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Parameter optimization decomposition and synthesis algorithm for a bundle of rotation shells connected with a ring frame

Dzyuba A. P.[1,2], Safronova I. A.[1], Sirenko V. N.[2], Torskyy A. R.[3,4]

[1] *Oles Honchar Dnipro National University, 72 Gagarina Av., Dnipro, Ukraine*
[2] *Yuzhnoye State Design Office, 3 Kryvorizka Str., Dnipro, Ukraine*
[3] *Pidstryhach Institute for Applied Problems of Mechanics and Mathematics, NASU, 3b Naukova Str., Lviv, Ukraine*
[4] *Lviv Polytechnic National University, 12 S. Bandera Str., Lviv, Ukraine*

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The method of weight optimization of a shell structure consisting of a power ring frame connected to it on each side of non-homogeneous shells of rotation with variable wall thickness under the action of a spatially asymmetric load is presented. The construction decomposition algorithm is applied. The optimization of shells is carried out based on the necessary Pontryagin's optimality conditions with phase constraints. Finite-dimensional optimization methods are used to seek the optimal configuration of the ring frame. The synthesis of the construction is carried out by the method of successive approximations. Numerical optimization results are presented.

Keywords: *bundle of rotation shells; ring frame; asymmetrical load; parameter optimization.*

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1. Introduction

Constituent elements of many machine-building structures in the aerospace industry are a system of several shells of rotation with irregular parameters connected by a power frame [1–6]. Solving the problems of weight optimization of such composite reinforced shell structures is rather complicated, which relates to the need to fulfill not only the requirements of strength reliability but also the conditions of joint deformation of structural components. Such shells of rotation with non-uniform parameters and the ring frame at each step need iterative search algorithm.

It should be noted that attempts to simultaneously use a sufficiently complex and cumbersome complete system of equations describing the behavior of both individual elements and the entire system of heterogeneous shells of rotation connected by a frame under arbitrary loading generally encounter significant difficulties [2]. They can be combined with finite-dimensional optimization methods [7, 8], taking into account the complexity of the optimization problem itself, which is usually non-linear and multi-parameter.

This is explained by the fact that the thicknesses of the shell walls are variable in order to reduce the weight indicators and (or) increase the strength reliability along the meridian of the shell wall thickness [9, 10] or the mechanical characteristics of the composite material are piecewise continuous functions.

So, the effective solution the problem requires the application of methods of the theory of optimal processes [12, 13], and the problem of determining the optimal geometric parameters of the cross sections of power elements is directly a problem of finite-dimensional optimization [7, 8].

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The application of finite element methods in the tasks of calculation and weight optimization of shell structures is presented in [14, 15].

The combination of the possibilities of joint application of the discrete-continuous approach, methods of the theory of optimal processes and finite-dimensional optimization in a single algorithm can be quite promising for solving applied problems.

2. Optimal design algorithm

Taking into account the mentioned circumstances, the presented article develops a method of optimal design of a shell structure [2, 5], which consists of a power frame with an arbitrary cross-sectional configuration and sets of several thin elastic shells connected to it on each side at arbitrary points (taking into account the eccentricity) rotation of variable stiffness under the action of spatially asymmetric loading through the frame and uneven pressure on the shells (Figure 1), where M_i, M_t, M_k, N_e, T_j, $q(s,\varphi)$, $q_1(s,\varphi)$, $q_2(s,\varphi)$ are external load components; $\bar{f}_i$, $\bar{f}_j$, $\bar{f}_k$ are force vectors of the interaction of the frame and the shells at the points C_1, C_2, C_3 of their connection; X_0OZ_0 are the main axes of inertia of the frame.

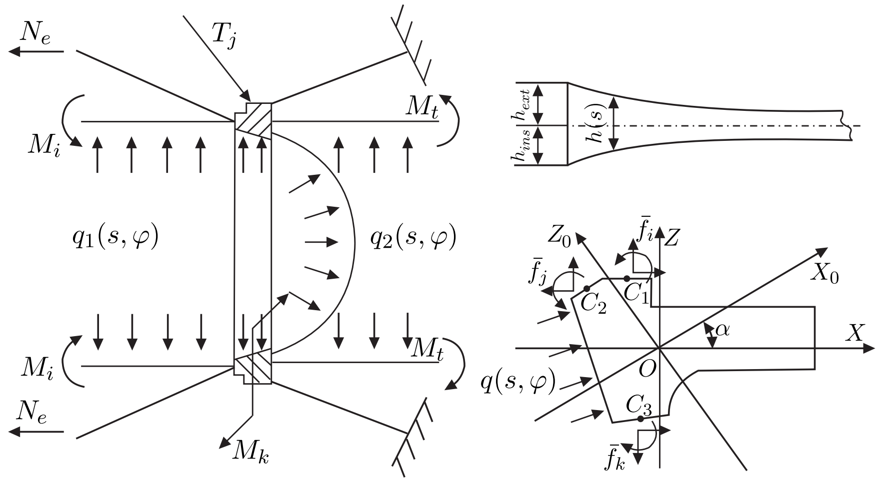

Fig. 1. Calculation scheme of a bundle of shells of rotation connected with a ring frame under the action of an asymmetric load.

In the general case, the wall thickness $h^t(s)$ $(t = 1, 2, \ldots)$ of the shells can be distributed asymmetrically relative to the drive surface and consist of external $h^t_{ext}(s)$ and internal $h^t_{ins}(s)$ parts [10]. The contact node of such a composite shell structure is divided into a ring with an arbitrary cross-sectional shape and a transition section with a thickness that varies smoothly and refers to the shell of rotation.

The essence of the proposed approach to solving the problem of weight optimization of such a complex structure consists in its separation (decomposition) into separate components (substructures) in the form of a power ring (frame) and shells of rotation with subsequent optimization of their parameters. For shells, this is the task of determining the optimal variable in the transition zones of the wall thickness, which is solved using the necessary optimality conditions of L. S. Pontryagin's maximum principle [12] in the presence of phase restrictions [13, 16] and the geometric dimensions of the cross sections of the frames by methods dimensional optimization [7, 8]. The synthesis of optimal substructures to clarify the forces of their joint deformation is carried out by the method of successive approximations [17].

The basic block diagram of the implementation of the proposed approach is shown in Figure 2. Here A is preparation and entry of input data; B is direct calculation of the composite shell structure (synthesis); C is calculation of interaction forces of individual substructures (decomposition of the structure); D_r, D^t_{sh} are determination of the stress-deformed state of the frame and t-th shells; E_r, E^t_{sh} are solving optimization problems for the frame and shells to determine the optimal geometric dimensions of the cross-section of the frame and the thickness of the walls $h^t(s)$ of the t-th shells; F

is comparative analysis of joint deformation efforts of substructures with optimal parameters for two consecutive iterations; G is exit conditions; H is processing results.

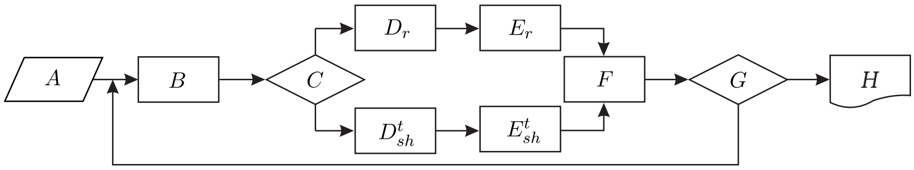

Fig. 2. The basic block diagram of the optimization of the composite shell structure.

Thus, it is proposed solving such a rather complex optimization problem by decomposing it into separate substructures in the form of reinforcing rings and shells of rotation of variable thickness $h(s)$ along the meridian, further optimizing their parameters and synthesizing the structure using iterative refinement of interaction forces individual elements.

At the same time, the task of optimal design of each of the shells of rotation, that are under the action of internal pressure and asymmetric marginal forces of interaction with the frame (applied along the line of their junction), is to find the variables along the meridian of the thicknesses of the shells $h^t(s)$ (transition sections) from the condition of minimum volume of their material

$$V^t = \min 2\pi\rho \int_{s_0}^{s_N} r^t(s) h^t(s)\, ds \tag{1}$$

(then, to reduce the cumbersomeness of the exposition, the index t is omitted) when performing equilibrium equations with boundary conditions of fastening [9,10] and limitations of strength, stiffness and design requirements, respectively

$$\bar{\sigma}(h, \bar{Y}, s) \leqslant [\bar{\sigma}];\ (a) \qquad \bar{w}(s) \leqslant [\bar{w}];\ (b) \qquad h(s) \geqslant h_0,\ (c) \tag{2}$$

where $s_0 \leqslant s \leqslant s_N$; $\bar{w}(s) = (\xi, \vartheta, \zeta)^T$; ξ, ς are radial and longitudinal movements; ϑ is the angle of rotation of the section, and the acting stresses σ_1, σ_2 are expressed through the components of the stress-deformed state in the form

$$\begin{gathered}
\sigma_1 = \frac{\cos\theta}{rh}(Nr) + \frac{12z}{rh^3}(M_1 r) + \frac{\sin\theta}{rh}\frac{F(s)}{2\pi};\\
\sigma_2 = \frac{E}{r} + \frac{Ez\cos\theta}{r}\vartheta + \frac{\mu\cos\theta}{rh}(Nr) + \frac{12\mu z}{rh^3}(M_1 r) + \frac{\mu\sin\theta}{rh}\frac{F(s)}{2\pi};\\
\tau_{1z} = \frac{3}{2h}\left(1 - \frac{4z^2}{h^2}\right)\left(\frac{\sin\theta}{r}(Nr) - \frac{\cos\theta}{r}\frac{F(s)}{2\pi}\right).
\end{gathered} \tag{3}$$

Here $r(s)$ is the radius of a parallel circle; $\theta(s)$ is the angle between the normal to the middle surface of the shell and the axis of rotation; Nr, $M_1 r$, $F(s)$ are tensile force, bending moment and load function [10,19]; z is shell thickness coordinate.

In the work, it is assumed that the greatest stresses occur on the surface of the shell $z = \pm h/2$, and strength constraints (2a) can be given in the form

$$\max_z \sigma_i \leqslant [\sigma], \quad \sigma_i = \sqrt{\sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2 + 3\tau_{1z}^2}. \tag{4}$$

The number of points where the optimal thickness must be determined is often quite large, even for a single shell, which makes it difficult to implement the problem using finite-dimensional optimization methods [7,8]. Therefore, solving the problem of weight optimization of the envelopes of rotation with variable wall thickness is carried out using the methods of the theory of optimal processes [16].

The problem of optimizing the geometric shape of the power frame, which is under the influence of external loads and forces of interaction with the shells, which are determined by solving the direct problem of calculating the stress-strain state of the composite structure and are considered known at each step of the search for the optimal design, is formulated as a problem of nonlinear programming and consists in finding a finite number of varied geometric dimensions of the cross-section of the frame, which provide a minimum volume of its material, while fulfilling the strength restrictions imposed at dangerous and (or) specified points of its cross-section and on the components of the displacement vector (linear, angular) of some cross-section points, as well as when fulfilling a number of structural (technological) requirements for the geometric parameters of the cross-section of the frame.

$$\begin{aligned} &V_{opt} = \min_{\delta_k} F(\delta_k) R_c; \text{ (a)} \quad \sigma \max_j (\delta_k) \leqslant [\sigma],\ j = \overline{1,n}; \quad \text{(b)} \\ &\bar{u}(\delta_k) \leqslant [\bar{u}]; \quad \text{(c)} \quad \psi_t(\delta_k) \leqslant 0,\ k = \overline{1,p}. \quad \text{(d)} \end{aligned} \tag{5}$$

Here F is cross-sectional area; R_c is the radius of the main axes of inertia of the cross-section of the ring frame (distance to the origin of coordinates).

It should be noted that such a step-by-step optimization not only allows the use of more adequate calculation schemes of composite shell structures, but also provides additional opportunities for considering a few different mechanical aspects.

Such as the asymmetric location (eccentricity) of the frame relative to the attached shells, its arbitrary, in accordance with the design requirements, shape, two- or one-sided (including stepwise, as more technological) change in the wall thickness of the shells and other design features. And although in the general case the correctness of solving the optimization problem with the help of such an approach cannot be proven (and therefore its use requires additional analysis of the obtained results and certain comparisons), such an approach allows you to significantly simplify the search for some rational design that satisfies the above requirements when minimum consumption of material for each of the substructures.

Solving the problem of finding both the optimal change in the thickness of the walls of the shells and the geometric dimensions of the cross-section of the frame can be carried out only with a known external load, which at the design stage includes the interaction forces $\bar{Q}(s_e, \varphi)$ of the substructures, which in turn depend on the sought-after geometric dimensions, stiffness ratios of shells and frames. Therefore, after solving the relevant optimization problems for individual substructures, their interaction efforts may change. The further task is to find out $\bar{Q}(s_e, \varphi)$ from the conditions the joint deformation of the entire structure. One of the possible ways of solving such a problem is the iterative refinement of the interaction efforts of individual substructures.

3. Algorithm of direct calculation of shells of rotation with variable stiffness

The algorithm for calculating the structure under consideration is based on the application of the discrete-continuous model [2,5], which is developed below for the case of irregular variable shell parameters [9,10,18]. Differential equations of the linear moment theory of shells in partial derivatives describing the state of the shells under asymmetric loading for the case of a change in the thickness of the shell wall only in the meridional direction $h = h(s)$, using the expansions of the external load, internal forces, and displacements in the Fourier series by circular coordinate in the form [2,10,18–20]

$$f = \sum_{k=0}^{\infty} f_k^c \cos k\varphi + \sum_{k=1}^{\infty} f_k^s \sin k\varphi; \quad \psi = \sum_{k=1}^{\infty} \psi_k^c \sin k\varphi - \sum_{k=0}^{\infty} \psi_k^s \cos k\varphi \tag{6}$$

are reduced to a system of ordinary differential equations for shells

$$\frac{d\bar{Y}_k}{ds} = \bar{C}_k(h,s) Y_k + \bar{F}_k(h,s); \quad s_0 \leqslant s \leqslant s_L \tag{7}$$

relative to the k-th ($k = 0, 1, \ldots$) harmonic of the vector of the main variables $\bar{Y}_k = (\zeta, \xi, \vartheta_1, v, Xr, Zr, M_1 r, S_1^* r)_k^T$ with the boundary conditions of fixing the edges and joint deformation along the line of junction of the shells with the frame in the form:

$$\bar{a}_{ke} \bar{Y}_k(s_e) + \bar{b}_{ke} = 0; \quad e = 0 \vee e = L. \tag{8}$$

Here f_k^A, f_k^s, ψ_k^c, ψ_k^s are coefficients of such expansions into trigonometric series, respectively, for symmetric and obliquely symmetric component movements, internal forces and loads; ζ, ξ, ϑ_1, v are components of movements, and Xr, Zr, $M_1 r$, $S_1^* r$ are internal efforts, respectively; $\bar{a}_{ke}$, $\bar{b}_{ke}$ are matrices of coefficients of Fourier series expansions of boundary conditions. The purpose and sequence of the components of the vector $\bar{Y}_k$ is adopted in accordance with [18].

To reduce the number of boundary value problems (7) ($k = 0, 1, \ldots$) necessary to achieve the given accuracy of the solution, and to reduce computational costs in general, the forecasting algorithm presented in [21] is effective.

Since the forces and displacements in the equations of state of the shell in [5, 9, 18] are related to the local coordinate system related to the normal and tangent to the meridian, the coefficients of the system may have gaps in the case when the meridian of the shell consists of several sections with angular points between them. This leads to the need to construct compatibility equations of deformation of different areas.

According to [19], these difficulties can be circumvented by switching to global coordinates. For this, internal forces and movements are projected not on the tangent and normal to the meridian, but on the normal to the axis of symmetry of the shell and on the axis itself. The coefficients of the Fourier series expansions of the corresponding functions for the radial and axial components of the load will be connected by the same transformations.

After such transformations, the coefficients of the resulting system of equations do not contain the curvature of the meridian $1/R_1$, so the main unknowns, assigned to a fixed coordinate system, remain continuous even for shells that have surface breaks, which makes it possible not to add conjugation equations for such cases. As for force unknowns $(X^*r)_k$, $(Z^*r)_k$, $(S^*r)_k$, $(M_1r)_k$, they can undergo ruptures of a predetermined magnitude only where concentrated forces are applied to the shells on a specific parallel of the load.

The equations of state of the shells are obtained with respect to some driving surface, the thickness of the shell is variable along the meridian and asymmetric (in the general case) with respect to this surface, and the shell itself is under the influence of an arbitrary spatial load $q(s,\varphi)$, the intensity vector of which can be decomposed into axial $q_{z(k)}(s)$, radial $q_{x(k)}(s)$, and circumferential $q_{2(k)}(s)$ components for each harmonic. The coefficients of system (7) for the case of an inhomogeneous orthotropic shell will be as follows, where only non-zero elements are given:

$$C_{11} = -d_{14}\frac{k^2}{r^2}\sin\theta\cos\theta; \quad C_{12} = \frac{\sin\theta}{r}\left(d_{13} + d_{14}k^2\frac{\sin\theta}{r}\right); \quad C_{13} = \cos\theta\left(1 + d_{14}\frac{\sin\theta}{r}\right);$$

$$C_{14} = k\frac{\sin\theta}{r}\left(d_{13} + d_{14}\frac{\sin\theta}{r}\right); \quad C_{15} = d_{11}\frac{\sin^2\theta}{r}; \quad C_{16} = d_{11}\frac{\sin\theta}{r}\cos\theta; \quad C_{17} = d_{12}\frac{\sin\theta}{r};$$

$$C_{21} = -d_{14}\frac{k^2}{r^2}\cos^2\theta; \quad C_{22} = \frac{cos\theta}{r}\left(d_{13} + d_{14}k^2\frac{\sin\theta}{r}\right); \quad C_{23} = d_{14}\frac{\cos^2\theta}{r} - \sin\theta;$$

$$C_{24} = k\frac{\cos\theta}{r}\left(d_{13} + d_{14}\frac{\sin\theta}{r}\right); \quad C_{25} = d_{11}\cos\theta\frac{\sin\theta}{r}; \quad C_{26} = d_{11}\frac{\cos^2\theta}{r}; \quad C_{27} = d_{12}\frac{\cos\theta}{r};$$

$$C_{31} = -d_{34}\frac{k^2}{r^2}\cos\theta; \quad C_{32} = \frac{1}{r}\left(d_{33} + d_{34}k^2\frac{\sin\theta}{r}\right); \quad C_{33} = d_{34}\frac{\cos\theta}{r}; \quad C_{34} = \frac{k}{r}\left(d_{33} + d_{34}\frac{\sin\theta}{r}\right);$$

$$C_{35} = d_{12}\frac{\cos\theta}{r}; \quad C_{36} = d_{12}\frac{\cos\theta}{r}; \quad C_{37} = d_{32}\frac{1}{r}; \quad C_{41} = \frac{k}{r}\left(d_{22}\frac{1}{r} + \sin\theta\right); \quad C_{42} = k\frac{\cos\theta}{r};$$

$$C_{43} = -d_{22}\frac{k}{r}; \quad C_{44} = \frac{\cos\theta}{r}; \quad C_{48} = \frac{d_{21}}{r}; \quad C_{51} = \frac{k^2}{r^3}\left(d_{62}2 + d_{54}k^2\cos^2\theta\right);$$

$$C_{52} = -\frac{k^2}{r^2}\cos\theta\left(d_{53} + d_{54}k^2\frac{\sin\theta}{r}\right); \quad C_{53} = -\frac{k^2}{r^2}\left(d_{62} + d_{54}\cos^2\theta\right);$$

$$C_{54} = -\frac{k^3}{r^3}\cos\theta\left(d_{53}r + d_{54}\sin\theta\right); \; C_{55} = d_{14}\frac{k^2}{r^2}\cos\theta\sin\theta; \; C_{56} = d_{14}\frac{k^2}{r^2}\cos^2\theta; \; C_{57} = d_{34}\frac{k^2}{r^2}\cos\theta;$$

$$C_{58} = \frac{k}{r^2}\left(-r\sin\theta + 2d_{61}\right);\ C_{61} = -d_{54}\frac{k^4}{r^3}\cos\theta\sin\theta;\ C_{62} = \frac{d_{43}}{r} + \frac{k^2}{r^2}\sin\theta\left(d_{53} + d_{54}k^2\frac{\sin\theta}{r}\right); \quad (9)$$

$$C_{63} = \frac{\cos\theta}{r}\left(d_{44} + d_{54}\frac{k^2}{r^2}\sin\theta\right); \quad C_{64} = \frac{k}{r}\left(d_{43} + d_{44}\frac{\sin\theta}{r} + d_{53}k^2\frac{\sin\theta}{r} + d_{54}k^2\frac{\sin^2\theta}{r^2}\right);$$

$$C_{65} = -\frac{\sin\theta}{r}\left(d_{13} + d_{14}k^2\frac{\sin\theta}{r}\right); \quad C_{66} = -\frac{\cos\theta}{r}\left(d_{13} + d_{14}k^2\frac{\sin\theta}{r}\right); \quad C_{67} = -\frac{d_{33}}{r} - d_{34}\frac{k^2}{r^2}\sin\theta;$$

$$C_{68} = -k\frac{\cos\theta}{r}; \quad C_{71} = -\frac{k^2}{r^2}\left(2d_{62} + d_{54}\cos^2\theta\right); \quad C_{72} = \frac{\cos\theta}{r}\left(d_{53} + d_{54}k^2\frac{\sin\theta}{r}\right);$$

$$C_{73} = d_{54}\frac{\cos^2\theta}{r} + d_{62}\frac{2k^2}{r}; \quad C_{74} = k\frac{\cos\theta}{r}\left(d_{53} + d_{54}\frac{\sin\theta}{r}\right); \quad C_{75} = -\cos\theta\left(1 + d_{14}\frac{\sin\theta}{r}\right);$$

$$C_{76} = \sin\theta - d_{14}\frac{\cos^2\theta}{r}; \quad C_{77} = -d_{34}\frac{\cos\theta}{r}; \quad C_{78} = d_{61}\frac{2k}{r}; \quad C_{81} = -\frac{k^3}{r^3}\cos\theta\left(d_{44}r + d_{54}\sin\theta\right);$$

$$C_{82} = \frac{k}{r^2}\left(d_{43}r + d_{53}\sin\theta + d_{44}k^2\sin\theta + d_{54}k\sin^2\theta\right); \quad C_{83} = \frac{k}{r^2}\cos\theta\left(d_{44}r + d_{54}\sin\theta\right);$$

$$C_{84} = \frac{k^2}{r^2}\left(d_{43}r + d_{53}\sin\theta + k^2\sin\theta\left(d_{44} + d_{54}\frac{\sin\theta}{r}\right)\right); \quad C_{85} = -k\frac{\sin\theta}{r}\left(d_{13} + d_{14}\frac{\sin\theta}{r}\right);$$

$$C_{87} = -\frac{k}{r}\left(d_{33} + d_{34}\frac{\sin\theta}{r}\right); \quad C_{88} = -\frac{\cos\theta}{r}; \quad \bar{F}_k = \{0, 0, 0, 0, -rq_x, -rq_z, 0, -rq_2\}_k^T,$$

where

$$d_{11} = \frac{D_{11}}{\Delta}; \quad d_{12} = -\frac{A_{11}}{\Delta}; \quad d_{13} = \frac{A_{11}A_{12} - B_{12}D_{11}}{\Delta}; \quad d_{14} = \frac{A_{11}D_{12} - D_{11}A_{12}}{\Delta}; \quad d_{21} = \frac{r}{\Delta_1};$$

$$d_{22} = -\frac{2A_{33}r + 4D_{33}\sin\theta}{\Delta_1}; \quad d_{32} = \frac{B_{11}}{\Delta}; \quad d_{33} = \frac{A_{11}B_{12} - B_{11}A_{12}}{\Delta}; \quad d_{34} = \frac{A_{11}A_{12} - B_{11}D_{12}}{\Delta};$$

$$d_{43} = d_{33}A_{12} + d_{13}B_{12} + B_{22}; \quad d_{44} = d_{34}A_{12} + d_{14}B_{12} + A_{22}; \quad d_{45} = -d_{12}A_{21} - d_{11}B_{12};$$

$$d_{46} = -d_{32}A_{12} - d_{11}B_{12}; \quad d_{53} = d_{13}A_{12} + d_{33}D_{12} + A_{22}; \quad d_{54} = d_{14}A_{12} + d_{34}D_{12} + D_{22};$$

$$d_{55} = -d_{11}A_{12} - d_{12}D_{12}; \quad d_{56} = -d_{11}A_{12} - d_{32}D_{12}; \quad d_{61} = d_{21}A_{33};$$

$$d_{62} = d_{22}A_{33} + 2D_{33}; \quad \Delta = B_{11}D_{11} - A_{11}^2; \quad \Delta_1 = 2A_{33}\sin\theta + B_{33}r;$$

$$A_{11} = \frac{E_1}{1 - \mu_1\mu_2}\frac{h_2^2 - h_=^2}{2}; \quad A_{12} = A_{21} = \frac{E_1\mu_2}{1 - \mu_1\mu_2}\frac{h_2^2 - h_=^2}{2}; \quad A_{22} = \frac{E_2}{1 - \mu_1\mu_2}\frac{h_2^2 - h_=^2}{2};$$

$$A_{33} = G\frac{h_2^2 - h_=^2}{2}; \quad B_{11} = \frac{E_1}{1 - \mu_1\mu_2}\left(h_2 + h_=\right); \quad B_{12} = B_{21} = \frac{E_1\mu_2}{1 - \mu_1\mu_2}\left(h_2 + h_=\right);$$

$$B_{22} = \frac{E_2}{1 - \mu_1\mu_2}\left(h_2 + h_=\right); \quad B_{33} = G\left(h_2 + h_=\right); \quad D_{11} = \frac{E_1}{1 - \mu_1\mu_2}\frac{h_2^3 + h_=^3}{3};$$

$$D_{12} = D_{21} = \frac{E_1\mu_2}{1 - \mu_1\mu_2}\frac{h_2^3 + h_=^3}{3}; \quad D_{22} = \frac{E_2}{1 - \mu_1\mu_2}\frac{h_2^3 + h_=^3}{3}; \quad D_{33} = G\frac{h_2^3 + h_=^3}{3};$$

μ_1, μ_2, E_1, E_2 G are Poisson's ratios and elastic moduli in the meridional and circumferential directions, respectively; $r(s)$ is the radius of a parallel circle; $\theta(s)$ is the angle between the normal to the middle surface of the shell and the axis of rotation.

The integration of the boundary value problems (7), (8) for each of the structural shells is carried out by the orthogonal sweep method [18, 19].

4. Shared deformation of the ring frame and shells

The known relations for the ring frame [1, 2, 22] after expansion into Fourier series (6) are given in matrix form

$$\begin{array}{ll} \bar{\varepsilon}_k = \bar{G}_{0k} \cdot \bar{\Delta}_k; \text{ (a)} & \bar{Q}_k = \bar{G}_3 \cdot \bar{\varepsilon}_k; \quad \text{(b)} \\ \bar{Q}_k = \bar{G}_{1k} \cdot \bar{\Delta}_k; \text{ (c)} & \bar{G}_{2k} \cdot \bar{Q}_k = \bar{\theta}_k \bar{f}_k; \text{ (d)} \end{array} \tag{10}$$

and presents [2], respectively: (a) is connection between the deformations $\bar{\varepsilon}_k = (\varepsilon, \chi_2, \chi_1, \chi)_k^T$ of the median line passing through the center of gravity of the cross-section of the frame and the displacements $\bar{\Delta}_k = (u, w, \vartheta, v)_k^T$ of the points of this line; (b) are vectors of generalized internal efforts $\bar{Q}_k = (M_2, M_1, M_{tur}, N)_k^T$ in the middle line with the deformation vector $\bar{\varepsilon}_k$ (physical relationships); (c) is connection between the effort vector $\bar{Q}_k$ and the vector of generalized movements $\bar{\Delta}_k$; (d) is ring equilibrium equation.

After substituting (10,c) into (10,d), we obtain the equation of the stress-deformed state of the ring taking into account the forces of interaction with the connected shells in the form of a system of linear algebraic equations with respect to the amplitudes of the displacements of the ring frame axis

$$\bar{G}_k \bar{\Delta}_k = \bar{\theta} \cdot \bar{f}_k, \tag{11}$$

where $\bar{G}_{1k} = \bar{G}_3 \cdot \bar{G}_{0k}$; $\bar{G}_k = \bar{G}_{2k} \cdot \bar{G}_{1k}$; $\bar{f}_k = (p_2, q, m_k, p_1, m_2, m_1)_k^T$ is vector of the spatial load of the ring, reduced to the line of centers of gravity of its cross-section, matrices $\bar{G}_{0k}, \bar{G}_{2k}, \bar{G}_3, \bar{\theta}_k$ are quite simple to write out of the general equations of state of the ring [22].

$$\bar{G}_{0k} = \begin{vmatrix} 0 & 1/R & 0 & k/R \\ k^2/R^2 & 0 & -1/R & 0 \\ 0 & k^2/R^2 & 0 & k/R^2 \\ k/R^2 & 0 & -k/R & 0 \end{vmatrix}; \quad \bar{G}_{2k} = \begin{vmatrix} k^2/R^2 & 0 & k/R^2 & 0 \\ 0 & k^2/R^2 & 0 & 1/R \\ -1/R & 0 & -k/R & 0 \\ 0 & k/R^2 & 0 & k/R \end{vmatrix};$$

$$\bar{G}_3 = \begin{vmatrix} 0 & EI_x & EI_{xz} & 0 \\ 0 & EI_{xz} & EI_z & 0 \\ 0 & 0 & 0 & GI \\ EF & 0 & 0 & 0 \end{vmatrix}; \quad \bar{\theta}_k = \begin{vmatrix} 1 & 0 & 0 & 0 & k/R & 0 \\ 0 & 1 & 0 & 0 & 0 & k/R \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1/R \end{vmatrix},$$

where R is the radius of the ring frame's center of gravity.

To consider the conditions of joint deformation of the ring and the shell along the contact line (Figure 3), the interaction forces of the frame and shells are interpreted as additional external loads acting on the frame in the form of reactions from each t-th shell, assigned to the line of centers of the cross section of the power ring. These dependencies can be represented as

$$\bar{f}_{tk}^r = \bar{\psi}_t^r \frac{\bar{Q}_{tk}^{*r}}{R}; \quad \bar{f}_{tk}^L = -\bar{\psi}_t^L \frac{\bar{Q}_{tk}^{*L}}{R}, \tag{12}$$

where $\bar{Q}_{jk}^{*r(L)} = \left\{ y_{5t}^{r(L)}, y_{6t}^{r(L)}, y_{7t}^{r(L)}, y_{8t}^{r(L)} \right\}_k^T$ is the vector has been built from the marginal efforts at the corresponding ends, r is right or L is left shells; $t = 1, \ldots, m_r$; $t = 1, \ldots, m_L$; m_r, m_L are number of right or left shells;

$$\bar{\psi}_t^{r(L)} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ z_t^{r(L)} & -x_t^{r(L)} & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & x_t^{r(L)} \\ 0 & 0 & 0 & z_t^{r(L)} \end{vmatrix}, \tag{13}$$

where $\bar{\psi}_t^{r(L)}$ is matrix, $z_t^{r(L)}$, $x_t^{r(L)}$, $\gamma_t^{r(L)}$ are coordinates of the points of contact of the right r (left L) of the shells and the ring frame in the local system, which has its origin in the center of gravity of the cross-section of the ring frame. In addition, the presence of different angles of junction of shells and ring frames can be taken.

The equation of the state of the ring (11) with consideration of (12) takes the form

$$\bar{G}_k \cdot \bar{\Delta}_k = \bar{\theta}_k \left(\bar{f}_{0k} + \sum_{t=1}^{m_r} \bar{f}_{tk}^r + \sum_{t=1}^{m_L} \bar{f}_{tk}^L \right), \tag{14}$$

After substituting (13) into (12) and then into (14), can get

$$\bar{G}_k \cdot \bar{\Delta}_k = \bar{\theta}_k \bar{f}_{0k} + \sum_{t=1}^{m_r} \bar{\eta}_{tk}^r \frac{\bar{Q}_{tk}^{*r}}{R} - \sum_{t=1}^{m_L} \bar{\eta}_{tk}^L \frac{\bar{Q}_{tk}^{*L}}{R}, \tag{15}$$

where $\bar{\eta}_{tk}^{r(L)} = \bar{\theta}_k \bar{\psi}_t^{r(L)}$.

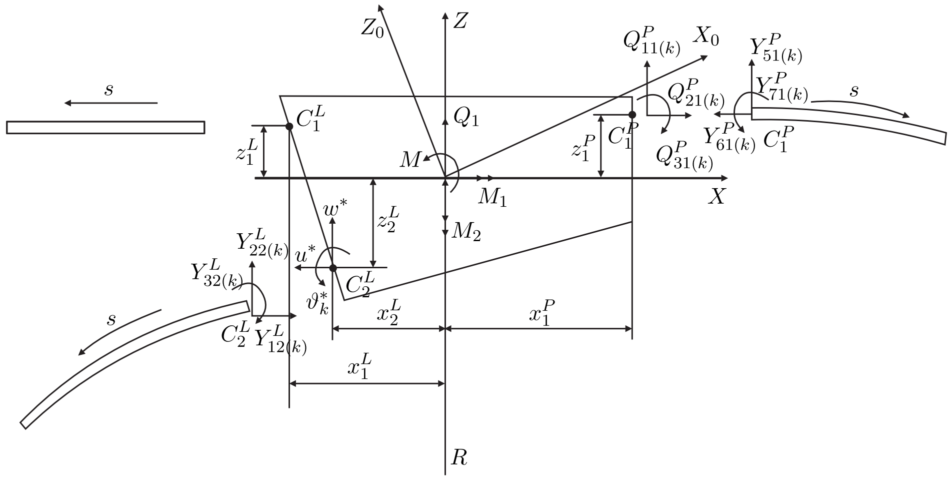

Fig. 3. Loades and movements along the lines of the junction of the shells and the ring frame.

Equation (15) is supplemented by the conditions of inseparability of the displacements of the points of the joint section of the shells and the frame, which establish the connection of the generalized marginal displacements $\bar{W}_{tk}^{*r(L)} = \{y_{1t}^{r(L)}, y_{2t}^{r(L)}, y_{3t}^{r(L)}, y_{4t}^{r(L)}\}_k^T$ of the ends of the t-th shells at ($s = s_0$ or $s = s_L$) with the corresponding displacements $\{u^*, w^*, \vartheta_k^*, v^*\}_k$ of the points $\left(z_t^{r(L)}, x_t^{r(L)}\right)$ of their connection with the frame (which in turn, are related to the generalized displacements $\bar{\Delta}_k$ of the center of gravity of the cross-section of the frame) as follows

$$\bar{W}_{tk}^{*r(L)}(s_e) = \bar{\varphi}_t^{r(L)} \cdot \bar{\beta}_{tk}^{r(L)} \cdot \bar{\Delta}_k = \bar{c}_{tk}^{r(L)} \cdot \bar{\Delta}_k, \quad (e = 0 \text{ or } e = L), \tag{16}$$

where

$$\bar{\beta}_{jk}^{r(L)} = \left| \begin{array}{cccc} 1 & 0 & z_t^{r(L)} & 0 \\ 0 & 1 & -x_t^{P(L)} & 0 \\ 0 & 0 & 1 & 0 \\ kx_t^{r(L)}/R & kz_t^{r(L)}/R & 0 & 1 + z_t^{r(L)}/R \end{array} \right|.$$

It should be noted that $\bar{\beta}_{tk}^{r(L)} = \left[\bar{\eta}_{tk}^{r(L)}\right]^T = \left[\bar{\theta}_k \cdot \bar{\psi}_{tk}^{r(L)}\right]^T$ depends on the eccentricities of the contact line of the ring and shell elements and the curvature of the middle line of the frame.

Between the marginal movements of the ends of the t-th shell

$$\bar{W}_0^{(t)} = \left(y_{1,0}^{(t)}, \dots, y_{n/2,0}^{(t)}\right)_k^T; \quad \bar{W}_L^{(t)} = \left(y_{1,L}^{(t)}, \dots, y_{n/2,L}^{(t)}\right)_k^T \tag{17}$$

and marginal efforts at the ends

$$\bar{Q}_0^{(t)} = \left(y_{n/2+1}^{(t)}, \dots, y_n^{(t)}\right)_k^T; \quad \bar{Q}_L^{(t)} = \left(y_{n/2+1}^{(t)}, \dots, y_n^{(t)}\right)_k^T \tag{18}$$

due to the linearity of differential equations (7), was established an unambiguous dependence in the form

$$\left| \begin{array}{c} \bar{Q}_0^t \\ \bar{Q}_L^t \end{array} \right|_k = \left| \begin{array}{cc} \bar{\Omega}_{00}^t & \bar{\Omega}_{0L}^t \\ \bar{\Omega}_{L0}^t & \bar{\Omega}_{LL}^t \end{array} \right|_k \left| \begin{array}{c} \bar{W}_0^t \\ \bar{W}_L^t \end{array} \right|_k + \left| \begin{array}{c} \bar{Q}_0^{*t} \\ \bar{Q}_L^{*t} \end{array} \right|_k. \tag{19}$$

Here $n = 8$ is the order of system (7), where $n/2$ values (17), (18) are given at $s = s_0$ and (or) $s = s_L$ and are used as boundary conditions for the t-th shell. The elements of the submatrix $\bar{\Omega}_{is}^t$ of each t-th shell element are given as follows: column n is the vector of generalized forces on the i-th edge, which arise when the n-th element of the vector of generalized movements on the j-th edge is equal to one in the absence of other external influences, and the vectors $\bar{Q}_0^{*t}$, $\bar{Q}_L^{*t}$ are vectors of the generalized edge forces acting on the ends of the t of a shell element when a specified external load acts on it with no edge movements.

Thus, the construction of stiffness matrices $\bar{\Omega}_{ij}^t$ and vectors $\bar{Q}_0^{*t}$, $\bar{Q}_L^{*t}$ for each shell element requires the solution of 9 boundary value problems (7), (8) by the method of orthogonal sweep [19].

Equations (16) together with (15), (19) (for each of the shells) form a solving system of linear algebraic equations with respect to the k-th harmonic of the stress-strain state vectors of the system. At the same time, a significant decrease in the order of such a system in comparison with the application of the finite element method is achieved due to the introduction of nodal lines of contact of shells and ring frames instead of nodal points of the finite element method.

After solving the resulting system, k-th harmonics of displacements and interaction forces of the frame and shells along the contact line are determined. The stress-strain state of each of the shells is determined by integrating the corresponding equations of state (7) under the currently found boundary conditions (efforts or displacements) at each of the ends of the shells. The general solution of the problem is a superposition of the individual harmonics' solutions.

5. Optimal design results

The capabilities of the developed algorithm are demonstrated on the example of solving the problem of weight optimization of a shell structure consisting of a set of three thin elastic shells of rotation (cylindrical, spherical, and conical), which are under the action of internal pressure q and are connected to each other by a power frame, loaded by four concentrated (from the frame plane) forces P (Figure $4a$). To solve the specific problem of calculating the stress-strain state and optimizing bundle of shells of rotation connected with a ring frame, the author's version of the application programs was used.

Limitations of strength, rigidity and structural requirements are considered. Frame parameters were taken as given. The evaluation of the stability of the shells and the frame was carried out according to approximate formulas [1, 2, 22], applied to the zones of maximum compression and maximum shear in the shells and according to the Euler stability criterion for the frame. The results of the numerical modeling showed that for the adopted parameters these restrictions are fulfilled with a certain margin and therefore were not directly considered when solving the optimization problem. During the iterative process, bilateral, internal, or only external thickness changes along the meridian of each of the shells varied.

The design parameters are as follows $R_c = 0.495\,\mathrm{m}$; $R_{sph} = 0.525\,\mathrm{m}$; $R_{ce} = 0.505\,\mathrm{m}$; $L_c = 1\,\mathrm{m}$; $L_{co} = 0.5\,\mathrm{m}$; $\theta_{co} = 0.75\,\mathrm{rad}$; $E_{sph} = E_r = 0.68 \cdot 10^{11}\,\mathrm{N/m^2}$; $\mu = 0.3$; $F_r = 0.8 \cdot 10^{-2}\,\mathrm{m^2}$; $I_x = 0.10^4 \cdot 10^{-6}\,\mathrm{m^4}$; $I_z = 0.27 \cdot 10^{-7}\,\mathrm{m^4}$; $P = 1.5 \cdot 10^4\,\mathrm{N}$; $q = 5 \cdot 10^{-3}\,\mathrm{N/m^2}$; $(x_c, z_c) = (-0.01, -0.01)$; $(x_{sph}, z_{sph}) = (0.01, -0.01)$; $(x_{co}, z_{co}) = (0.01, 0.01)$; $[\sigma]_{sph} = 1.6 \cdot 10^8\,\mathrm{N/m^2}$; $[\sigma]_c = 1 \cdot 10^8\,\mathrm{N/m^2}$; $h_{c\cdot min} = 1.2 \cdot 10^{-2}\,\mathrm{m}$; $h_{sph\cdot min} = 1.2 \cdot 10^{-2}\,\mathrm{m}$; $h_{ca\cdot min} = 0.6 \cdot 10^{-2}\,\mathrm{m}$; $\varepsilon = 10^{-2}$.

In Figure $4b$, $4c$, $4d$ graphs of changes along the meridian of the outer and inner parts of the thicknesses are shown: b – spherical, c – cylindrical and d – conical shell.

From the analysis of the numerical results, it follows that the most dangerous places in this design are the places where the shells are clamped and connected to the frame. Therefore, it is expedient to provide for them a transition area much larger than in other places, variable according to a certain law, the thickness of the wall of the shells. The carried-out optimization made it possible to achieve a significant level of savings in shell material for each individual substructure. Thus, material savings for a cylindrical shell were almost achieved 71.2%, for a conical shell 72.2%, for a spherical shell 84.2%. It also should be noted that the shells obtained by varying the thickness only from the outer side of the drive surface turned out to be lighter (by $2 \div 4\%$) than projects where only the inner thickness was varied. The convergence of the process was $12 \div 15$ iterations of the structure decompositions.

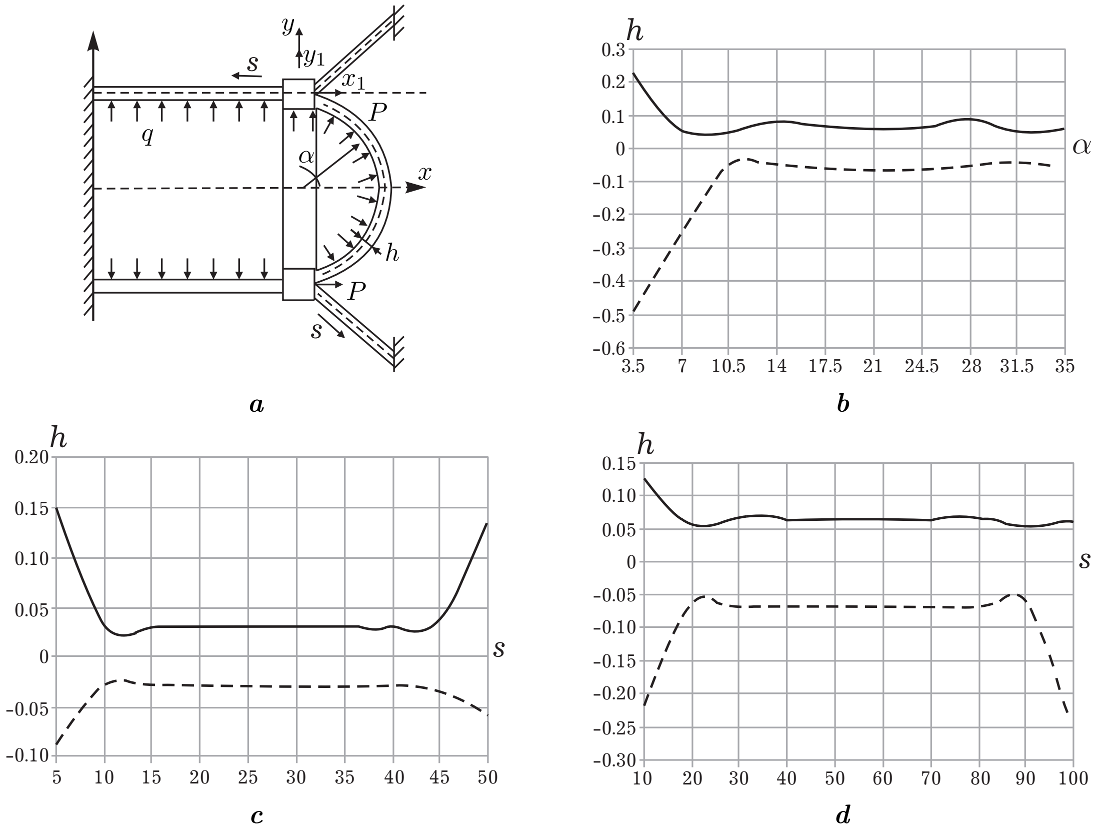

Fig. 4. The optimal design results of cylindrical, spherical, and conical shells connected with a ring frame under the action of an asymmetric load.

6. Conclusions

The method of optimal design of a composite shell structure, consisting of a power frame with an arbitrary cross-sectional shape and elastic shells of variable stiffness connected to it on each side under the action of an asymmetric load, was developed and tested.

It is demonstrated that the application of algorithms for the direct calculation of a bundle of shells of rotation connected with a ring frame based on a discrete-continuous model and solving optimization problems by decomposing the structure into separate substructures and its subsequent synthesis with optimal parameters allows the best using of the possibilities of modern methods of the theory of optimal processes and finite-dimensional optimization. This approach is promising for solving the problem of reducing material consumption.

The possibilities of the approach are tested by solving the problems of optimal material distribution of several (up to three on each side) shells of rotation connected with a ring frame under asymmetric loading.

The results of optimal design are presented and analyzed.

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Алгоритм декомпозиції та синтезу для оптимізації параметрів сполученого шпангоутом пучка оболонок обертання

Дзюба А. П.[1,2], Сафронова І. А.[1], Сіренко В. М.[2], Торський А. Р.[3,4]

[1] *Дніпровський національний університет імені Олеся Гончара, вул. Гагаріна, 72, Дніпро, Україна*
[2] *Конструкторське бюро "Південне", вул. Криворізька, 3, Дніпро, Україна*
[3] *Інститут прикладних проблем механіки і математики ім. Я. С. Підстригача НАН України, вул. Наукова, 3б, Львів, Україна*
[4] *Національний університет "Львівська політехніка", вул. С. Бандери, 12, Львів, Україна*

Подана методика вагової оптимізації оболонкової конструкції, що складається із силового шпангоута та приєднаних до нього з кожного боку неоднорідних оболонок обертання зі змінною товщиною стінки під дією просторового несиметричного навантаження. Застосовується алгоритм декомпозиції конструкції. Оптимізація оболонок здійснюється на основі необхідних умов оптимальності Л. С. Понтрягіна з фазовими обмеженнями. Оптимальна конфігурація шпангоута відшукується методами скінчено-вимірної оптимізації. Синтез конструкції здійснюється методом послідовних наближень. Подані числові результати оптимізації.

Keywords: *пучок оболонок обертання; шпангоут; несиметричне навантаження; оптимізація параметрів.*