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FROM NEWTON'S BINOMIAL AND PASCAL’S TRIANGLE TO COLLATZ'S PROBLEM

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Abstract It is shown that: 1. The sequence $\{2^0, 2^1, 2^2, 2^3, 2^4, 2^5, 2^6, 2^7, 2^8, ...\}$ that forms the main graph m=1 of Collatz is related to the power transformation of Newton's binomial $(1+1)^\xi$, $\xi=0,1,2,3,...$ 2. The main K_{main} and side m >1 graphs and their corresponding sequences $\{K_{main}\}$ and $\{K_m\}$ are related by the relation $\{K_m\}=m\cdot\{K_{main}\}$. 3. Side graphs generated by prime odd numbers 5, 7, 11, 13, 17, 19, 23, 25, 29, 31,…are not divisible by three, are formed without nodes. Side graphs, which are generated by compozite of odd numbers 3, 9, 15, 21, 27, 33, 39, 45,… are divisible by three, are formed with nodes. 4. The trajectories of transformations of odd numbers, through 3, 6, 8,…. iterations from the beginning of calculations, merge with a trajectory of calculations of the first smaller number on value of the number.

Keywords: Collatz conjecture, recurrent sequence, transformation 3n + 1, natural numbers

Introduction and problem statement

Classical Collatz’s algorithm [1], converts any natural number *n* according to the function

$$f_n = \begin{cases} \frac{1}{2} f_{n-1} \ if f_{n-1} \ is \ even \quad (a) \\ 3f_{n-1} + 1 \ if \ f_{n-1} \ is \ odd \quad (b) \end{cases}, \tag{1}$$

which forms an iterative sequence A006370 [2] that is the cycle according to Collatz’s hypothesis

$$... \to 4 \to 2 \to 1. \tag{2}$$

Many works were devoted to Collatz’s task [3,4]. Much attention was paid to the Collatz’ graphs dynamics [5-7]. The given article has formulated the task to research graphs forming, covers the laws of elevation to the degree of Newton's binomial

$$(x+y)^\xi, \ x = y \ \rangle \ 1, \ \xi = 0,1,2,3,4, ... \tag{3}$$

That gave possibility to generalize the algorithm (1b) as follows

$$f(n) = \quad if \quad n \equiv 0 \quad mod\, 2 \quad then \quad \frac{n}{2} \ (a) \quad else \quad (2x-1)n + 1 \ (b). \tag{4}$$

Transformations (1a) as well as stop cycle (2), in algorithm (4), are unchanged as in contrast to the division of even numbers by 4, 8, 16 etc. when dividing an even number by 2, there is no fractional remainder.

The Collatz graph is shown in Fig.1. It is based on the sequence A000079 [2] of numbers with a binary base

$$K_{m=1} = \{2^0, \ 2^1, \ 2^2, . 2^3, \ 2^4, \ 2^5, \ 2^6, \ 2^7, 2^8, .., 2^\xi, ...\}, \quad \xi = 0,1,2,3, ..., \tag{5}$$

which will be called $K_{m=1} = K_{main}$. The sequence (5) is formed by the power transformation of

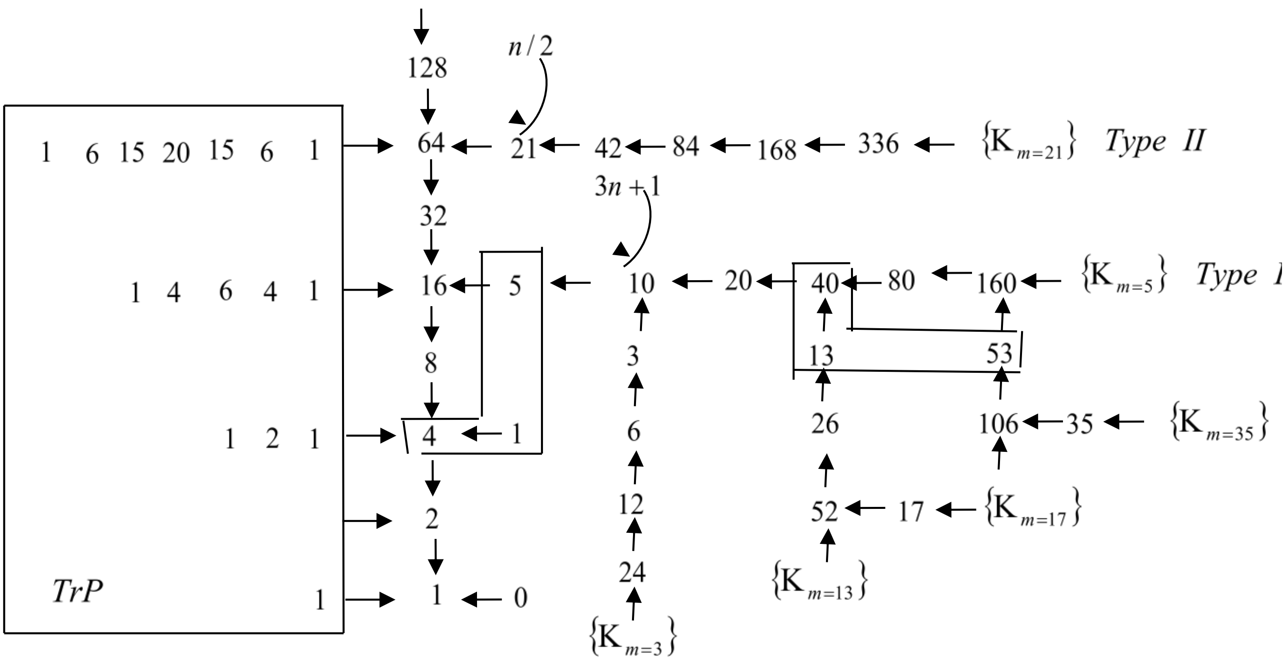

Fig.1. Collatz graphs with nodes (*Type I*) and without nodes (*Type II*).

Newton's binomial $(1+1)^{\xi}=\sum_{i=0}^{\xi} C_{\xi}^{i}$, whose binomial coefficients C_{ξ}^{i} form Pascal's triangle (TrP) [8]. The rows of a Pascal's triangle with an odd number (OdN) of binomial coefficients correspond to the nodes on the main graph.

The total numbers $\sum_{i=0}^{\xi} C_{\xi}^{i}$ with an OdN of binomial coefficients form even numbers of type $2^{2\xi}$ and OdN of the type $2^{2\xi}-1$. The first OdN of the type $2^{2\xi}-1$ equals $2^{2}-1$. Thus on the main graph the sequence of numbers

$$m=\frac{2^{2\xi}-1}{2^{2}-1}=\frac{2^{2\xi}-1}{3} \quad (a) \quad \Rightarrow \quad m: \quad \{0,\ 1,\ 5,\ 21,\ 85,\ 341,\ 1365,\ 45461, \ldots\} \quad (b), \tag{6}$$

known as A002450 [2], form nodes, from which further side graphs with numbers are generated m >1 (Fig.1). The sequence of numbers (6b), in its turn is divided into two types: numbers of the type 5, 85, 341, 5461, 21845,…, which are not divided by three, and numbers of the type 21, 1365, 87381, … which are divided by three. Thus, on the main Collatz's graph side graphs are also formed of two types: in the first case with nodes of $Type\ I$, in the second case without nodes of $Type\ I$I. The repetition period of these nodes is nonlinear, with increasing position ξ forms the sequence of numbers

$$T_{\xi+1,\xi}=(4-1)\cdot\{4^{\xi}\} \quad \Longrightarrow \quad 1 \overset{T_{1,0}=3}{\longleftrightarrow} 4 \overset{T_{2,1}=12}{\longleftrightarrow} 16 \overset{T_{3,2}=48}{\longleftrightarrow} 64 \overset{T_{4,3}=192}{\longleftrightarrow} 256 \overset{T_{5,4}=768}{\longleftrightarrow} 1024 \longleftarrow \tag{7}$$

We will show that the main graph $\{K_{main}\}=\{K_{m=1}\}$ and side graphs $\{K_{m}\}$, are connected by the correlation:

$$\{K_{m}\}=m\cdot\{K_{main}\}=m\cdot\{K_{m=1}\}. \tag{8}$$

To do this, consider any side graph with a number m = 5. The graph shows the sequence of numbers

$$\{m\}: \quad \{5,\ 10,\ 20,\ 40,\ 80,\ 160, \ldots\}, \tag{9}$$

known as A020714 [2]. Writing (8) as $5\cdot\{2^{0}, 2^{1},\ 2^{2},\ 2^{3},\ 2^{4},\ 2^{5},\ 2^{6},\ 2^{7}, 2^{8}, \ldots\}$, we will get the formula confirmation (8). For all other side graphs with numbers $m \rangle 1$, the formula (8) is grounded similarly. The fact that formula (8) is universal is confirmed by results of the below mentioned calculations according to the formula (8) for any nodes on any side graphs

$$\begin{cases} m=1: \quad \{K_{m=1}\} = 1\cdot\{K_{main}\} = \{1{,}2,\ 4,\ 8,\ 16,\ 32,\ 64,\ 128, 256,\ 512,\ 1024,\ 2048,\ 4096,\dots\} \\ m=5: \quad \{K_{m=5}\} = 5\cdot\{K_{main}\} = \{5,\ 10,\ 20,\ 40,\ 80,\ 160,\ 320,\ 640,\ 1280,\ 2560,\ 5120,\dots\} \\ m=:21: \quad \{K_{m=21}\} = 21\cdot\{K_{main}\} = \{21{,}42,\ 84,\ 168,\ ,\ 336,\ 672,\ 1344, 2688, 5376,\ 10752,\dots\} \\ m=85: \quad \{K_{m=85}\} = 85\cdot\{K_{main}\} = \{85{,}170,\ 340,\ 680,\ 1360,\ 2720,\ 5440,\ 10880,\ 21760\dots\} \\ m=341: \quad \{K_{m=341}\} = 341\cdot\{K_{main}\} = \{341,\ 682,\ 1364,\ 2728,\ 5456,\ 10912,\ 21824,\ 43648\dots\} \\ m=1365: \quad \{K_{m=1365}\} = 1365\cdot\{K_{main}\} = \{1365,\ 2730,\ 5460,\ 10920,\ 21840,\ 43680,\ 87360,\dots\} \\ m=5461: \quad \{K_{m=5461}\} = 5461\cdot\{K_{main}\} = \{5461,\ 10922,\ 21844,\ 43688,\ 87376,\ 174752,\ ,\dots\} \\ m=:21845\ \{K_{m=21845}\} = 21845\cdot\{K_{main}\} = \{21845,\ 43690,\ 87389,\ 174760,\ 349520, 699040,\dots\} \\ \dots \end{cases}, \qquad (10)$$

where the numbers of new nodes are calculated according to the formula:

$$m_S = \frac{m\cdot 2^S - 1}{3} \quad (a), \quad \Rightarrow$$

$$\begin{cases} m=3 \quad \textit{has no nodes} \\ m=5 \quad \begin{cases} s: & 1, & 3, & 5, & 7, & 9, & 11, & 13, \\ m_S: & 3, & 13, & 53, & 213, & 853, & ,3413, & 13653 \end{cases} \\ m=13 \quad \begin{cases} s: & 2, & 4, & 6, & 8, & 10, & 12, & 14, \\ m_S: & 17, & 69, & 277, & 1109, & 4437, & 17749, & 70997,.. \end{cases} \\ m=17 \quad \begin{cases} s: 1, & 3, & 5, & 7, & 9, & 11, & 13,\ . \\ m_S: 11, & 45, & 181, & 725, & 2901, & 11605, & 46421,.. \end{cases} \\ m=53 \quad \begin{cases} s: & 1, & 3, & 5, & 7, & 9, & 11, & 13,.\ .. \\ m_S: 35, & 141, & 565, & 2261, & 9045, & 36181, & 144725,,.. \end{cases} \\ m=85 \quad \begin{cases} s: & 2, & 4, & 6, & 8, & 10, & 12, & .. \\ m_S: & 113, & 453, & 1813, & 7253, & 29013, & 116053, & .. \end{cases} \quad (b), \\ m=113 \quad \begin{cases} s: & 1, & 3, & 5, & 7, & 9, & 11, & .. \\ m_S: & 75, & 301, & 1205, & 4821, & 19285, & 77141, & .. \end{cases} \\ m=213 \quad \textit{has no nodes} \\ m=227 \quad \begin{cases} s: & 1, & 3, & 5, & 7, & 9, & 11, & \dots \\ m_S: ,151, & 605, & 2421, & 9685, & 38741,. & 154965,\dots & . \end{cases} \\ m=853 \quad \begin{cases} s: & 2, & 4, & 6, & 8, & 10, & \dots \\ m_S: & 1137, & 4549, & 18197, & 72789, & 291157,. & . \end{cases} \\ m=1024 \quad \begin{cases} s: & 2, & 4, & 6, & 8, & 10, & .. \\ m_S: & 1365, & 5461, & 21845, & 87381, & 349525 & .. \end{cases} \\ \dots \end{cases} \qquad (11)$$

In formula (11), the number value m_S is the same as the value number m. Thus, the formula of connection (8), together with formulas (11), allows the process of side graphs multiplication in the Collatz's tree to continue indefinitely.

In Fig.1, with the dotted line in the form of a complex rectangular shape, the rule of addition is highlighted

$$1 = 0+1, \quad 5 = \begin{cases} 4^1+1 \\ 4\cdot 1+1 \end{cases}, \quad 21 = \begin{cases} 4^2+1 \\ 4\cdot 5+1 \end{cases}, \quad 85 = \begin{cases} 4^3+1 \\ 4\cdot 21+1 \end{cases}, \quad 341 = \begin{cases} 4^4+1 \\ 4\cdot 85+1 \end{cases}, \dots \qquad (12)$$

We ground the universal rule (12). To do this we consider the side graph m = 5 on which the nodes with numbers are formed

$$\{m\}: \quad \{3,\ 13,\ 53,\ 213,\dots\}, \qquad (13)$$

that form the sequence A072197 [2]. For nodes with numbers (13), the rule (12) is written in the following way:

$$3 = 0+3, \quad 13 = \begin{cases} 3+10 \\ 4\cdot 3+1 \end{cases}, \quad 53 = \begin{cases} 13+40 \\ 4\cdot 13+1 \end{cases}, \quad 213 = \begin{cases} 53+160 \\ 4\cdot 53+1 \end{cases}, \quad 853 = \begin{cases} 213+640 \\ 4\cdot 213+1 \end{cases}, \dots \qquad (14)$$

Let's take from the sequence (11) other any side graph, for example with number m=4549 that is generated from the side graph with number m=853. On the given graph according to the rule (11a) the nodes are formed

$$nodes_S = 4549 \quad \begin{cases} s: \quad 2, \quad 4, \quad 6, \quad 8, \quad 10, \\ m_S: 6065, \quad 24261, \quad 97045, \quad 388181, \quad 1552725, \dots \end{cases} \tag{15}$$

As according to the formula (8) we have $\{K_{m=4549}\} = 4549 \cdot \{K_{main}\} =$ 4549, 9098, 18196, 388181, 1552725,…, for the given sequence of nodes the rule (12) is written in the following way: $8196 + 6065 = 24261$ (element s=4 in (15)), $72784 + 24261 = 97045$ (element s=6 in (15)), $291136 + 97045 = 388181$ (element s=8 in (15)) etc. So, the rule (12) of numbers composition in the nodes of the graphs is universal.

Now consider the laws of Collatz graphs, built on the laws of elevation to the degree of Newton's binomial

$$(x + y)^{\xi}, \; x = y = 2,4,8, \dots \; \xi = 0,1,2,3,4, \dots \tag{16}$$

Thus, Newton's binomial

$$(2 + 2)^{\xi}, \; \xi = 0,1,2,3, \dots \tag{17}$$

Forms the main graph $\theta = 1$, which corresponds to the following sequence

$$\aleph_{\theta=1} \coloneqq \{16^0, 16^1, \; 16^2, \; 16^3, \; 16^4, \; 16^5,,, \dots\}. \tag{18}$$

Results and discussion

On the graph (18), the nodes are formed at points with numbers

$$\{\theta\}: \quad \{0, \; 1, \; 17, \; 273, \; 4369, \; 69905, \; 1118481, \; 17895697, \dots\}, \tag{19}$$

which are calculated according to the recurrent formula

$$\theta_s = \frac{1 \cdot 16^s - 1}{15}, \quad s = 0,1,2,3, \dots \tag{20}$$

For the nodes (18) the rule is true (12):

$$1 = 0 + 1, \quad 17 = \begin{cases} 1 + 16^1 \\ 16 \cdot 1 + 1 \end{cases}, \quad 273 = \begin{cases} 17 + 16^2 \\ 16 \cdot 17 + 1 \end{cases}, \quad 4369 = \begin{cases} 273 + 16^3 \\ 16 \cdot 273 + 1 \end{cases}, \quad 69905 = \begin{cases} 4369 + 16^4 \\ 16 \cdot 4369 + 1 \end{cases}, \dots \tag{21}$$

In the nodes (19) as well as in the nodes with other numbers the model (17) forms both types of side graphs $Type\ I$ and $Type\ I$I (Fig.2). For side graphs of $Type\ I$, the numbers of the nodes are calculated according to the formula:

$$\Omega = \frac{2\theta \cdot 4^S - 1}{15},$$

$$\Rightarrow \begin{cases} \theta = 17 \quad \begin{cases} s: 1, \quad 3, \quad 5, \quad 7, \quad 9, \quad 11, \quad . \\ \Omega: 9, \quad 145, \quad 2321, \quad 37137, \quad 594193, \quad 9507089,,.. \end{cases} \\ \theta = 273 \quad has\ no\ nodes \\ 2\theta = 4369 \quad \begin{cases} s: \quad 1, \quad 3, \quad 5, \quad 7, \quad 9,, \quad . \\ \Omega: 1165, \quad 18641, \quad 298257, \quad 4772113, \quad ,76353809,,.. \end{cases} \\ \theta = 69905 \quad has\ no\ nodes \\ \theta = 1118481 \quad has\ no\ nodes \\ \dots \end{cases} \tag{22}$$

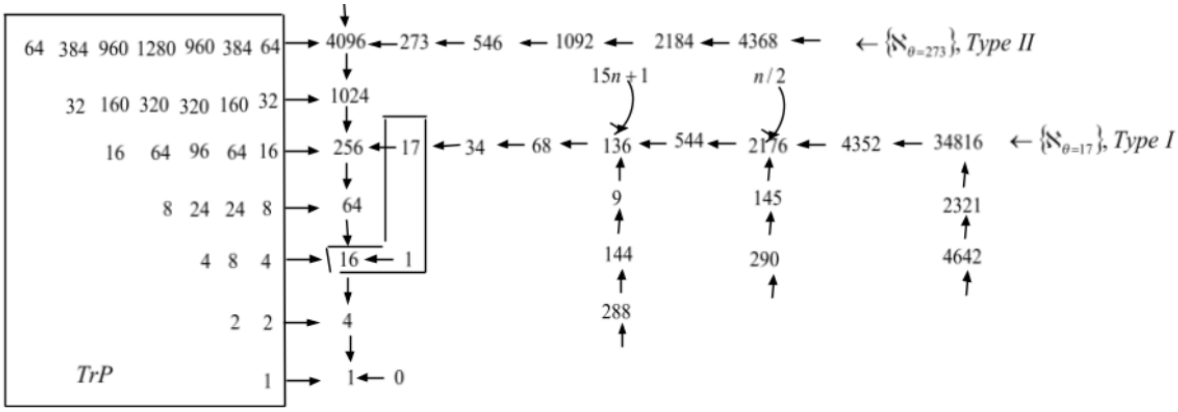

Fig.2. Plot of Pascal's triangle TrP and Collatz's graph in the model (17)

In the model (17) the main graph $\{G_{m=1}\} = \{K_{m=1}\}$ and side graphs $\{G_{m\,\rangle\,1}\}$ are connected between similar (8) correlation:

$$\{G_m\} = \theta \cdot \{K_{m=1}\}, \tag{23}$$

and algorithm (1) in the given case is formulated as follows:

$$\ell_n = \begin{cases} \frac{1}{2}\ell_{n-1} \; if \; \ell_{n-1} \; is \; even \quad (a) \\ 15\ell_{n-1} + 1 \; if \; \ell_{n-1} \; is \; odd \quad (b) \end{cases}. \tag{24}$$

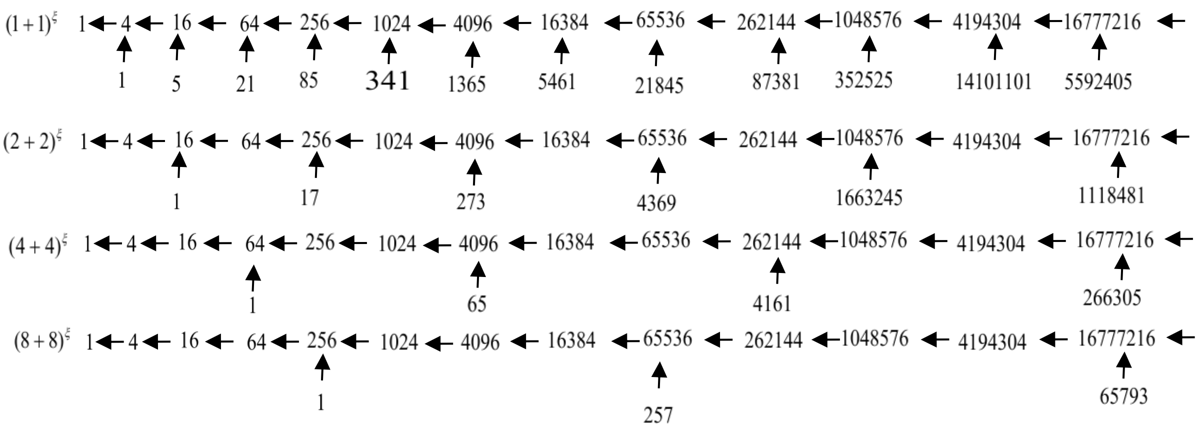

Fig.3. Plot of the main graphs Collatz in the model (16)

The Collatz's graphs can be formed on the basis of power transformations of Newton's binomial with larger values of arguments $x = 4,8,16,,,$ To compare, in Fig.3 the main graphs for first four meanings of $x = 1,2,4,8$ are given, the numbers of the nodes are calculated according to the recurrent formula:

$$m_S = \frac{4^S x^{2S} - 1}{4x^2 - 1}, \quad s = 1,2,3,4,... \tag{25}$$

The first node on each of the graphs is formed at the point with the number $m_1 = (2x)^2$.

With the growing of ξ in (17), periods of recurrence of nodes in the graphs of Fig.3 increase as power functions so that their values form the sequence:

$$T_{\xi+1,\xi} = (4x^2 - 1) \cdot \{(2x)^{2\xi}\}: \quad \Rightarrow \begin{cases} x = 1, \quad T_\xi = 3 \cdot \{4^0, 4^1, 4^2, 4^3, ...\}, \\ x = 2, \quad T_\xi = 15 \cdot \{16^0, 16^1, 16^2, 16^3, ...\}, \\ x = 4, \quad T_\xi = 63 \cdot \{64^0, 64^1, 64^2, 64^3, ...\}, \\ \quad ... \end{cases} \tag{26}$$

In general case $x = 1,2,4,8$, The Collatz's algorithm (1) is formulated in the following way:

$$\ell_n = \begin{cases} \frac{1}{2}\ell_{n-1}\ if\ \ell_{n-1}\ is\ even \quad (a) \\ (4x^2-1)\ell_{n-1}+1\ if\ \ell_{n-1}\ is\ odd \quad (b) \end{cases}. \tag{27}$$

In other words, for $x = 2,4,8$ odd numbers are transformed by the following functions (28)

$$\begin{cases} x=2, \quad f(x)=15x+1, \\ x=4, \quad f(x)=63x+1, \\ x=8, \quad f(x)=255x+1, \\ \quad\quad ... \end{cases}. \tag{28}$$

In (27a) even numbers are divisible by half because the division of even numbers by 4, 8, 16 etc., may be accompanied by the appearance of a fractional residue.

Conclusions

In an infinite set of natural integers $n \in \mathbb{N}$, for any natural number there is another natural number bigger than it is. Therefore, each other natural integer corresponds to a different graph (trajectory) of Collatz. The Collatz's graphs are formed of two types, with nodes and without them. The trajectories of the odd numbers transformations in 3,6,8,…. Iterations from the beginning of the calculations merge with the trajectory of calculations of the first smaller by the value of the number. Therefore, the index m is a universal characteristic of the trajectory of Collatz transformations of even and odd numbers.

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ВІД БІНОМА НЬЮТОНА ТА ТРИКУТНИКА ПАСКАЛЯ ДО ЗАДАЧІ КОЛЛАТЦА

Отримано: жовтень 01, 2023 / Переглянуто: жовтень 10, 2023 / Прийнято: жовтень 14, 2023

Анотація. Показано, що: 1. Послідовність $\{2^0, 2^1, 2^2, 2^3, 2^4, 2^5, 2^6, 2^7, 2^8, ...\}$, яка утворює головний графік $m = 1$ Коллатца, пов'язана зі степеневим перетворенням бінома

Ньютона $(1+1)^{\xi},\ \xi = 0,1,2,3,\ldots$ 2. Головний K_{main} і бічний m >1 графіки та відповідні їм послідовності $\{K_{main}\}$ і $\{K_m\}$ пов'язані співвідношенням $\{K_m\} = m \cdot \{K_{main}\}$. 3. Бічні графи, породжені простими непарними числами 5, 7, 11, 13, 17, 19, 23, 25, 29, 31,…не діляться на три, утворюються без вузлів. Бічні графи, які генеруються композицією непарних чисел 3, 9, 15, 21, 27, 33, 39, 45,… діляться на три, утворюються з вузлами. 4. Траєкторії перетворень непарних чисел, через 3, 6, 8,…. ітерації від початку обчислень, зливаються з траєкторією обчислень першого меншого за значенням числа.

Ключові слова: гіпотеза Коллатца, рекурентна послідовність, перетворення 3n + 1, натуральні числа.