^2 \left(R_H(t) - R_H(t_0)\right)}} \sqrt{\frac{S_0}{S^3}} \exp\left(-\frac{1}{2\sigma^2} \frac{\left(r(t - t_0) - \ln \frac{S}{S_0}\right)^2}{\left(R_H(t) - R_H(t_0)\right)} \right). \tag{37}$$

The formula for the European Call option we obtain is based on (37) by integrating over pay function

$$C(S_0, t) = \int_{K}^{\infty} K(S, S_0, t)(S - K)dS,$$

and K is the strike price.

After calculations, we obtain the Black–Scholes formula in the case of fBm, which is given in [18].

CEV model. The CEV model introduces a more complicated dependency of volatility on price magnitude in a model of geometric Brownian motion. For fBm it was considered in [14, 15]

$$dS(\tau) = rS(\tau) + \sigma S(\tau)^{\alpha} dB(\tau).$$

Name of the model is related to the fact that $\frac{d\ln(S(\tau)^{\alpha})}{d\ln S(\tau)} = \alpha$ is a constant (volatility elasticity) $0 < \alpha < 2$ and also $r > 0$, $\sigma > 0$. Based on (58) we shall find for transition probability density that

$$K(S,S_0,t) = \exp\Big(\frac{r}{2}(2\alpha-1)\Big)\sqrt{\frac{S_0}{\sigma^2 S^3}}\int_{S_0}^{S}\mathcal{D}S(\tau)\exp\bigg(-\frac{1}{2\sigma^2}\int_{t_0}^{t}\frac{\big(\dot{S}(\tau)-rS(\tau)\big)^2}{S(\tau)^{2\alpha}\dot{R}_H(\tau)}d\tau\bigg) \times \exp\bigg(-\frac{1}{8}(2-\alpha)\alpha\sigma^2\int_{t_0}^{t}\frac{\dot{R}_H(\tau)}{S^{2-2\alpha}}d\tau\bigg). \tag{38}$$

Here measure element $\mathcal{D}S(\tau)$ is given by expression (34) with substitution under square root $S(\tau)^2 \to S(\tau)^{2\alpha}$. The calculation of path integral is performed in two steps. First we perform varialbe substitution $S(\tau) \to ((1-\alpha)\sigma x(\tau))^{\frac{1}{1-\alpha}}$ (we consider the case $0 < \alpha < 1$). The path integral in (38) will convert to

$$\int_{x_0}^{x}\mathcal{D}x(\tau)\exp\bigg(-\frac{1}{2}\int_{t_0}^{t}\frac{(\dot{x}(\tau)+(\alpha-1)rx(\tau))^2}{\dot{R}_H(\tau)}d\tau - \frac{(2-\alpha)\alpha}{8(\alpha-1)^2}\int_{t_0}^{t}\frac{\dot{R}_H(\tau)}{x(\tau)^2}d\tau\bigg), \tag{39}$$

where measure element $\mathcal{D}x(\tau)$ is defined in formula (36). Next step let us perform variable substitution $x(\tau) \to x_1(\tau)e^{-r(\alpha-1)\tau}$ in integral (39). As a result we obtain

$$e^{-\frac{r}{2}(1-\alpha)(t+t_0)}\int_{x_{10}}^{x_1}\mathcal{D}x_1(\tau)\exp\bigg(-\frac{1}{2}\int_{t_0}^{t}\frac{\dot{x}_1(\tau)^2}{\dot{D}(\tau)}d\tau - \frac{(\lambda^2-\frac{1}{4})}{2}\int_{t_0}^{t}\frac{\dot{D}(\tau)}{x_1(\tau)^2}d\tau\bigg). \tag{40}$$

In formula (40) following notations were introduced $\lambda^2 = \frac{1}{4(\alpha-1)^2}$, $\dot{D}(\tau) = e^{2(\alpha-1)r\tau}\dot{R}_H(\tau)$. Measure element $\mathcal{D}x_1(\tau)$ defined as in formula (36) with substitution $\dot{R}_H(\tau) \to \dot{D}(\tau)$ and also integral (40) is considered in bounds $x_{10} = \frac{e^{-(1-\alpha)rt_0}}{\sigma(1-\alpha)}S_0^{1-\alpha}$, $x_1 = \frac{e^{-(1-\alpha)rt}}{\sigma(1-\alpha)}S^{1-\alpha}$.

Next, we will change the parametrization of "trajectories" in integral (40) by time variable substitution $\eta_1 = \int \dot{D}(\tau)d\tau = D(\tau)$ and will introduce new variable $x_1(\tau) = \xi(\eta_1)$ ($\tau \in [t_0, t]$, $\eta_1 \in [\eta_0, \eta]$). Since for $\dot{D}(\tau) > 0$ an unequivocal transform of interval $[t_0, t]$ into $[\eta_0, \eta]$ exists. As a result of the mentioned transformation, we obtain path integral with a known value [26]

$$\int_{\xi_0}^{\xi}\mathcal{D}\xi(\eta)\exp\bigg(-\frac{1}{2}\int_{\eta_0}^{\eta}\dot{\xi}(\eta_1)^2 d\eta_1 - \frac{(\lambda^2-\frac{1}{4})}{2}\int_{\eta_0}^{\eta}\frac{d\eta_1}{\xi(\eta_1)^2}\bigg) = \frac{\sqrt{\xi\xi_0}}{\eta-\eta_0}\exp\bigg(-\frac{\xi^2+\xi_0^2}{2(\eta-\eta_0)}\bigg)I_\lambda\bigg(\frac{\xi\xi_0}{\eta-\eta_0}\bigg),$$

where $I_\lambda(x)$ is a modified Bessel function. We also take into account that $\eta = D(t)$, $\eta_0 = D(t_0)$, $x_1 = \xi$, $x_{10} = \xi_0$.

To sum up, given transformations for transition probability density, we obtain the following expression

$$K(S,S_0,t) = \frac{e^{-\frac{1}{2}r(t_0+(3-4\alpha)t)}S^{-2\alpha}\sqrt{SS_0}}{(1-\alpha)\sigma^2(D(t)-D(t_0))}\exp\bigg(-\frac{S^{2-2\alpha}e^{-2(1-\alpha)rt}+S_0^{2-2\alpha}e^{-2(1-\alpha)rt_0}}{2(1-\alpha)^2\sigma^2(D(t)-D(t_0))}\bigg) \times I_\lambda\bigg(\frac{e^{-(1-\alpha)r(t+t_0)}(SS_0)^{1-\alpha}}{(\alpha-1)^2\sigma^2(D(t)-D(t_0))}\bigg). \tag{41}$$

After calculations we obtain that $D(t) - D(t_0) = \frac{2H}{(2r(1-\alpha))^{2H}}\big(\Gamma(2H, 2r(1-\alpha)t_0) - \Gamma(2H, 2r(1-\alpha)t)\big)$, $\Gamma(\nu, x)$ is a partial Gamma function. Transition probability density (41) was obtained in [14] by a slightly different approach.

Vasicek model. The stochastic differential equation of the Vasicek model is the following

$$dr(\tau) = \beta(\mu - r(\tau)) + \sigma dB(\tau), \tag{42}$$

were β, μ, $\sigma > 0$ are model parameters. The stochastic equation is used in modelling the time structure of interest rate [1]. Besides that, (42) is known as Ornstein–Uhlenbeck equation [22, 27] and has a wide range of applications. For transition probability density after substituting respective values of the model into formula (58), we obtain

$$K(r,r_0,t) = \exp\left(\frac{1}{2}\beta(t-t_0)\right)\int_{r_0}^{r}\mathcal{D}r(\tau)\exp\left(-\frac{(\dot{r}(\tau)+\beta r(\tau)-\beta\mu)^2}{2\sigma^2\dot{R}_H(\tau)}\right). \tag{43}$$

Here measure element $\mathcal{D}r(\tau)$ is given by formula (36). In the path integral inside (43), we shall perform a variable substitution $r(\tau) \to r_1(\tau)\exp(-\beta\tau)+\mu$. As a result we obtain

$$\int_{r_0}^{r}\mathcal{D}r(\tau)\exp\left(-\frac{(\dot{r}(\tau)+\beta r(\tau)-\beta\mu)^2}{2\sigma^2\dot{R}_H(\tau)}\right) = e^{\frac{1}{2}\beta(t+t_0)}\int_{r_{10}}^{r_1}\mathcal{D}r_1(\tau)\exp\left(-\frac{1}{2}\int_{t_0}^{t}\frac{\dot{r}_1(\tau)^2}{\sigma^2\dot{D}(\tau)}d\tau\right). \tag{44}$$

In formula (44) the following notation was introduced $\dot{D}(\tau) = e^{2\beta\tau}\dot{R}_H(\tau)$, $r_{10} = e^{\beta t_0}(r_0-\mu)$, $r_1 = e^{\beta t}(r-\mu)$, where measure element $\mathcal{D}r(\tau)$ is given by formula (36) with substitution $\dot{R}_H(\tau) \to \dot{D}(\tau)$. For path integral in the right part (44), we shall use a known value as in (35). As a result, we obtain

$$K(r,r_0,t) = \frac{e^{\beta t}}{\sqrt{2\pi\sigma^2(D(t)-D(t_0))}}\exp\left(-\frac{1}{2}\frac{\big((r-\mu)e^{\beta t}+(\mu-r_0)e^{\beta t_0}\big)^2}{\sigma^2(D(t)-D(t_0))}\right). \tag{45}$$

Here we need to take into account that $D(t)-D(t_0) = \int_{t_0}^{t}\exp(2\beta\tau)\dot{R}_H(\tau)d\tau$.

By direct verification, one can validate that given solutions (37), (41), (45) satisfy Fokker–Planck equation (18).

6. Conclusions

This work considers some specifics of fBm application to models of financial engineering. A known Fokker–Planck equation is researched for stochastic differential equations based on fBm. For the specified equation, a solution was built for transition probability density in the form of the path integral. The measure of the fBm stochastic process was researched. It was shown that transition probability density determined based on process measure and based on mentioned Fokker–Planck equation do not match. It emerges that the Fokker–Planck equation corresponds to a measure with different covariance. A form of mentioned covariance and its respective measure were found, and the transition probability density of the stochastic process in the form of path integral was determined. Path integral found with the two approaches match. Characteristics of a stochastic process given by fBm covariance approximation were found. An approach of specification of approximation by perturbation series for inverse covariance matrix is shown. Examples of solutions for transition probability density for known models generalized for the fBm case in the method of path integral are given.

Appendix A

Let us consider some fBm implementation on time interval $\tau \in [0,t]$. Then equation (5) for a discrete case will have the form

$$B(t_i) = \sum_{j<i}^{n} K_H(t_i,t_j)\,dW_j.$$

Here time moments t_i, $i \in \{1,\dots,n\}$ are ordered on an interval $(0 < t_1 < t_2 < \dots t_{n-1} < t_n = t)$, $dW_i = W_i - W_{i-1}$, $i \in \{1,\dots,n\}$ are values of variables of Brownian motion on that interval. The Gaussian measure of fBm we shall determine by averaging the Wiener measure

$$\mu(\mathbf{B}) = \left\langle \prod_{i=1}^{n}\delta\big(B_i - B(t_i)\big)\right\rangle_W. \tag{46}$$

The following is denoted:

$$\Big\langle(\dots)\Big\rangle_W = \int_{-\infty}^{\infty}\exp\left(-\frac{1}{2}\sum_{i=1}^{n}\frac{dW_i^2}{dt_i}\right)(\dots)\prod_{i=1}^{n}\frac{dW_i}{\sqrt{2\pi dt_i}}, \quad dt_i = t_i - t_{i-1}, \quad i \in \{1,\dots,n\}. \tag{47}$$

Using integral representations for δ-functions in (46)

$$\mu(\mathbf{B}) = \int_{-\infty}^{\infty} \prod_{i=1}^{n} \frac{dX_i}{2\pi} e^{\mathrm{i}\sum_{i=1}^{n} X_i B_i} \left\langle e^{-\mathrm{i}\sum_{i=1}^{n} X_i B(t_i)} \right\rangle_W, \tag{48}$$

and also transformation in a term in exponent (48)

$$\sum_{i=1}^{n} X_i B(t_i) = \sum_{i=1}^{n} X_i \sum_{j<i}^{n} K_H(t_i, t_j) dW_j = \sum_{j=1}^{n} \left(\sum_{i \geqslant j}^{n} X_i K_H(t_i, t_j) \right) dW_j,$$

after averaging over Wiener measure for the multiplier in (48), we obtain

$$\left\langle e^{-\mathrm{i}\sum_{i=1}^{n} X_i B(t_i)} \right\rangle_W = e^{-\frac{1}{2}\sum_{j=1}^{n} \left(\sum_{i \geqslant j}^{n} X_i K_H(t_i, t_j) \right)^2 dt_j}. \tag{49}$$

Expression in exponent (49) we shall transform in the following way

$$\sum_{j=1}^{n} \left(\sum_{i \geqslant j}^{n} X_i K_H(t_i, t_j) \right)^2 dt_j = \sum_{i=1}^{n} \sum_{l=1}^{n} \left(\sum_{j=1}^{i} K_H(t_i, t_j) K_H(t_l, t_j) \theta(l-j) dt_j \right) X_i X_l. \tag{50}$$

Here we use the Heaviside function $\theta(l-j)$ that selects elements from $l > j$.

Expression in the right side of equation (50) one should consider for $l > i$ and $l < i$. It is then easy to see according to (6) that specified expression is an integral sum for $R_H(t_i, t_j)$ for $n \to \infty$, $\max(dt_i) \to 0$, $i \in \{1, \ldots, n\}$. As a result we obtain

$$\sum_{j=1}^{i} K_H(t_i, t_j) K_H(t_l, t_j) \theta(l-j) dt_j \to R_H(t_i, t_l). \tag{51}$$

Substituting (51) into (50) and accordingly into (49) after integrating over variables X_i, $i \in \{1, \ldots, n\}$ transition probability fBm (46), we obtain expression (8).

Appendix B

As before let us set the time interval breakdown $\{0 < t_1 < t_2 < \ldots < t_n = t\}$, value of conditional quantity of fBm $\mathbf{B} = \{B_1, B_2, \ldots, B_n\}$ at certain moments of time and their increments $d\mathbf{B} = \{B_1, B_2 - B_1, \ldots, B_n - B_{n-1}\}$. To set the transition probability density (8) based on increments $d\mathbf{B}$, it is convenient to convert to matrix transformation. Obviously, the following equation is valid

$$\mathbf{B} = \hat{L}\, d\mathbf{B}, \tag{52}$$

where $\hat{L}$ is a lower triangular matrix with all elements equal to zero,

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ . & . & . & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix},$$

vectors $\mathbf{B}$, $d\mathbf{B}$ are considered as vector columns. Quadratic form in the exponent of formula (8) we shall write down in matrix form

$$\mathbf{B}^T \hat{R}^{-1} \mathbf{B} = d\mathbf{B}^T \hat{L}^T \hat{R}^{-1} \hat{L}\, d\mathbf{B} = d\mathbf{B}^T (\delta^2 \hat{R})^{-1} d\mathbf{B}.$$

Here the following matrix is denoted

$$\delta^2 \hat{R} = \hat{L}^{-1} \hat{R} (\hat{L}^{-1})^T, \tag{53}$$

where symbol T denotes matrix transpose. Inverse matrix $\hat{L}^{-1}$ is easy to find based on correlation (52). As a result $\hat{L}^{-1}$ is a lower triangular matrix for which $\hat{L}_{ii}^{-1} = 1$, $\hat{L}_{i+1,i}^{-1} = -1$, $i \in \{1, \ldots, n\}$,

$$L^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ . & . & . & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}.$$

This way for the matrix elements $(\delta^2 \hat{R})_{ij}$ we obtain

$$(\delta^2 \hat{R})_{ij} = \hat{R}_{ij} - \hat{R}_{i-1,j} - \hat{R}_{i,j-1} + \hat{R}_{i-1,j-1}. \tag{54}$$

To sum up, for probability density (8) we obtain

$$\mu(d\mathbf{B}) = (2\pi)^{-\frac{n}{2}} \sqrt{\det(\delta^2 R)^{-1}} \exp\bigg(-\frac{1}{2} \sum_{i,j} dB_i (\delta^2 \hat{R})^{-1}_{ij} dB_j \bigg). \tag{55}$$

From formula (53) it results that $\det \delta^2 \hat{R} = \det \hat{R}$.

We shall note that for Brownian motion ($H = \frac{1}{2}$) we have

$$\hat{R}^0_{ij} = R_{\frac{1}{2}}(t_i, t_j) = \min(t_i, t_j).$$

It is easy to show then that for $\delta \hat{R}^0_{ij}$ using formula (54) we obtain

$$(\delta^2 \hat{R}^0)_{ij} = dt_i \delta_{ij}; \quad (\delta^2 \hat{R}^0)^{-1}_{ij} = \frac{1}{dt_i} \delta_{ij}; \quad dt_i = t_i - t_{i-1}, \quad (i,j) \in \{1, \dots, n\}.$$

It is obvious that after substituting matrix elements $\delta \hat{R}^{0-1}_{ij}$ into (8) we obtain Wiener measure that is used in formula (47).

Appendix C

Equation (18) we shall rewrite using time variable substitution $\tau = R_H(t)$. For "time" variable τ we obtain the following Fokker–Planck equation

$$\frac{\partial \tilde{K}(r,\tau)}{\partial \tau} = \frac{1}{2} \frac{\partial^2 \Sigma(r)^2 \tilde{K}(r,\tau)}{\partial r^2} - \frac{\partial \tilde{A}(r,\tau) \tilde{K}(r,\tau)}{\partial r}, \tag{56}$$

where $\tilde{K}(r,\tau) = K(r,t)$, $\tilde{A}(r,\tau) = \frac{A(r)}{2H\tau^{1-\frac{1}{2H}}}$. Let us consider solution to equation (56) on "time" interval $[0,\tau]$ that corresponds to interval $[0,t]$ (we consider case when $\frac{1}{2} < H < 1$). In work, [19] solution to transition probability density for an equation of type (56) in the form of path integral is given

$$\tilde{K}(r, r_0, \tau) = \sqrt{\frac{\Sigma(r_0)}{\Sigma(r)^3}} \int_{r_0}^{r} \tilde{\mathcal{D}} r(\tau_1) \exp\bigg(-\frac{1}{2} \int_0^\tau \Big(\frac{\dot{r}(\tau_1) - \tilde{A}(r(\tau_1), \tau_1)}{\Sigma(r(\tau_1))} \Big)^2 d\tau_1 - \int_0^\tau u_0(r(\tau_1))\, d\tau_1 \bigg) \tag{57}$$

with the following notations

$$u_0(r(\tau)) = \frac{1}{2} \tilde{A}'_r(r(\tau), \tau) - \tilde{A}(r(\tau), \tau) \frac{\Sigma'(r(\tau))}{\Sigma(r(\tau))} + \frac{1}{8} \Sigma'(r(\tau))^2 - \frac{1}{4} \Sigma(r(\tau)) \Sigma''(r(\tau)).$$

A measure element (57) becomes the following

$$\tilde{\mathcal{D}} r(\tau) = \prod_\tau \frac{dr(\tau)}{\sqrt{2\pi \Sigma(r(\tau))^2 d\tau}}.$$

Inversing in formula (57) to time variable t we obtain solution to the Fokker–Planck equation (18) in path integral form

$$K(r, r_0, t) = \sqrt{\frac{\Sigma(r_0)}{\Sigma(r)^3}} \int_{r_0}^{r} \mathcal{D} r(\tau) \exp\bigg(-\frac{1}{2} \int_0^t \frac{\big(\dot{r}(\tau) - A(r(\tau))\big)^2}{\dot{R}_H(\tau) \Sigma(r(\tau))^2} d\tau - \int_0^t u(r(\tau)) d\tau \bigg), \tag{58}$$

where

$$u(r(\tau)) = \frac{1}{2} A'(r(\tau)) - A(r(\tau)) \frac{\Sigma'(r(\tau))}{\Sigma(r(\tau))} + \dot{R}_H(\tau) \frac{1}{8} \Big(\Sigma'(r(\tau))^2 - 2\Sigma(r(\tau)) \Sigma''(r(\tau)) \Big) \tag{59}$$

and also measure element (58) is equal to

$$\tilde{\mathcal{D}} r(\tau) = \prod_\tau \frac{dr(\tau)}{\sqrt{2\pi \dot{R}_H(\tau) \Sigma(r(\tau))^2 d\tau}}. \tag{60}$$

[1] Lyuu Y.-D. Financial Engineering and Computation: Principles, Mathematics, and Algorithms. Cambridge University Press (2004).

[2] Gardiner C. Handbook of Stochastic Methods for Springer Berlin, Heidelberg. Springer Berlin, Heidelberg (2004).

[3] Paul W., Baschnagel J. Stochastic Processes. From Physics to Finance. Springer Cham (2013).

[4] Øksendal B. Stochastic Differential Equations: An Introduction with Applications. Springer Berlin, Heidelberg (2003).

[5] Sawal A. S., Ibrahim S. N., Laham M. F. The valuation of knock-out power calls under Black–Scholes framework. Mathematical Modeling and Computing. **9** (1), 57–64 (2022).

[6] Sawal A. S., Ibrahim S. N. I., Roslan T. R. N. Pricing equity warrants with jumps, stochastic volatility, andstochastic interest rates. Mathematical Modeling and Computing. **9** (4), 882–891 (2022).

[7] Yanishevskyi V. S., Nodzhak L. S. The path integral method in interest rate models. Mathematical Modeling and Computing. **8** (1), 125–136 (2021).

[8] Hassler U. Stochastic Processes and Calculus: An Elementary Introduction with Applications. Springer Cham (2016).

[9] Umarov S., Hahn M., Kobayashi K. Beyond the triangle: Brownian motion, Ito calculus, and Fokker–Planck equation — Fractional Generalizations. World Scientific Publishing Co. Pte. Ltd. (2018).

[10] Nourdin I. Selected Aspects of Fractional Brownian Motion. Springer Milano (2012).

[11] Bender C. An Ito formula for generalized functionals of a fractional Brownian motion with arbitrary Hurst parameter. Stochastic Processes and their Applications. **104** (1), 81–106 (2003).

[12] Nualart D. The Malliavin Calculus and Related Topics. Springer Berlin, Heidelberg (2006).

[13] Ahmadian D., Ballestra L. V. Pricing geometric Asian rainbow options under the mixed fractional Brownian motion. Physica A: Statistical Mechanics and its Applications. **555**, 124458 (2020).

[14] Araneda A. A. The fractional and mixed-fractional CEV model. Journal of Computational and Applied Mathematics. **363**, 106–123 (2020).

[15] Araneda A. A., Bertschinger N. The sub-fractional CEV model. Physica A: Statistical Mechanics and its Applications. **573**, 125974 (2021).

[16] Ibrahim S. N. I., Laham M. F. Call warrants pricing formula under mixed-fractional Brownian motion with Merton jump-diffusion. Mathematical Modeling and Computing. **9** (4), 892–897 (2022).

[17] Herzog B. Adopting Feynman–Kac Formula in Stochastic Differential Equations with (Sub-)Fractional Brownian Motion. Mathematics. **10** (3), 340 (2022).

[18] Araneda A. A. European option pricing under generalized fractional Brownian motion. ArXiv:2108.12042v1 (2021).

[19] Yanishevskyi V. S., Baranovska S. P. Path integral method for stochastic equations of financial engineering. Mathematical Modeling and Computing. **9** (1), 166–177 (2022).

[20] Osu B. O., Ifeoma C. A. Fractional Black Scholes Option Pricing with Stochastic Arbitrage Return. International Journal of Partial Differential Equations and Applications. **4** (2), 20–24 (2016).

[21] Calvo I.,Sánchez R., Carreras B. A. Fractional Lévy motion through path integrals. Journal of Physics A: Mathematical and Theoretical. **42** (5), 055003 (2009).

[22] Mishura Y., Zili M. Stochastic Analysis of Mixed Fractional Gaussian Processes. ISTE Press Ltd, London, and Elsevier Ltd, Oxford (2018).

[23] Yan L., Shen G., He K. Itô's formula for a sub-fractional Brownian motion. Communications on Stochastic Analysis. **5** (1), 135–159 (2011).

[24] Chaichian M., Demichev A. Path integrals in physics. Stochastic processes and quantum mechanics. CRC Press (2001).

[25] Goovaertsa M., Schepper A. D., Decampsa M. Closed-form approximations for diffusion densities: an integral path approach. Journal of Computational and Applied Mathematics. **164–165**, 337–364 (2004).

[26] Kleinert H. Path Integrals in Quantum Mechanics, Statistics, Polymer Physics, and Financial Markets. World Scientific (2004).

[27] Ascione G., Mishura Y., Pirozzi E. Fractional Ornstein-Uhlenbeck Process with Stochastic Forcing, and its Applications. Methodology and Computing in Applied Probability. **23**, 53–84 (2021).

Фрактальний броунівський рух в моделях фінансової інженерії

Янішевський В. С., Ноджак Л. С.

Національний університет "Львівська політехніка", вул. С. Бандери, 12, 79013, Львів, Україна

Розглянуто застосування фрактального броунівського руху (ФБР) в стохастичних моделях фінансової інженерії. Для відомого з літературних джерел рівняння Фоккера–Планка для випадку ФБР побудовано розв'язок для густини умовної ймовірності у методі функціонального інтегрування. Показано, що зазначений розв'язок не випливає з гаусової міри ФБР з точною коваріацією. Знайдено вигляд апроксимації коваріації ФБР, для якої розв'язки знайдені на основі гаусової міри ФБР і відомого рівняння Фоккера–Планка співпадають.

Ключові слова: *фрактальний броунівський рух; стохастичне рівняння; густина умовної ймовірності; рівняння Фоккера–Планка; функціональний інтеграл.*