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# Complex dynamics and chaos control in a nonlinear discrete prey–predator model

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The dynamics of prey–predator interactions are often modeled using differential or difference equations. In this paper, we investigate the dynamical behavior of a two-dimensional discrete prey-predator system. The model is formulated in terms of difference equations and derived by using a nonstandard finite difference scheme (NSFD), which takes into consideration the non-overlapping generations. The existence of fixed points as well as their local asymptotic stability are proved. Further, it is shown that the model experiences Neimark–Sacker bifurcation (NSB for short) and period-doubling bifurcation (PDB) in a small neighborhood of the unique positive fixed point under certain parametric conditions. This analysis utilizes bifurcation theory and the center manifold theorem. The chaos produced by NSB and PDB is stabilized. Finally, we use numerical simulations and computer analysis to check our theories and show more complex behaviors.

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## 1. Introduction

Predator–prey interactions are the most well-studied processes in ecological systems, and they are crucial to understanding community dynamics and ecosystems. Interactions in populations with overlapping generations are typically described by differential equations due to the continuous nature of birth and death processes in such populations [1–4]. Births occur in regular breeding seasons for species with synchronized generations, such as insects and birds. As a result, discrete-time models with difference equations are best suited to represent their interactions [5–11]. To examine the analytical aspects of a solution that is difficult to calculate, different schemes can be implemented for discretizing a continuous dynamical system, and discussing the numerical solution may be performed. Usually, the most commonly used methods are piecewise constant arguments and the forward Euler scheme to achieve the desired discrete-time counterparts of continuous-time models. Usually, the most commonly used methods are piecewise constant arguments and the forward Euler scheme to achieve the desired discrete-time counterparts of continuous-time models. These methods, however, are not dynamically consistent with their continuous counterparts. Some recent works on discrete-time models can be found in, among many others, [12–18].

Depending on the interactions between different species and the availability of nutrients, different functional responses are considered in ecological population models [19]. A realistic system model relies heavily on the selection of a functional response [20–22]. In this paper, we consider the following Leslie–Gower model, introduced and investigated in [23]:

$$\frac{dx}{dt} = rx\left(1-\frac{x}{K}\right) - \frac{mx}{A+x}y, \tag{1}$$

$$\frac{dy}{dt} = s\left(1-\frac{hy}{x}\right)y, \tag{2}$$

where $x \geqslant 0$ and $y \geqslant 0$ represent the prey and predator densities, respectively. The parameters $r$, $K$, $m$, $s$, $h$ are positive constants that respectively represent the intrinsic growth rate, the carrying capacity of the prey, the rate of predation on the prey, the intrinsic growth rate of the predator, and the number of prey required to support one predator at equilibrium. In Eq. (1), the functional response ($\frac{mx}{A+x}$) is type 2, proposed by Holling [19] for *non-learning* predators.

Regarding the dynamical consistency of dynamical properties, the authors in [24] investigated some biological systems by applying nonstandard finite difference schemes (NFDS) of Mickens type [25]. In the same way, in [26], Lui and Elaydi proposed and investigated discrete-time prey–predator models of competitive and comparative systems, derived following NFDS Mickens scheme. Additional interesting papers on the NSFD scheme can be found in [27–31] and the bibliographies therein.

Motivated by the previous cited works, we apply the NSFD scheme to (1)–(2), we obtain:

$$\frac{x_{t+1}-x_t}{\phi_1(\delta)} = rx_t - \frac{r}{K}x_t x_{t+1} - \frac{mx_t}{A+x_t}y_t, \tag{3a}$$

$$\frac{y_{t+1}-y_t}{\phi_2(\delta)} = sy_t - sh\frac{y_t y_{t+1}}{x_t}, \tag{3b}$$

with

$$\phi_1(\delta) = \frac{\exp(r\delta)-1}{r}, \quad \phi_2(\delta) = \frac{\exp(s\delta)-1}{s}.$$

Developing the system (3), one gets

$$x_{t+1} = \frac{\exp(r\delta)x_t - \frac{mx_t(\exp(r\delta)-1)}{r(A+x_t)}y_t}{1+\beta x_t}, \tag{4}$$

$$y_{t+1} = \frac{\exp(s\delta)x_t y_t}{x_t + (\exp(s\delta)-1)hy_t}. \tag{5}$$

The goal of this research is to find the system's fixed points (4)–(5) and analyze the asymptotic stability conditions of these fixed points. Furthermore, the interesting aspect of this study, is to prove rigorously, by using center manifold theory, that the system possesses NSB and PDB. In particular, we employ two chaos control techniques to completely eliminate or delay the chaos induced by bifurcation.

To sum up, the paper is organized as follow: in Section 2, the existence and the asymptotic stability of the fixed points are investigated. In Section 3, the PDB is established analytically by using center manifold theory. The existence of NSB is proved analytically in Section 4. The chaos control is developed in Section 5. Detailed numerical simulations are developed to support the analytical findings in Section 6. Finally, Section 7 makes the conclusion to this paper.

## 2. Stability analysis of system (4)–(5)

To achieve the fixed point of the system (4)–(5), we need to solve the following algebraic system

$$x = \frac{\exp(r\delta)x - \frac{mx(\exp(r\delta)-1)}{r(A+x)}y}{1+\beta x}, \quad y = \frac{\exp(s\delta)xy}{x+(\exp(s\delta)-1)hy}. \tag{6}$$

The following Lemma summarizes the fixed points obtained from (6), as well as their conditions of existence.

**Lemma 1.** *The system (4)–(5) has a boundary fixed point and a unique positive fixed point in* $\mathbb{R}^2_+$. *More precisely,*

- *For all positive parametric values, the system has a boundary fixed point noted* $A(K,0)$.
- *If* $\frac{m}{hr}+\frac{A}{K}>1$ *then system has a unique positive fixed point* $B\left(\left(K(1-\frac{m}{hr})-A\right)+\frac{K\sqrt{\Delta}}{hr}, \frac{x^*}{h}\right)$, *where*

$$\Delta = \left(m + hr\left(\frac{A}{K}-1\right)\right)^2 + 4\frac{(hr)^2}{K}A.$$

The Jacobian matrix of the system (4)–5) evaluated at any fixed point $(x,y)$ is given by

$$J(x,y) = \begin{pmatrix} j_{11} \equiv \frac{\exp(r\delta)(A+x)^2+\phi_1(\delta)my(\beta x^2-A)}{(A+x)^2(1+\beta x)^2} & j_{12} \equiv \frac{-\phi_1(\delta)p(x)}{1+\beta x} \\ j_{21} \equiv \frac{\exp(s\delta)(\exp(s\delta)-1)hy^2}{(x+(\exp(s\delta)hy)^2)} & j_{22} \equiv \frac{x}{x+(\exp(s\delta)-1)hy} \end{pmatrix}. \tag{7}$$

The characteristic equation of this Jacobian matrix (7) is $\gamma^2 - T\gamma + D = 0$. The eigenvalues of the matrix (7) are $\gamma_1$ and $\gamma_2$. The trace $(\operatorname{tr} J(x,y))$ and the determinant $(\det J(x,y))$ are the trace of (7) are given by:

$$T \doteq \operatorname{tr} J(x,y) = j_{11} + j_{22}, \quad D \doteq \det J(x,y) = j_{11}j_{22} - j_{21}j_{12}.$$

The following Lemma is required to analyze stability and bifurcation using the eigenvalues analysis method [5] (see also [32–34]).

**Lemma 2.** *Let $\mathcal{F}(\gamma) = \gamma^2 - T\gamma + D$ where $T$ and $D$ are constants. Suppose that $\mathcal{F}(1) > 0$, $\gamma_1$, $\gamma_2$ are two roots of $F(\gamma) = 0$. Then*

**(i)** $|\gamma_1| < 1$ *and* $|\gamma_2| < 1$ *iff* $\mathcal{F}(-1) > 0$ *and* $D < 1$;
**(ii)** $(|\gamma_1| > 1$ *and* $|\gamma_2| < 1)$ *or* $(|\gamma_1| < 1$ *and* $|\gamma_2| > 1)$ *iff* $\mathcal{F}(-1) < 0$;
**(iii)** $|\gamma_1| > 1$ *and* $|\gamma_2| > 1$ *iff* $\mathcal{F}(-1) > 0$ *and* $D > 1$;
**(iv)** $\gamma_1 = -1$ *and* $|\gamma_2| \neq 1$ *iff* $\mathcal{F}(-1) = 0$ *and* $D \neq 1$;
**(v)** $\gamma_1$ *and* $\gamma_2$ *are complex and* $|\gamma_1| = 1$ *and* $|\gamma_2| = 1$ *iff* $T^2 - 4D < 0$ *and* $D = 1$.

Now, the eigenvalues of the boundary fixed point $A(K,0)$ are $\gamma_1 = \exp(-r\delta)$ and $\gamma_2 = 1$. Hence, the boundary fixed point $A(K,0)$ is non-hyperbolic. The Jacobian matrix about $B(x^*,y^*)$ is given by

$$J(B) = \begin{pmatrix} \frac{h\exp(r\delta)(A+x^*)^2 + \phi_1(\delta)mx^*(\beta x^{*2} - A)}{h(A+x^*)^2(1+\beta x^*)^2} & \frac{-\phi_1(\delta)p(x^*)}{1+\beta x^*} \\ \frac{1-\exp(-s\delta)}{h} & \exp(-s\delta) \end{pmatrix}. \tag{8}$$

The characteristic equation of (8) is

$$\gamma^2 - \operatorname{tr} J(B)\gamma + \det J(B) = 0, \tag{9}$$

where

$$\operatorname{tr} J(B) = \frac{h\exp(r\delta)(A+x^*)^2 + \phi_1(\delta)mx^*(\beta x^{*2} - A)}{h(A+x^*)^2(1+\beta x^*)^2} + \exp(-s\delta), \tag{10}$$

$$\det J(B) = \frac{h\exp(r\delta)(A+x^*)^2 + \phi_1(\delta)mx^*(\beta x^{*2} - A)}{h(A+x^*)^2(1+\beta x^*)^2}\exp(-s\delta) + \frac{\phi_1(\delta)p(x^*)(1-\exp(-s\delta))}{h(1+\beta x^*)}. \tag{11}$$

**Lemma 3.** *Assume that* $\frac{m}{hr} + \frac{A}{K} > 1$.

— *The positive fixed point $B(x^*,y^*)$ is locally asymptotically stable if*

$$\left|\frac{h\exp(r\delta)(A+x^*)^2 + \phi_1(\delta)mx^*(\beta x^{*2} - A)}{h(A+x^*)^2(1+\beta x^*)^2} + \exp(-s\delta)\right| - 1$$
$$< \left(\frac{h\exp(r\delta)(A+x^*)^2 + \phi_1(\delta)mx^*(\beta x^{*2} - A)}{h(A+x^*)^2(1+\beta x^*)^2}\right)\exp(-s\delta) + \frac{\phi_1(\delta)p(x^*)(1-\exp(-s\delta))}{h(1+\beta x^*)} < 1.$$

— *The positive fixed point $B(x^*,y^*)$ is non-hyperbolic if*

$$s = \frac{1}{\delta}\ln\left(\frac{\Theta h(1+\beta x^*) - \phi_1(\delta)p(x^*)}{h(1+\beta x^*) - \phi_1(\delta)p(x^*)}\right), \tag{12}$$

$$s = \frac{1}{\delta}\ln\left(\frac{\phi_1(\delta)p(x^*) - h(1+\beta x^*)(\Theta+1)}{h(1+\beta x^*)(\Theta+1) + \phi_1(\delta)p(x^*)}\right). \tag{13}$$

If the non-hyperbolic condition (12) holds, then one of the eigenvalues related to $B(x^*,y^*)$ is $-1$ and the other is neither 1 nor $-1$. Thus (12) can be written as

$$P_d = \left\{(r,\delta,m,K,s,A) > 0, \operatorname{tr}^2 J(B) > 4\det J(B), s = \frac{1}{\delta}\ln\frac{\phi_1(\delta)p(x^*) - h(1+\beta x^*)(\Theta+1)}{h(1+\beta x^*)(\Theta+1) + \phi_1(\delta)p(x^*)}\right\}. \tag{14}$$

If the non-hyperbolic condition (13) holds, then the eigenvalues related to $B(x^*,y^*)$ are a pair of complex conjugate numbers with modulus 1. Thus (13) can be written as

$$N_s = \left\{(r,\delta,m,K,s,A) > 0, \operatorname{tr}^2 J(B) < 4\det J(B), s = \frac{1}{\delta}\ln\frac{\theta h(1+\beta x^*) - \phi_1(\delta)p(x^*)}{h(1+\beta x^*) - \phi_1(\delta)p(x^*)}\right\}. \tag{15}$$

## 3. Period-doubling bifurcation

For the fixed point $(x^*, y^*)$ associated to the system (4)–(5). The space (14) can be written as

$$P_d = \{s = \widehat{s}, \operatorname{tr}^2 J(B) < 4 \det J(B), (r, \delta, m, K, s, A) > 0\},$$

one of the eigenvalues of $J(x^*, y^*)$ is $-1$ and the other is neither 1 nor $-1$. Therefore the system (4)–(5) undergoes PDB at the fixed point $(x^*, y^*)$ if $s$ varies in the small neighborhood of $s = \widehat{s}$ and $(r, \delta, m, K, s, A) \in P_d$. Giving a perturbation $s^*$ (where $s^* \ll 1$) of the parameter $s$ in the neighborhood of $s = \widehat{s}$ to the system (4)–(5), then

$$x_{t+1} = \frac{\exp(r\delta)x_t - \phi_1(\delta)\frac{m\, x_t y_t}{A+x_t}}{1+\beta x_t} = f(x_t, y_t, s^*), \tag{16a}$$

$$y_{t+1} = \frac{\exp((s+s^*)\delta)x_t y_t}{x_t + h(\exp((s+s^*)\delta)-1)y_t} = g(x_t, y_t, s^*). \tag{16b}$$

Let $v_t = x_t - x^*$, $w_t = y_t - y^*$, then (16) becomes

$$v_{t+1} = \frac{\exp(r\delta)(v_t + x^*) - \phi_1(\delta)\frac{m\,(v_t+x^*)(w_t+y^*)}{A+(v_t+x^*)}}{1+\beta(v_t+x^*)} - x^*, \tag{17a}$$

$$w_{t+1} = \frac{\exp(s+s^*)(v_t+x^*)(w_t+y^*)}{(v_t+x^*) + h(\exp(s+s^*)-1)(w_t+y^*)} - y^*. \tag{17b}$$

Expanding (17) in Taylor series about $(v_t, w_t, s^*) = (0,0,0)$, and considering the terms up to second order, we have

$$v_{t+1} = \alpha_1 v_t + \alpha_2 w_t + \alpha_{12} v_t w_t + \alpha_{11} v_t^2, \tag{18a}$$

$$\begin{aligned} w_{t+1} = {} & \beta_1 v_t + \beta_2 w_t + \beta_{12} v_t w_t + \beta_{11} v_t^2 + \beta_{22} w_t^2 + \beta_{13} s^* v_t + \beta_{23} s^* w(t) + \beta_{123} s^* v(t) w(t) \\ & + \beta_{113} s^* v^2(t) + \beta_{223} s^* w^2(t), \end{aligned} \tag{18b}$$

where

$$\alpha_1 = \frac{h\exp(r\delta)(A+x^*)^2 + \phi_1(\delta) m x^*(\beta x^{*2} - A)}{h(A+x^*)^2(1+\beta x^*)^2}, \quad \alpha_2 = -\frac{x^*\phi_1(\delta) m}{(1+\beta x^*)(A+x^*)},$$

$$\alpha_{12} = -\frac{\phi_1(\delta) m (A - \beta x^{*2})}{(1+\beta x^*)^2 (A+x^*)^2}, \quad \alpha_{11} = -\frac{\beta\exp(r\delta)}{(1+\beta x^*)^2} - \phi_1(\delta) m \frac{2x^{*2}y^*\beta - Ay^* - A^2\beta y^*}{(1+\beta x^*)^2(A+x^*)^3},$$

$$\beta_1 = \frac{1-\exp(-s\delta)}{h}, \quad \beta_2 = \exp(-s\delta),$$

$$\beta_{12} = \frac{1}{x^*}\left(-1 + 2\frac{\exp(s\delta)-1}{\exp(2s\delta)}\right), \quad \beta_{11} = \frac{1}{x^*}\left(\frac{\exp(-s\delta)}{h} + \frac{\exp(-2s\delta)}{h^2}\right),$$

$$\beta_{13} = \frac{\delta\exp(-s\delta)}{h}, \quad \beta_{22} = \frac{h}{x^*}\big(\delta\exp(-s\delta) + 2\delta(\exp(s\delta)-1)\exp(-2s\delta)\big), \quad \beta_{23} = -\delta\exp(-s\delta),$$

$$\beta_{123} = \frac{2\delta}{x^*}\frac{(2-\exp(s\delta))}{\exp(2s\delta)}, \quad \beta_{113} = -\frac{\delta}{x^*}\left(\frac{\exp(-s\delta)}{h} + \frac{\exp(-2s\delta)}{h^2}\right),$$

$$\beta_{223} = \frac{h}{x^*}\big(\delta\exp(-s\delta) + 2\delta(\exp(s\delta)-1)\exp(-2s\delta)\big).$$

Now we define an invertible matrix $T = \begin{pmatrix} \alpha_2 & \alpha_2 \\ -1-\alpha_1 & \gamma_2 - \alpha_1 \end{pmatrix}$, and use the transformation $\begin{pmatrix} v_t \\ w_t \end{pmatrix} = T\begin{pmatrix} X_t \\ Y_t \end{pmatrix}$. Writing $v_t = \alpha_2(X_t + Y_t)$, $w(t) = -(1+\alpha_1)X_t + (\gamma_2 - \alpha)Y_t$. Thus, the system (18) becomes

$$\begin{pmatrix} X_{t+1} \\ Y_{t+1} \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & \gamma_2 \end{pmatrix}\begin{pmatrix} X_t \\ Y_t \end{pmatrix} + \begin{pmatrix} F_1(X_t, Y_t, s^*) \\ G_1(X_t, Y_t, s^*) \end{pmatrix}, \tag{19}$$

where

$$F_1(X_t, Y_t, s^*) = \frac{\gamma_2 - \alpha_1}{1+\gamma_2}\Big(\big(-\alpha_{12}\alpha_2(1+\alpha_1) + \alpha_{11}\alpha_2^2\big)X^2(t)$$

$$
\begin{aligned}
&+ \big(\alpha_{12}\alpha_2(\gamma_2-\alpha_1) - \alpha_{12}\alpha_2(1+\alpha_1) + 2\alpha_{11}\alpha_2^2\big)X_tY_t + \big(\alpha_{12}\alpha_2(\gamma_2-\alpha_1) + \alpha_{11}\alpha_2^2\big)Y_t^2\Big)\\
&- \frac{1}{\gamma_2+1}\Big(\big(-\beta_{12}\alpha_2(1+\alpha_1) + \beta_{11}\alpha_2^2 + \beta_{22}(1+\alpha_1)^2\big)X_t^2\\
&+ \big(\beta_{113}\alpha_2^2 + \beta_{223}(1+\alpha_1)^2 - \beta_{123}\alpha_2(1+\alpha_1)\big)X_t^2 s^*\\
&+ \big(\beta_{12}\alpha_2(\gamma_2-\alpha_1) + \beta_{11}\alpha_2^2 + \beta_{22}(\gamma_2-\alpha_1)^2\big)Y_t^2\\
&+ \big(\beta_{223}\alpha_2^2 + \beta_{223}(\gamma_2-\alpha_1)^2 + \beta_{123}\alpha_2(\gamma_2-\alpha_1)\big)Y_t^2 s^*\\
&+ \big(\beta_{12}\alpha_2(\gamma_2-\alpha_1) - \beta_{12}\alpha_2(1+\alpha_1) + 2\beta_{11}\alpha_2^2 - 2\beta_{22}(1+\alpha_1)(\gamma_2-\alpha_1)\big)X_tY_t\\
&+ \big(2\beta_{113}\alpha_2^2 + 2\beta_{223}(1+\alpha_1)(\gamma_2-\alpha_1) + \beta_{123}\alpha_2(\gamma_2-\alpha_1) - \beta_{123}\alpha_2(1+\alpha_1)\big)X_tY_ts^*\\
&+ \big(\beta_{13}\alpha_2 - \beta_{23}(1+\alpha_1)\big)X_ts^* + \big(\beta_{13}\alpha_2 + \beta_{23}(\gamma_2-\alpha_1)\big)Y_ts^*\Big),
\end{aligned}
$$

and

$$
\begin{aligned}
G_1(X_t,Y_t,s^*) = {}& \frac{1+\alpha_1}{1+\gamma_2}\Big(\big(-\alpha_{12}\alpha_2(1+\alpha_1) + \alpha_{11}\alpha_2^2\big)X^2(t)\\
&+ \big(\alpha_{12}\alpha_2(\gamma_2-\alpha_1) - \alpha_{12}\alpha_2(1+\alpha_1) + 2\alpha_{11}\alpha_2^2\big)X_tY_t + \big(\alpha_{12}\alpha_2(\gamma_2-\alpha_1) + \alpha_{11}\alpha_2^2\big)Y_t^2\Big)\\
&+ \frac{1}{\gamma_2+1}\Big(\big(-\beta_{12}\alpha_2(1+\alpha_1) + \beta_{11}\alpha_2^2 + \beta_{22}(1+\alpha_1)^2\big)X_t^2\\
&+ \big(\beta_{113}\alpha_2^2 + \beta_{223}(1+\alpha_1)^2 - \beta_{123}\alpha_2(1+\alpha_1)\big)X_t^2 s^*\\
&+ \big(\beta_{12}\alpha_2(\gamma_2-\alpha_1) + \beta_{11}\alpha_2^2 + \beta_{22}(\gamma_2-\alpha_1)^2\big)Y_t^2\\
&+ \big(\beta_{223}\alpha_2^2 + \beta_{223}(\gamma_2-\alpha_1)^2 + \beta_{123}\alpha_2(\gamma_2-\alpha_1)\big)Y_t^2 s^*\\
&+ \big(\beta_{12}\alpha_2(\gamma_2-\alpha_1) - \beta_{12}\alpha_2(1+\alpha_1) + 2\beta_{11}\alpha_2^2 - 2\beta_{22}(1+\alpha_1)(\gamma_2-\alpha_1)\big)X_tY_t\\
&+ \big(2\beta_{113}\alpha_2^2 + 2\beta_{223}(1+\alpha_1)(\gamma_2-\alpha_1) + \beta_{123}\alpha_2(\gamma_2-\alpha_1) - \beta_{123}\alpha_2(1+\alpha_1)\big)X_tY_ts^*\\
&+ \big(\beta_{13}\alpha_2 - \beta_{23}(1+\alpha_1)\big)X_ts^* + \big(\beta_{13}\alpha_2 + \beta_{23}(\gamma_2-\alpha_1)\big)Y_ts^*\Big).
\end{aligned}
$$

Hereafter, we determine the center manifold $\mathcal{W}_c(0,0)$ of (19) about $(0,0)$ in a small neighborhood of $s^*$. Thus, there exists a center manifold [35] $\mathcal{W}_c(0,0)$ that can be represented as follows:

$$\mathcal{W}_c(0,0) = \big\{(X_t,Y_t)\colon Y_t = h(X_t,s^*) = a_1X_t^2 + a_2X_ts^* + a_3s^{*2} + O\big((|X_t|+|s^*|)^2\big)\big\},$$

where $O\big((|X_t|+|s^*|)^2\big)$ is a function with order at least three in their variables $(X_t,s^*)$. Moreover, the center manifold must satisfy

$$h\big(-X_t + F_1(X_t,h(X_t,s^*)),s^*),s^*\big) - \gamma_2 h(X_t,s^*) - G_1\big(X_t,h(X_t,s^*),s^*\big) = 0. \tag{20}$$

By equating (20), we obtain

$$
\begin{aligned}
a_1 &= \frac{1+\alpha_1}{1-\gamma_2^2}\big(-\alpha_{12}\alpha_2(1+\alpha_1) + \alpha_{11}\alpha_2^2\big) + \frac{1}{1-\gamma_2^2}\big(-\beta_{12}\alpha_2(1+\alpha_1) + \beta_{11}\alpha_2^2 + \beta_{22}(1+\alpha_1)^2\big),\\
a_2 &= \frac{-1}{1+\gamma_2}\big(\beta_{13}\alpha_2 - \beta_{23}(1+\alpha_1)\big), \quad a_3 = 0.
\end{aligned}
$$

Therefore, we consider the map which is the map (19) restricted to the center manifold $\mathcal{W}_c(0,0)$

$$f = X_{t+1} = -X_t + h_1X_ts^* + h_2X_t^2 + h_3X_t^2s^* + h_4X_t^3, \tag{21}$$

where

$$
\begin{aligned}
h_1 &= -\frac{1}{1+\gamma_2}\big(\beta_{13}\alpha_2 - \beta_{23}(1+\alpha_1)\big),\\
h_2 &= \frac{\gamma_2-\alpha_1}{1+\gamma_2}\big(-\alpha_{12}\alpha_2(1+\alpha_1) + \alpha_{11}\alpha_2^2\big) - \frac{1}{\gamma_2+1}\big(-\beta_{12}\alpha_2(1+\alpha_1) + \beta_{11}\alpha_2^2 + \beta_{22}(1+\alpha_1)^2\big),\\
h_3 &= \frac{-1}{1+\gamma_2}\big(\beta_{13}\alpha_2 - \beta_{23}(1+\alpha_1)\big)\bigg[\frac{\gamma_2-\alpha_1}{1+\gamma_2}\big(\alpha_{12}\alpha_2(\gamma_2-\alpha_1) - \alpha_{12}\alpha_2(1+\alpha_1) + 2\alpha_{11}\alpha_2^2\big)
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{1}{1+\gamma_2}\big(\beta_{12}\alpha_2(\gamma_2-\alpha_1) - \beta_{12}\alpha_2(1+\alpha_1) + 2\beta_{11}\alpha_2^2 - 2\beta_{22}(1+\alpha_1)(\gamma_2-\alpha_1)\big)\Big] \\
&- \frac{1}{1+\gamma_2}\big(\beta_{113}\alpha_2^2 + \beta_{223}(1+\alpha_1)^2 - \beta_{123}\alpha_2(1+\alpha_1)\big) \\
&- \Bigg(\frac{1+\alpha_1}{(1-\gamma_2^2)(1+\gamma_2)}\big(-\alpha_{12}\alpha_2(1+\alpha_1)+\alpha_{11}\alpha_2^2\big) \\
&+ \frac{1}{(1-\gamma_2^2)(1+\gamma_2)}\big(-\beta_{12}\alpha_2(1+\alpha_1)+\beta_{11}\alpha_2^2+\beta_{22}(1+\alpha_1)^2\big)\Bigg)\big(\beta_{13}\alpha_2+\beta_{23}(\gamma_2-\alpha_1)\big),
\end{aligned}
$$

$$
\begin{aligned}
h_4 = &\left(\frac{1+\alpha_1}{1-\gamma_2^2}\big(-\alpha_{12}\alpha_2(1+\alpha_1)+\alpha_{11}\alpha_2^2\big) + \frac{1}{1-\gamma_2^2}\big(-\beta_{12}\alpha_2(1+\alpha_1)+\beta_{11}\alpha_2^2+\beta_{22}(1+\alpha_1)^2\big)\right) \\
&\times \Big[\frac{\gamma_2-\alpha_1}{1+\gamma_2}\big(\alpha_{12}\alpha_2(\gamma_2-\alpha_1) - \alpha_{12}\alpha_2(1+\alpha_1) + 2\alpha_{11}\alpha_2^2\big) \\
&- \frac{1}{1+\gamma_2}\big(\beta_{12}\alpha_2(\gamma_2-\alpha_1) - \beta_{12}\alpha_2(1+\alpha_1) + 2\beta_{11}\alpha_2^2 - 2\beta_{22}(1+\alpha_1)(\gamma_2-\alpha_1)\big)\Big].
\end{aligned}
$$

In order for the map (21) to undergoes a period-doubling bifurcation, we require that the following discriminatory quantities are non zero [36]:

$$
\pi_1 = \left(\frac{\partial^2 f}{\partial X_t \partial s^*} + \frac{1}{2}\frac{\partial f}{\partial s^*}\frac{\partial^2 f}{\partial^2 X_t}\right)\Big|_{(0,0)} = h_1 \neq 0,
$$

$$
\pi_2 = \left(\frac{1}{6}\frac{\partial^3 f}{\partial X_t^3} + \big(\frac{1}{2}\frac{\partial^2 f}{\partial X_t^2}\big)^2\right)\Big|_{(0,0)} = h_4 + h_2^2 \neq 0.
$$

From the above analysis we have the following theorem.

**Theorem 1.** *If $\pi_2 \neq 0$, and $\pi_1 \neq 0$ the model (4)–(5) undergoes a period-doubling bifurcation about the positive fixed point $B(x^*, y^*)$ when $s^*$ varies in a small neighborhood of $O(0,0)$. Moreover, if $\pi_2 > 0$ (resp $\pi_2 < 0$), then the period 2 points that bifurcate from $B(x^*, y^*)$ are stable (unstable).*

## 4. Neimark–Sacker bifurcation

The roots of the characteristic equation (9) at $B(x^*, y^*)$ are a pair of complex conjugate numbers $\gamma_1$, $\gamma_2$ given by

$$
\gamma_{1,2} = \frac{\operatorname{tr} J(B) \pm i\sqrt{4\det J(B) - (\operatorname{tr} J(B))^2}}{2},
$$

with $trJ(B)$ and $\det J(B)$ are given in (10)and (11) respectively. Now NSB occurs when the roots of the above equation are complex conjugates with unit modulus. It occurs for $s = \overline{s}$, we construct then a set $N_b = \big\{(r, \delta, m, K, s, A) > 0, s = \overline{s}, \operatorname{tr}^2 J(B) < 4\det J(B)\big\}$. If we vary $s$ in the neighborhood of $s = \overline{s}$ keeping other parameters in NSB constant, then the positive fixed point $B$ undergoes NSB.

Taking a perturbation $s^*$ where ($s^* \ll 1$) of the parameter $s$ in the neighborhood of $s = \overline{s}$ in the system (4)–(5), we have

$$
x_{t+1} = \frac{\exp(r\delta)x_t - \phi_1(\delta)\frac{m\,x_t y_t}{A+x_t}}{1+\beta x_t} = f(x_t, y_t, s^*), \tag{22a}
$$

$$
y_{t+1} = \frac{\exp((s+s^*)\delta)x_t y_t}{x_t + h(\exp((s+s^*)\delta)-1)y_t} = g(x_t, y_t, s^*). \tag{22b}
$$

Let $v_t = x_t - x^*$, $w_t = y_t - y^*$, then from (22) we set

$$
v_{t+1} = \frac{\exp(r\delta)(v_t+x^*) - \phi_1(\delta)\frac{m\,(v_t+x^*)(w_t+y^*)}{A+(v_t+x^*)}}{1+\beta(v_t+x^*)} - x^*, \tag{23a}
$$

$$
w_{t+1} = \frac{\exp(s+s^*)(v_t+x^*)(w_t+y^*)}{(v_t+x^*) + h(\exp(s+s^*)-1)(w_t+y^*)} - y^*. \tag{23b}
$$

Expanding the above in Taylor series at $(v_t, w_t) = (0, 0)$ considering the terms up to second order, we have

$$v_{t+1} = \alpha_1 v_t + \alpha_2 w_t + \alpha_{12} v_t w_t + \alpha_{11} v_t^2 + O\big((|v_t| + |w_t|)^2\big), \tag{24a}$$

$$w_{t+1} = \beta_1 v_t + \beta_2 w_t + \beta_{12} v_t w_t + \beta_{11} v_t^2 + \beta_{22} w_t^2 + O\big((|v_t| + |w_t|)^2\big), \tag{24b}$$

where

$$\alpha_1 = \frac{h\exp(r\delta)(A+x^*)^2 + \phi_1(\delta)mx^*(\beta x^{*2} - A)}{h(A+x^*)^2(1+\beta x^*)^2}, \quad \alpha_2 = -\frac{x^*\phi_1(\delta)m}{(1+\beta x^*)(A+x^*)},$$

$$\alpha_{12} = -\frac{\phi_1(\delta)m(A - \beta x^{*2})}{(1+\beta x^*)^2(A+x^*)^2}, \quad \alpha_{11} = -\frac{\beta\exp(r\delta)}{(1+\beta x^*)^2} - \phi_1(\delta)m\frac{2x^{*2}y^*\beta - Ay^* - A^2\beta y^*}{(1+\beta x^*)^2(A+x^*)^3},$$

$$\beta_1 = \frac{1 - \exp(-s\delta)}{h}, \quad \beta_2 = \exp(-s\delta),$$

$$\beta_{12} = \frac{1}{x^*}\left[-1 + 2\frac{\exp(s\delta) - 1}{\exp(2s\delta)}\right], \quad \beta_{11} = \frac{1}{x^*}\left[\frac{\exp(-s\delta)}{h} + \frac{\exp(-2s\delta)}{h^2}\right],$$

$$\beta_{22} = \frac{h}{x^*}\Big[\delta\exp(-s\delta) + 2\delta(\exp(s\delta) - 1)\exp(-2s\delta)\Big].$$

The roots of the characteristic equation associated with the linearized map (24) at $(v_t, w_t) = (0, 0)$ are given by

$$\gamma_{1,2}(s*) = \frac{\operatorname{tr} J(s^*) \pm i\sqrt{4\det J(s^*) - (\operatorname{tr}(s^*))^2}}{2}, \quad |\gamma_{1,2}(s^*)| = \sqrt{\det J(s^*)},$$

when $s^* = 0$, we have

$$\det(J(0)) = 1 \quad \text{and} \quad \left.\frac{d|\gamma_{1,2}|}{ds^*}\right|_{s^*=0} \neq 0. \tag{25}$$

Additionally, we required that when $s^* = 0$, $\gamma_{1,2}^m \neq 1$, $m = 1, 2, 3, 4$. This is equivalent to $\operatorname{tr} J(0) \neq -2, -1, 1, 2$. Let $\eta = \operatorname{Re}(\gamma_{1,2})$, and $\xi = \operatorname{Im}(\gamma_{1,2})$. The model (24) is written as

$$\begin{pmatrix} v_{t+1} \\ w_{t+1} \end{pmatrix} = \begin{pmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{pmatrix}\begin{pmatrix} v_t \\ w_t \end{pmatrix} + \begin{pmatrix} \alpha_{12} v_t w_t + \alpha_{11} v_t^2 \\ \beta_{12} v_t w_t + \beta_{11} v_t^2 + \beta_{22} w_t^2 \end{pmatrix}.$$

Let consider the invertible matric $P = \begin{pmatrix} \alpha_2 & 0 \\ \eta - \alpha_1 & -\xi \end{pmatrix}$, associated to the eigenvalue $\gamma_{1,2} = \eta \pm i\xi$. Using the following translation

$$\begin{pmatrix} v_t \\ w_t \end{pmatrix} = \begin{pmatrix} \alpha_2 & 0 \\ \eta - \alpha_1 & -\xi \end{pmatrix}\begin{pmatrix} X_t \\ Y_t \end{pmatrix}.$$

Therefore, one gets

$$\begin{pmatrix} X_{t+1} \\ Y_{t+1} \end{pmatrix} = \begin{pmatrix} \eta & -\xi \\ -\xi & \eta \end{pmatrix}\begin{pmatrix} X_t \\ Y_t \end{pmatrix} + \begin{pmatrix} F(X_t, Y_t) \\ G(X_t, Y_t) \end{pmatrix}, \tag{26}$$

with

$$F(X_t, Y_t) = \frac{1}{\alpha_2}\big(\alpha_{12}\alpha_2(\eta - \alpha_1) + \alpha_{11}\alpha_2^2\big)X_t^2 - \frac{1}{\alpha_2}\big(\xi\alpha_{12}\alpha_2\big)X_t Y_t,$$

and

$$G(X_t, Y_t) = \left(\frac{\eta - \alpha_1}{\xi\alpha_2}\big(\alpha_{12}\alpha_2(\eta - \alpha_1) + \alpha_{11}\alpha_2^2\big) - \frac{1}{\xi}\big(\beta_{12}\alpha_2(\eta - \alpha_1) + \beta_{11}\alpha_2^2 + \beta_{22}(\eta - \alpha_1)^2\big)\right)X_t^2$$
$$- \left(\frac{\eta - \alpha_1}{\xi\alpha_2}\big(\xi\alpha_{12}\alpha_2\big) - \frac{1}{\xi}\big(\beta_{12}\alpha_2\xi + 2(\eta - \alpha_1)\beta_{22}\xi\big)\right)X_t Y_t + \beta_{22}\xi Y_t^2.$$

In order for (26) to undergoes a NSB, it is mandatory that the following discriminatory quantity, (i.e, $L \neq 0$ [36]),

$$L = -\operatorname{Re}\left[\frac{(1 - 2\overline{\gamma})\overline{\gamma}^2}{1 - \gamma}\rho_{11}\rho_{20}\right] - \frac{1}{2}|\rho_{11}|^2 - |\rho_{02}|^2 + \operatorname{Re}(\overline{\gamma}\rho_{21}), \tag{27}$$

where

$$\rho_{02} = \frac{1}{8}\big[F_{X_tX_t} - F_{Y_tY_t} - 2G_{X_tY_t} + i(G_{X_tX_t} - G_{Y_tY_t} + 2F_{X_tY_t})\big]_{(0,0)},$$

$$\rho_{11} = \frac{1}{4}\big[F_{X_tX(t)} + F_{Y_tY_t} + i(G_{X_tX_t} + G_{Y_tY_t})\big]_{(0,0)},$$

$$\rho_{20} = \frac{1}{8}\big[F_{X_tX_t} - F_{Y_tY_t} + 2G_{X_tY_t} + i(G_{X_tX_t} - G_{Y_tY_t} - 2F_{X_tY_t})\big]_{(0,0)},$$

$$\rho_{21} = \frac{1}{16}\big[F_{X_tX_tX_t} + F_{X_tY_tY_t} + G_{X_tX_tY_t} + G_{Y_tY_tY_t} + i(G_{X_tX_tX_t} + G_{X_tY_tY_t} - F_{X_tX_tY_t} - F_{Y_tY_tY_t})\big]_{(0,0)}.$$

After calculating, one gets

$$\begin{aligned}\rho_{02} = \frac{1}{4}\Bigg[&\left(\frac{1}{\alpha_2}\big(\alpha_{12}\alpha_2(\eta-\alpha_1) + \alpha_{11}\alpha_2^2\big) + \left(\frac{\eta-\alpha_1}{\xi\alpha_2}(\xi\alpha_{12}\alpha_2) - \frac{1}{\xi}\big(\beta_{12}\alpha_2\xi + (\eta-\alpha_1)\beta_{22}\xi\big)\right)\right)\\ &+ i\Bigg(\left(\frac{\eta-\alpha_1}{\xi\alpha_2}\big(\alpha_{12}\alpha_2(\eta-\alpha_1) + \alpha_{11}\alpha_2^2\big) - \frac{1}{\xi}\big(\beta_{12}\alpha_2(\eta-\alpha_1) + \beta_{11}\alpha_2^2 + \beta_{22}(\eta-\alpha_1)^2\big)\right)\\ &- \beta_{22}\xi - \frac{1}{\alpha_2}\big(\xi\alpha_{12}\alpha_2\big)\Bigg)\Bigg],\\
\rho_{11} = \frac{1}{2}\Bigg[&\frac{1}{\alpha_2}\big(\alpha_{12}\alpha_2(\eta-\alpha_1) + \alpha_{11}\alpha_2^2\big) + i\Bigg(\bigg(\frac{\eta-\alpha_1}{\xi\alpha_2}\big(\alpha_{12}\alpha_2(\eta-\alpha_1) + \alpha_{11}\alpha_2^2\big)\\ &- \frac{1}{\xi}\big(\beta_{12}\alpha_2(\eta-\alpha_1) + \beta_{11}\alpha_2^2 + \beta_{22}(\eta-\alpha_1)^2\big)\bigg) + \beta_{22}\xi\Bigg)\Bigg],\\
\rho_{20} = \frac{1}{4}\Bigg[&\left(\frac{1}{\alpha_2}\big(\alpha_{12}\alpha_2(\eta-\alpha_1) + \alpha_{11}\alpha_2^2\big) - \left(\frac{\eta-\alpha_1}{\xi\alpha_2}(\xi\alpha_{12}\alpha_2) - \frac{1}{\xi}\big(\beta_{12}\alpha_2\xi + 2(\eta-\alpha_1)\beta_{22}\xi\big)\right)\right)\\ &+ \Bigg(\left(\frac{\eta-\alpha_1}{\xi\alpha_2}\big(\alpha_{12}\alpha_2(\eta-\alpha_1) + \alpha_{11}\alpha_2^2\big) + \frac{1}{\xi}\big(\beta_{12}\alpha_2(\eta-\alpha_1) + \beta_{11}\alpha_2^2 + \beta_{22}(\eta-\alpha_1)^2\big)\right)\\ &- \beta_{22}\xi + \frac{1}{\alpha_2}\big(\xi\alpha_{12}\alpha_2\big)\Bigg)\Bigg],\\
\rho_{21} = 0&.\end{aligned}$$

Based on the above analysis, we state the following result on NSB.

**Theorem 2.** *If the condition (25) holds and $L$ defined in (27) is nonzero, then the model (4)–(5) undergoes a NSB at the positive fixed point $B(x^*, y^*)$ when $s^*$ varies near the origin and $(r, \delta, m, K, s, A) \in N_b$. Moreover, if $L < 0$ ($L > 0$) then an attracting (respectively repelling) invariant closed curve bifurcates from the fixed point $B(x^*, y^*)$ for $s > \overline{s}$ (respectively, $s < \overline{s}$).*

## 5. Chaos control

In this section, we control the chaos influenced by NSB and PDB. Chaos can be controlled by various methods (see, e.g. [14, 28, 33]). In this paper, we apply the state feedback method [14] to stabilize the chaotic orbits at an unstable fixed point of system (4)–(5). Thus, we introduce a feedback control force $P_t$ such that

$$x_{t+1} = \frac{\exp(r\delta)x_t - \phi_1(\delta)\frac{m\,x_t y_t}{A+x_t}}{1+\beta x_t}, \tag{28}$$

$$y_{t+1} = \frac{\exp(\widetilde{s}\delta)x_t y_t}{x_t + h(\exp(\widetilde{s}\delta)-1)y_t} - \underbrace{\mu(x_t - x^*) + \nu(y_t - y^*)}_{P_t}, \tag{29}$$

where $\mu$, $\nu$ are feedback gains and $\widetilde{s}$ is the nominal value for $s$ which belongs to some chaotic regions. The Jacobian matrix of (28)–(29) at $B(x^*, y^*)$ is

$$J(B) = \begin{pmatrix} \frac{\exp(r\delta)}{(1+\beta x^*)^2} - \frac{\phi_1(\delta)my^*(\beta x^{*2}-A)}{(1+\beta x^*)^2(A+x^*)^2} & -\frac{\phi_1(\delta)mx^*}{(1+\beta x^*)(A+x^*)} \\ \frac{1}{h}\big(1-\exp(-\widetilde{s}\delta)\big) - \mu & \exp(-\widetilde{s}\delta) - \nu \end{pmatrix}. \tag{30}$$

The corresponding characteristic equation of (30) is

$$\kappa^2 - \left(\frac{\exp(r\delta)}{(1+\beta x^*)^2} - \frac{\phi_1(\delta) m y^*(\beta x^{*2} - A)}{(1+\beta x^*)^2 (A+x^*)^2} + \exp(-\widetilde{s}\delta) - \nu\right)\kappa + \left(\frac{\exp(r\delta)}{(1+\beta x^*)^2} - \frac{\phi_1(\delta) m y^*(\beta x^{*2} - A)}{(1+\beta x^*)^2 (A+x^*)^2}\right)\big(\exp(-\widetilde{s}\delta) - \nu\big) + \frac{\phi_1(\delta) m x^*}{(1+\beta x^*)(A+x^*)}\left(\frac{1}{h}\big(1-\exp(-\widetilde{s}\delta)\big) - \mu\right) = 0. \quad (31)$$

Let $\kappa_1$, $\kappa_2$ are the eigenvalues of the characteristic Eq. (31) then sum and the product of their roots are given by

$$\kappa_1 + \kappa_2 = \frac{\exp(r\delta)}{(1+\beta x^*)^2} - \frac{\phi_1(\delta) m y^*(\beta x^{*2} - A)}{(1+\beta x^*)^2 (A+x^*)^2} + \exp(-\widetilde{s}\delta) - \nu, \quad (32)$$

$$\kappa_1\kappa_2 = \left(\frac{\exp(r\delta)}{(1+\beta x^*)^2} - \frac{\phi_1(\delta) m y^*(\beta x^{*2} - A)}{(1+\beta x^*)^2 (A+x^*)^2}\right)\big(\exp(-\widetilde{s}\delta) - \nu\big) + \frac{\phi_1(\delta) m x^*}{(1+\beta x^*)(A+x^*)}\left(\frac{1}{h}\big(1-\exp(-\widetilde{s}\delta)\big) - \mu\right). \quad (33)$$

**Lemma 4.** *The system (28)–(29) is locally asymptotically stable if all the eigenvalues of the characteristic Eq. (31) lie in an open unit disc.*

**Proof.** The marginal stability lines can be obtained from the conditions $\kappa_1 = \pm 1$, $\kappa_1\kappa_2 = 1$. For the conditions $\kappa_1\kappa_2 = 1$, Eq. (33) gives

$$L_1\colon \frac{\phi_1(\delta) m x^*}{(1+\beta x^*)(A+x^*)}\mu + \left(\frac{\exp(r\delta)}{(1+\beta x^*)^2} - \frac{\phi_1(\delta) m y^*(\beta x^{*2} - A)}{(1+\beta x^*)^2 (A+x^*)^2}\right)\nu = -1 + \exp(-\widetilde{s}\delta)\left(\frac{\exp(r\delta)}{(1+\beta x^*)^2} - \frac{\phi_1(\delta) m y^*(\beta x^{*2} - A)}{(1+\beta x^*)^2 (A+x^*)^2}\right) + \frac{\phi_1(\delta) m x^*\big(1-\exp(-\widetilde{s}\delta)\big)}{h(1+\beta x^*)(A+x^*)}. \quad (34)$$

The Eq. (34) expresses the first condition for marginal stability. For $\kappa_1 = 1$, the Eq. (32) yields

$$L_2\colon -\frac{\phi_1(\delta) m x^*}{(1+\beta x^*)(A+x^*)}\mu + \left(1 - \frac{\exp(r\delta)}{(1+\beta x^*)^2} + \frac{\phi_1(\delta) m y^*(\beta x^{*2} - A)}{(1+\beta x^*)^2 (A+x^*)^2}\right)\nu = -1 + \left(\frac{\exp(r\delta)}{(1+\beta x^*)^2} - \frac{\phi_1(\delta) m y^*(\beta x^{*2} - A)}{(1+\beta x^*)^2 (A+x^*)^2}\right)\big(1-\exp(-\widetilde{s}\delta)\big) + \exp(-\widetilde{s}\delta) - \frac{\phi_1(\delta) m x^*\big(1-\exp(-\widetilde{s}\delta)\big)}{h(1+\beta x^*)(A+x^*)},$$

similarly for $\lambda_1 = -1$, it gives

$$L_3\colon \frac{\phi_1(\delta) m x^*}{(1+\beta x^*)(A+x^*)}\mu + \left(1 + \frac{\exp(r\delta)}{(1+\beta x^*)^2} - \frac{\phi_1(\delta) m y^*(\beta x^{*2} - A)}{(1+\beta x^*)^2 (A+x^*)^2}\right)\nu = 1 + \big(\exp(-\widetilde{s}\delta) + 1\big)\left(\frac{\exp(r\delta)}{(1+\beta x^*)^2} - \frac{\phi_1(\delta) m y^*(\beta x^{*2} - A)}{(1+\beta x^*)^2 (A+x^*)^2}\right) + \exp(-\widetilde{s}\delta) + \frac{\phi_1(\delta) m x^*\big(1-\exp(-\widetilde{s}\delta)\big)}{h(1+\beta x^*)(A+x^*)}.$$

The lines $L_1$, $L_2$, $L_3$ give the conditions for the eigenvalues to have absolute value less than 1. The triangular region bounded by these lines accommodates stable eigenvalues. ∎

## 6. Numerical simulations

In this section, we give some illustrative simulations to our theoretical findings. We choose $(r, \delta, m, A, K, h) = (0.5, 3, 5, 3.5, 2.5)$ and initial conditions $(x_0, y_0) = (0.8, 0.5)$ for the system (4)–(5). All orbits are attracted to the positive fixed point $B(0.7, 0.28)$, which is locally asymptotically stable, see Figure 1$a$. Increasing the value of $s$, from $s = 0.64$ to $s = 0.69$ (see Figures 1$b$–1$e$), the system starts to lose its asymptotic stability. Based on the theorem 2, the value $L = 0.0961179847 > 0$. This proves the existence of an attracting closed invariant curve, which indicates that the system undergoes a NSB about the positive fixed point $B$.

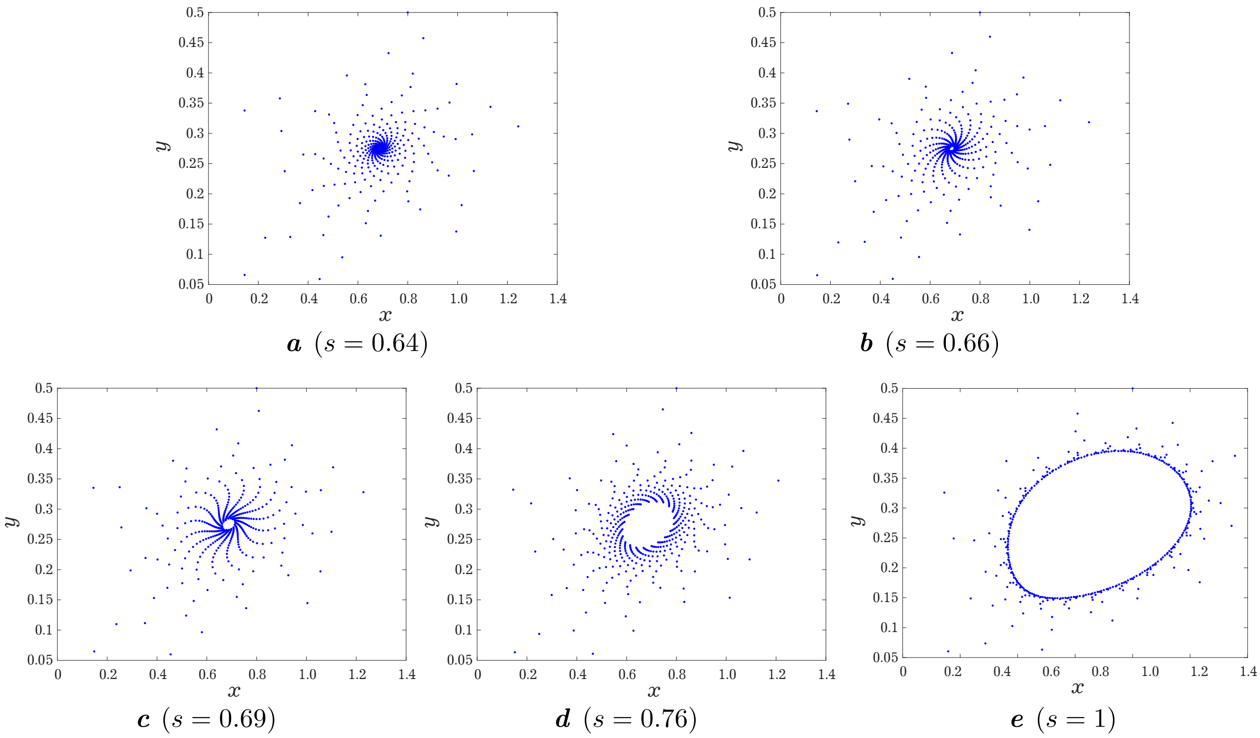


**Fig. 1.** Phase portraits for the discrete model (4)–(5) for different values of $s$.

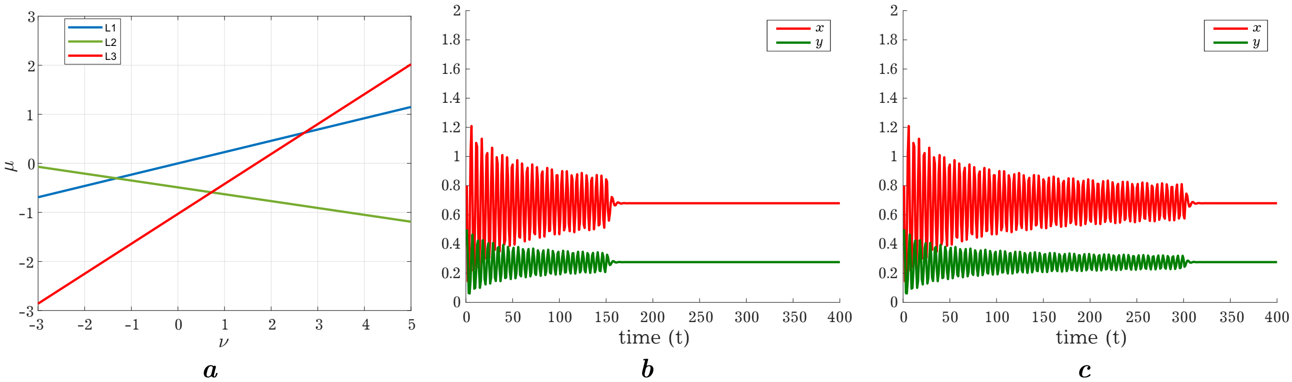


**Fig. 2.** ($\boldsymbol{a}$) Stability region of the controlled system (28)–(29). ($\boldsymbol{b}$) and ($\boldsymbol{c}$) Stable time series for $x$ and $y$ for the controlled system (28)–(29) $\widetilde{s} = 0.76$.

For exploring complexity in the system (4)–(5), a bifurcation diagram with respect to $s$ is plotted in Figure 3$a$. It is observed that the system (4)–(5) exhibits a range of period-doubling bifurcation. To stabilize chaos influenced by NSB in the system (4)–(5), we implement the state feedback control method. We choose a chaotic value of $s = 0.76$,

$$L_1\colon 2.61496844\mu - 0.6135971594\nu = -0.001760686,$$
$$L_2\colon 2.61496844\mu + 0.386402841\nu = -1.28587932,$$
$$L_3\colon 2.61496844\mu - 1.6135971594\nu = -2.717642052.$$

Using Lemma 4, one gets the lines $L_1$, $L_2$ and $L_3$ of the asymptotic stability for the system (28)–(29). These lines determines a triangle region (see Figure 2$a$), such that, for every values of $\mu$ and $\nu$ chosen from this triangle, the system (28)–(29) is controllable in the sense that the asymptotic stability is verified. Toward this, we chose $\mu = 0.2$, $\nu = -0.1$ (i.e., the feedback controlling force is $P_t = 0.2(x_t - 0.7) - 0.1(y_t - 0.28)$). For these values, the efficiency of the this method is proved. In Figures 2$b$–2$c$, time series are plotted which show that our derived system (28)–(29) converges asymptotically to the positive equilibrium point $B(0.7, 0.28)$.

Now, in order to control the chaos produced by PDB in Figure 3$a$, we take into account the triangle region (Figure 2$a$). We set $\nu = 1$ and $\mu \in (0.57, 1)$ (Figure 2$b$), and the fixed point $B$ is locally asymptotically stable. Additional simulations are carried out to stabilize the chaotic behavior produced at $s = 2$ in Figure 3$a$. We set $\mu = -0.8$ and $\nu \in (-2, -1.8)$ (Figure 3$c$), and $\mu = 0$ and $\nu \in (-2, 0.5)$ (Figure 3$d$). The dense chaotic region is reduced to periodic and quasi-periodic window in (Figure 3$d$) with respect to Figure 3$a$. Hence, the above method controls chaos with respect to different parameters.

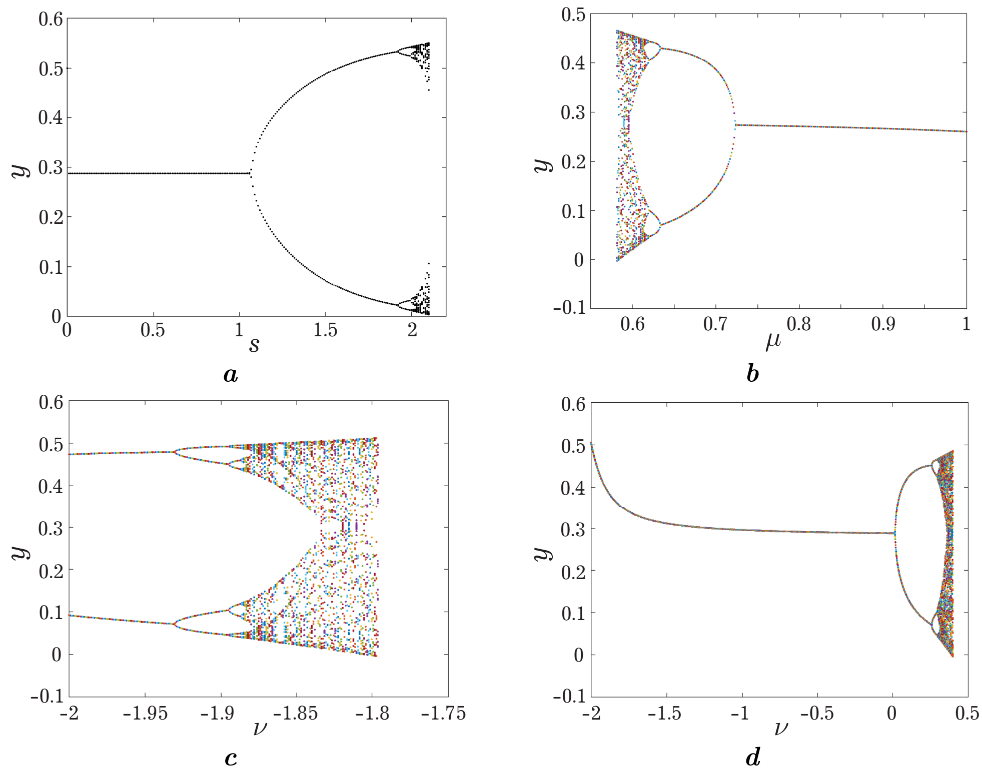


**Fig. 3.** (***a***) Bifurcation diagrams for the discrete model (4)–(5) with respect to $s$. (***b***) Bifurcation diagram of the controlled system (28)–(29) for $\nu = 1$ and $\mu \in (0.57, 1)$. (***c***) Bifurcation diagram of the controlled system (28)–(29) for $\mu = -0.8$ and $\nu \in (-2, -1.8)$. (***d***) Bifurcation diagram of the controlled system (28)–(29) for $\mu = 0$ and $\nu \in (-2, 0.5)$.

## 7. Conclusion

In this paper, we explore the rich dynamical properties of a discrete-time, two-dimensional prey–predator system. The model is developed by discretizing a differential predator-prey model by using a nonstandard finite difference scheme. The existence and local asymptotic stability of the fixed points are investigated. In order to support the complexity of (4)–(5), the presence of NSB and PDB for the positive fixed point $B(x^*, y^*)$ is proved analytically by using bifurcation and center manifold theories. Further numerical simulations are performed. Through these simulations, we showed that the model (4)–(5) goes through NSB and PDB when the parameters vary in the neighborhood of (14) and (15). We implemented the state feedback method to avoid unstable orbits, and the provided numerical plots give evidence of the successful implementation of this method.

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## Складна динаміка та керування хаосом у нелінійній дискретній моделі здобич–хижак

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Динаміку взаємодії жертви та хижака часто моделюють за допомогою диференціальних або різницевих рівнянь. У запропонованій роботі досліджується динамічна поведінка двовимірної дискретної системи “жертва–хижак”. Модель сформульована в термінах різницевих рівнянь і виведена за допомогою нестандартної скінченно-різницевої схеми (NSFD), яка враховує покоління, що не перекриваються. Доведено існування нерухомих точок, а також їх локальну асимптотичну стійкість. Далі показано, що модель зазнає біфуркацію Неймарка–Саккера (скорочено NSB) та біфуркацію подвоєння періоду (PDB) у невеликому околі нерухомої точки співіснування за певних параметричних умов. Цей аналіз використовує теорію біфуркацій та теорему центрального многовиду. Хаос, на який впливають NSB і PDB, стабілізується за допомогою методу зворотного зв'язку стану. Чисельне моделювання та комп'ютерний аналіз використовуються, щоб перевірити запропоновану теорію та показати складніші випадки.

**Ключові слова:** *різницеві рівняння; асимптотична стабільність; біфуркаційний аналіз; керування хаосом.*