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Mathematical modeling of impurity diffusion process under given statistics of a point mass sources system. I

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The model of the impurity diffusion process in the layer where a system of random point mass sources acts, is proposed. Mass sources of various power are uniformly distributed in a certain internal interval of the body. Statistics of random sources are given. The solution of the initial-boundary value problem is constructed as a sum of the homogeneous problem solution and the convolution of the Green's function and the system of the random point mass sources. The solution is averaged over both certain internal subinterval and the entire body region. Simulation units are designed for modeling of the behavior of the averaged concentration function with acting system of point mass sources of various power. On this basis, the averaged concentration field is investigated depending on the internal interval length, power and number of sources in the system as well as the concentration values at the layer boundaries.

Keywords: *mathematical modeling; diffusion; random point source; Green's function; uniform distribution; software.*

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1. Introduction

Statistical modeling as a method of solving probabilistic and deterministic problems based on the use of random variables and laws of probability theory has gained wide application for a large number of problems related to the analysis, synthesis, and optimization of parameters, forecasting of complex physical and technical objects. A characteristic feature of these mathematical models is their indeterminacy, meaning that in a statistical model, defined through mathematical relationships, some variables do not have specific values, but have a probability distribution only. The main problem of statistical modeling is the selection of an appropriate statistical model to represent the formation process. Typically, data preparation can be extremely complex, necessitating both an understanding of the process and the appropriate statistical analysis. Research in this scientific field has been vigorously advancing over the past two decades.

Chen et al. [1] studied linear models for regression analysis with compositional variables. These regression models assume a certain mathematical relationship between variables and suppose that data errors are in a specific form. They are widely used in statistical mathematical modeling. For modeling systems under uncertainty, a method for constructing interval models has been developed [2, 3]. In the paper [4], sequential regression approaches were used for analyzing processes where covariates are revealed stage by stage. This sequential approach involves fitting a regression model for the next stage using covariates identified at the end of the current stage, thus considering the history of the process up to the stage under consideration. The authors of the paper [5] proposed a statistical model that combines several linear regression models and presented algorithms for estimating the maximum likelihood of such a model. Wilcox [6] obtained results that show the feasibility of calculating a confidence interval using a robust regression estimator in case when the property of heteroscedasticity is satisfied. The obtained method is applicable for a small sample using estimates of the least squares method, as well as robust regression estimates.

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Gong et al. [7] considered the inverse problem of finding a source governed by a random Gaussian field for a stochastic time-fractional diffusion equation. It was shown that the Fourier modulus of the diffusion coefficient of the random source is uniquely determined by the variance of the Fourier transform of boundary data. In the article [8], a time partitioning method is presented for degenerate convection-diffusion equations stochastically perturbed by white noise. Numerical modeling was conducted for the case of fluid diffusion in porous media. Zhang et al. [9] constructed and investigated the numerical solution of the inverse source problem for the time-fractional diffusion equation. In [10], the authors consider the inverse problem to find the parameters of a stochastic time-fractional diffusion equation. Theorems on the solution correctness of the direct problem are proven, and a scheme for finding the unknown parameters using Tikhonov regularization is described. Reis et al. [11] studied the model of random walk infiltration in homogeneous and fractal environments with localized sources on their boundaries. Exact solutions were obtained in two- and three-dimensional cases. The article [12] deals with the inverse source problem for the time-fractional diffusion equation. A solution is constructed using the regularization method, and conditions for the convergence of the approximate solution to the exact one are obtained. In [13], the diffusion process described by a continuous-time random walk model is investigated. An analytical exact solution was found, and the Green's function was constructed. Budhiraja et al. [14], using stochastic partial differential equations driven by Poisson random measures, studied a system that models the spread of pollutants in a waterway. Basic qualitative properties of the solution for the indicated stochastic system were established. The work [15] provides a qualitative and quantitative description of polymeric membranes with dispersed magnetic powder (magnetic membranes) for air separation. Diffusion processes are described, and mathematical models based on the diffusion equation in such type membranes are proposed. In [16], the non-Gaussian spread of dissolved substances subject to advection, dispersion, and kinetic adsorption is analyzed. A formulated mathematical model combines the application of Markov chain theory to describe kinetic sorption with the consideration of a system of mass transfer differential equations. The paper [17] presents an Euler-Lagrange approach for analyzing flows and transport processes in connected hydrosystems. A feature is the use of stochastic Lagrange particles for the numerical study of connected hydrosystems.

In [18], theoretical models were developed for the concentration and velocity of fluid along the flow in an unstable, uniform, one-dimensional flow saturated with sediments. The effect of stratification through diffusion coefficients was investigated. Yu et al. [19] considered a model of fractional diffusion with heterogeneous Dirichlet boundary conditions in a finite area. They explored various variants of weight functions of the continuous distribution with mean μ and standard deviation σ, and studied their impact on the behavior of the local and global time solution. Semi-analytical solutions were proposed for the analysis of the model. In [20], the convergence of a multilevel Monte Carlo Markov chain algorithm for solving a linear parabolic partial differential equation containing a logarithmic undefined Gauss' diffusion coefficient was analyzed.

This paper studies diffusion processes in a layer under the action of a system of randomly located point mass sources. The main goal of the proposed study is to develop a mathematical model of impurity diffusion under the action of a system of random point sources and to construct a solution to the corresponding initial-boundary value problem, and to perform a numerical analysis of the concentration field of migrating particles also.

2. Mathematical model

Let us consider an impurity substance diffusing in a layer of thickness x_0. Additionally, in the body, there is a set of point mass sources $\omega_i\delta(x-\hat{x}_i)$, where ω_i is the power of the i-th source, $\delta(x)$ is the Dirac delta function [21], we treat these as a system of sources at random points $x=\hat{x}_i$, $\hat{x}_i \in [\bar{x}_1, \bar{x}_2]$, and $0 \leqslant \bar{x}_1 < \bar{x}_2 \leqslant x_0$ (Figure 1).

Let the sources statistics be given as $\langle\delta(x-\hat{x}_i)\rangle$, $\langle\delta(x-\hat{x}_i)\delta(x-x_j)\rangle$, $\langle\delta(x-\hat{x}_i)\delta(x-x_j)\delta(x-\hat{x_k})\rangle$, $\ldots$, $i,j,k=1,\ldots,N$.

The diffusion equation of the impurity substance under the action of a system of point sources in the case of one-dimensional spatial variable takes the following form [22, 23], based on Fick's laws,

$$\rho\frac{\partial c(t,x)}{\partial t} = d\frac{\partial^2 c(t,x)}{\partial x^2} + \sum_{i=1}^{N}\omega_i\,\delta(x-\hat{x}_i), \tag{1}$$

where $c(t,x)$ is the concentration of the migrating substance, ρ is the body density, d is the kinetic coefficient of transfer, N is the random point sources number; t is time, x is the spatial coordinate.

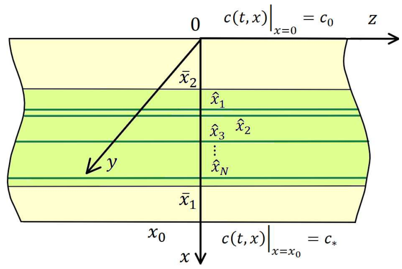

Fig. 1. The layer where the impurity diffuses under action of a system of random point mass sources.

Assume that the first-order initial and boundary conditions are given. Specifically, the initial condition is zero, and constant particle concentration values are maintained on the both surfaces of the layer,

$$c(t,x)\big|_{t=0} = 0, \tag{2}$$

$$c(t,x)\big|_{x=0} = c_0 \equiv \text{const}, \quad c(t,x)\big|_{x=x_0} = c_* \equiv \text{const}. \tag{3}$$

The impact of the system of random point-mass sources within the body results in the stochasticity of the function being sought, that is, the randomness of the impurity concentration field.

3. Construction of the solution

We aim to find the solution to the initial-boundary value problem (1)–(3) as a sum of the solution to the homogeneous initial-boundary value problem and the convolution of the Green's function with the system of random point sources [22]. We have

$$c(t,x) = c^h(t,x) + \sum_{i=1}^{N}\omega_i\int_0^t\int_0^{x_0} G(t,t',x,x')\,\delta(x'-\hat{x}_i)\,dx'dt', \tag{4}$$

where $c^h(t,x)$ is the solution to the following homogeneous problem

$$\rho\,\frac{\partial c^h(t,x)}{\partial t} = d\,\frac{\partial^2 c^h(t,x)}{\partial x^2}, \tag{5}$$

$$c^h(t,x)\big|_{t=0} = 0, \tag{6}$$

$$c^h(t,x)\big|_{x=0} = c_0 \equiv \text{const}, \quad c^h(t,x)\big|_{x=x_0} = c_* \equiv \text{const}; \tag{7}$$

and $G(t,t',x,x')$ is the Green's function of the problem (1)–(3), that can be found by solving the following problem with zero initial and boundary conditions

$$\rho\,\frac{\partial G(t,t',x,x')}{\partial t} - d\,\frac{\partial^2 G(t,t',x,x')}{\partial x^2} = \delta(t-t')\,\delta(x-x'), \tag{8}$$

$$G(t,t',x,x')\big|_{t=0} = 0, \tag{9}$$

$$G(t,t',x,x')\big|_{x=0} = 0, \quad G(t,t',x,x')\big|_{x=x_0} = 0. \tag{10}$$

The solution to the homogeneous initial-boundary value problem (5)–(7)can be obtained by reducing to a problem with zero boundary conditions and applying the Laplace integral transform with respect to the time variable and the finite Fourier sine-transform with respect to the spatial variable [24]. As a result,

$$c^h(t,x) = c_0\left(1-\frac{x}{x_0}\right) + c_*\frac{x}{x_0} - \frac{2}{x_0\rho}\sum_{n=1}^{\infty}\tfrac{1}{y_n}\left(c_0+(-1)^{n+1}c_*\right)e^{-dy_n^2 t/\rho}\sin(y_n x), \tag{11}$$

where $y_n = \pi n/x_0$, $n = 1, 2, \ldots$.

One can find the Green's function by solving the initial-boundary value problem (8)–(10) again by applying Laplace and Fourier integral transforms. The Green's function is obtained in the form

$$G(t,t',x,x') = \frac{2}{x_0\rho}\,\theta(t-t')\sum_{n=1}^{\infty} e^{-dy_n^2(t-t')/\rho}\sin(y_n x')\sin(y_n x), \tag{12}$$

where $\theta(t-t')$ is the Heaviside step function [25].

4. Averaging the concentration over random spatial coordinates of source locations

Let us average the function $c(t,x)$ over the random coordinates of the source locations $\hat{x}_i$. Taking into account that $c^h(t,x)$ (11) is a deterministic function, we can state that $\langle c^h(t,x)\rangle = c^h(t,x)$. Hence,

$$\langle c(t,x)\rangle = c^h(t,x) + \left\langle \sum_{i=1}^{N} \omega_i \int_0^t \int_0^{x_0} G(t,t',x,x')\,\delta(x'-\hat{x}_i)\,dx'\,dt' \right\rangle.$$

We consider the set of random point sources $\omega_i\,\delta(x-\hat{x}_i)$ as a system of sources acting within the body's region. Let $f(x_i)$ represents the distribution density function of the random variable $\hat{x}_i$. Assume that the internal mass sources comprise a system of sources, and that the contribution of each source to the system is equally probable. Then the function $f(\sum_{i=1}^{N}\hat{x}_i)$ is the sum of the density functions $f(\hat{x}_i)$ with unit weights p_i:

$$f\left(\sum_{i=1}^{N}\hat{x}_i\right) = \sum_{i=1}^{N} p_i f(\hat{x}_i), \quad p_i \equiv 1. \tag{13}$$

Let the interval of sources action be $[\bar{x}_1;\bar{x}_2] \subseteq [0,x_0]$. Taking into account relation (13), the averaged concentration field $\langle c(t,x)\rangle$ can be presented in the form

$$\langle c(t,x)\rangle = c^h(t,x) + \sum_{i=1}^{N}\omega_i \int_{\bar{x}_1}^{\bar{x}_2}\int_0^t\int_0^{x_0} f(\hat{x}_i)\,G(t,t',x,x')\,\delta(x'-\hat{x}_i)\,dx'dt'd\hat{x}_i. \tag{14}$$

Let us substitute the expression for the Green's function (12) into relation (14). After integrating,

$$\langle c(t,x)\rangle = c^h(t,x) + \frac{2}{dx_0}\sum_{n=1}^{\infty}\frac{1}{y_n^2}\big(1-e^{-dy_n^2 t/\rho}\big)\sin(y_n x)\sum_{i=1}^{N}\omega_i\int_{\bar{x}_1}^{\bar{x}_2} f(\hat{x}_i)\sin(y_n\hat{x}_i)\,d\hat{x}_i.$$

Let us consider a certain configuration of the powers of sources that are part of the system. Suppose each of the random variables $\hat{x}_i$ is uniformly distributed [26] over the interval $[\bar{x}_1;\bar{x}_2]$. Taking into consideration that $f\big(\sum_{i=1}^{N}\hat{x}_i\big)$ is the distribution function of the sources, and the disposition of the source at the any given point is equally probable, then $\int_{-\infty}^{\infty} f\big(\sum_{i=1}^{N}\hat{x}_i\big)d\hat{x}_i = 1$. Thus, $f(\hat{x}_i) = 1/(N(\bar{x}_2-\bar{x}_1))$.

If the averaging interval covers the entire layer thickness ($[\bar{x}_1,\bar{x}_2]=[0,x_0]$), then $f(\hat{x}_i) = 1/(Nx_0)$.

Let us find the specific form of $\langle c(t,x)\rangle$ at $0<x_1<x_2<x_0$. We obtain

$$\langle c(t,x)\rangle = c^h(t,x) + \frac{2\Omega}{dx_0 N(\bar{x}_2-\bar{x}_1)}\sum_{n=1}^{\infty} S_{12}(y_n)\big(1-e^{-dy_n^2 t/\rho}\big)\sin(y_n x), \tag{15}$$

where

$$S_{12}(y_n) = \frac{\cos(y_n\bar{x}_1)-\cos(y_n\bar{x}_2)}{y_n^3}, \quad \Omega = \sum_{i=1}^{N}\omega_i. \tag{16}$$

In the case where the powers of the point sources are identical, that is $\omega_i=\omega$, $\forall i=1,\ldots,N$, and $\Omega = N\omega$, relation (15) takes the form

$$\langle c(t,x)\rangle = c^h(t,x) + \frac{2\omega}{dx_0(\bar{x}_2-\bar{x}_1)}\sum_{n=1}^{\infty}\frac{\cos(y_n\bar{x}_1)-\cos(y_n\bar{x}_2)}{y_n^3}\big(1-e^{-dy_n^2 t/\rho}\big)\sin(y_n x). \tag{17}$$

In addition, if $\bar{x}_1=0$, and $\bar{x}_2=x_0$,

$$\langle c(t,x)\rangle = c^h(t,x) + \frac{2\omega}{dx_0^2}\sum_{n=1}^{\infty}\frac{1-\cos(y_n x_0)}{y_n^3}\big(1-e^{-dy_n^2 t/\rho}\big)\sin(y_n x). \tag{18}$$

If we treat point mass sources that act in a certain region of the body as a united system of sources, and each of these sources acts with the same power, then the averaged solution (17) for the system takes the form of the averaged solution to the problem with a single source of the same power [27].

5. Numerical analysis of the averaged concentration field

Further calculations are performed in the dimensionless variables

$$\xi = x/x_0, \quad \tau = d/(\rho x_0^2)t. \tag{19}$$

The following are adopted as the base values of the problem's parameters: $d = 1$, $\rho = 1$, $N = 2$, $\omega_1 = \omega_2 = 3$, $\xi_0 = 1$, $\bar{\xi}_1 = \bar{x_1}/x_0 = 0.4$, $\bar{\xi}_2 = \bar{x_2}/x_0 = 0.6$, $c_0 = 1$, $c_* = 0.1$. Graphs of the averaged concentration of the impurity substance in a layer of dimensionless thickness ξ_0 for the uniform distribution of the random placement of the point sources system are shown in Figures 2–7. Calculations were carried out using formulas (15)–(18). The series in these formulas were summed with the precision 10^{-12}.

Figure 2 illustrates the distributions of the averaged concentration of impurity particles at different moments of dimensionless time $\tau = 0.02, 0.06, 0.1, 0.5$ (curves 1–4 respectively). Figure 2a presents the graphs of the function $\langle c(\tau, \xi)\rangle / c_0$, for a system consisting of two equally powerful point sources acting within the interval $[0.4, 0.6]$. The calculations here were conducted using formula (17). Figure 2b shows the distributions of the averaged concentration while the action interval of the point source system coincides with the body's region. For this case, calculations were performed using the formula (18).

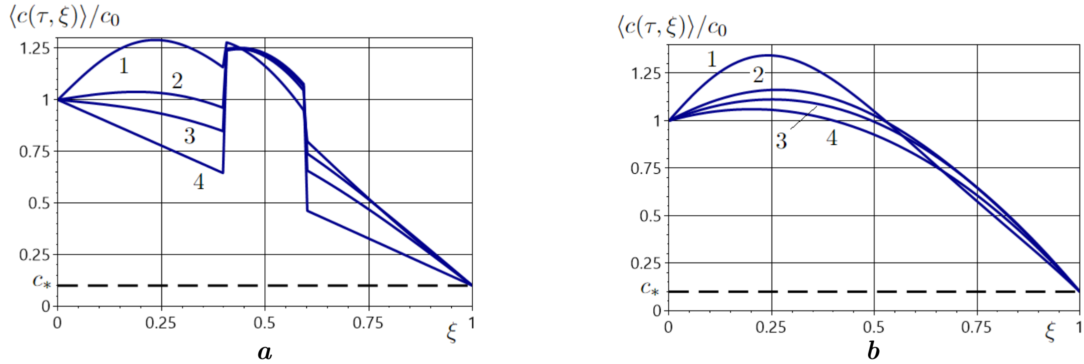

Fig. 2. Graphs of the averaged concentration of the impurity under the action of two equally powerful sources at different moments in time for $[\bar{\xi_1}, \bar{\xi_2}] = [0.4, 0.6]$ (***a***) and $[\bar{\xi_1}, \bar{\xi_2}] = [0, \xi_0]$ (***b***).

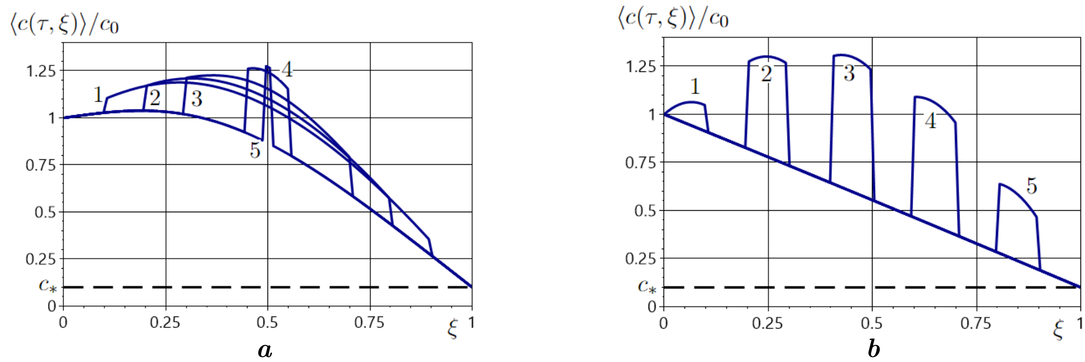

Fig. 3. Graphs of the averaged concentration under the action of two equally powerful sources for different lengths of the interval $[\xi_1, \xi_2]$ (***a***) and for different positions of the interval in the body (***b***).

Figure 3a shows the behavior of the function $\langle c(\tau, \xi)\rangle / c_0$ depending on the length of the interval $[\xi_1, \xi_2]$: $[\xi_1, \xi_2] = [0.1, 0.9]$, $[\xi_1, \xi_2] = [0.2, 0.8]$, $[\xi_1, \xi_2] = [0.3, 0.7]$, $[\xi_1, \xi_2] = [0.45, 0.55]$, $[\xi_1, \xi_2] = [0.49, 0.51]$ (curves 1–5) at time $\tau = 0.06$. Figure 3b demonstrates plots of the averages concentration function for different placements of the interval of action of point mass sources $[\xi_1, \xi_2]$ within the domain: $[\xi_1, \xi_2] = [0, 0.1]$, $[\xi_1, \xi_2] = [0.2, 0.3]$, $[\xi_1, \xi_2] = [0.4, 0.5]$, $[\xi_1, \xi_2] = [0.6, 0.7]$, $[\xi_1, \xi_2] = [0.8, 0.9]$ (curves 1–5) at the time $\tau = 0.5$.

Figure 4 shows the distributions of the normalized averaged concentration $\langle c(\tau, \xi)\rangle / c_0$ for different values of impurity concentration maintained at the boundary of the layer $\xi = \xi_0$: $c_* = 0, 0.1, 0.5, 0.75, 1.2$ (curves 1–5).

Here and further Figures ***a*** are presented for a small time interval of diffusion process $\tau = 0.06$, and Figures ***b*** are for larger time interval $\tau = 0.5$. Figure 5 illustrates the distributions of the averaged concentration normalized by its value at the boundary $\xi = \xi_0$, i.e. c_*, depending on the value $c(\tau, \xi)$ at the upper boundary $\xi = 0$: $c_0 = 0, 0.1, 0.5, 0.75, 1.2$ (curves 1–5).

Figure 6 presents the graphs of the function $\langle c(\tau,\xi)\rangle/c_0$ for different numbers of point sources in the interval $[0.4, 0.6]$ with equal fixed power $\Omega = 6$: $\omega = 6, 3, 1.5, 0.2, 0.05$ (curves 1–4). Figure 7 demonstrates the behavior of the averaged concentration function under the action of a system with five point sources, including one source of dominant power. Curve 1 corresponds to the system $\{1, 1, 1, 1, 40\}$, curve 2 — $\{1, 1, 1, 1, 20\}$, curve 3 — $\{1, 1, 1, 1, 10\}$, curve 4 — $\{1, 1, 1, 1, 1\}$.

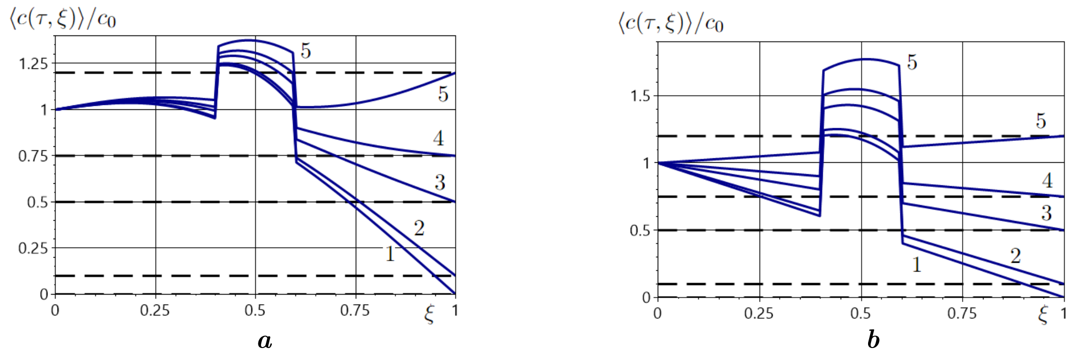

Fig. 4. Graphs of the averaged concentration for different values of concentration at the lower boundary of the layer for small (***a***) and large (***b***) times.

It should be noted that the presence of a system of point mass sources in the layer increases the concentration of the migrating substance throughout the body area. In this case, a characteristic increase in the function of averaged concentration from the surface of the body (Figures 2*b*, 3*a*, 4*a*, 5*a* and 5*b*) or in the middle of the layer (Figures 2*a*, 3*b*, 4*a*, 4*b* and 5–7) is observed. For all considered cases, the dimensionless time to reach a steady state is the same, $\tau = 0.5$. The presence of the interval $[\bar{\xi}_1, \bar{\xi}_2]$ of the action of the system of point sources significantly affects the behavior of the averaged concentration of the impurity (Figures 2–7). At the same time, for small diffusion times, there is a significant increase in impurity concentration from the surface $\xi = 0$ (curves 1 in Figure 2*a* and 2*b*). As the process time increases, the concentration values in the upper part of the layer decrease (curves 2–4 in Figure 2). Moreover, these changes occur more slowly with larger τ. In the case where the distribution interval of point sources coincides with the entire body area, the function of the averaged concentration is always smooth (Figure 2*b*). However, if the interval of action of the point sources is less than the thickness of the layer, then a sharp increase in the averaged concentration near the point $\xi < \bar{\xi}_1$ and a sharp decrease near $\xi = \bar{\xi}_2$ (Figures 2*a* and 3–7) are observed. In this case, such an increase in $\langle c(\tau,\xi)\rangle$ is smaller for smaller times and increases with the growth of τ (Figure 2*a*).

It should be noted that the impact of the system of point sources on the averaged concentration of the impurity substance is significant, as is the influence of the width of the interval of possible source locations (Figure 3*a*). The narrower the interval of action of the point source system, the higher the values the averaged concentration reaches within this interval (Figure 3*a*). Furthermore, the closer the interval $[\bar{\xi}_1, \bar{\xi}_2]$ is to the boundaries of the body, the smaller the increase in $\langle c(\tau,\xi)\rangle$ over this range (curves 2–5 in Figure 3*b*). When the interval of action of the point sources is located near the upper boundary of the layer, then the boundary condition has a noticeable effect on the averaged concentration (curve 1 in Figure 3*b*).

The influence of the concentration value at the lower surface of the layer, c_*, is significant, and with the increase of time, the greatest impact of this parameter occurs in the middle of the body (Figures 4*a* and 4*b*). A similar situation is observed when varying the concentration value of the impurity at the upper boundary of the layer, c_0 (Figure 5). It should be noted that here, curves 1 demonstrate how the diffusion process develops in the interval $[0, 0.4]$, when sources are active in the interval $[0.4, 0.6]$ and a constant mass source acts at the body boundary $\xi = \xi_0$ (Figure 5), but there is no source at the upper surface of the body $\xi = 0$.

The analysis of the number of point sources in the system shows changes in the values of $\langle c(\tau,\xi)\rangle$ only in the interval $[\xi_1, \xi_2]$ (Figure 6). The fewer the number of sources N for the same Ω, the higher the concentration function values on this interval (Figure 6*a*). Moreover, over time, the difference between the values of $\langle c(\tau,\xi)\rangle|_{\xi\in[\xi_1,\xi_2],N=1}$ and $\langle c(\tau,\xi)\rangle|_{\xi\in[\xi_1,\xi_2],N=4}$ increases, particularly for $\tau = 0.5$, reaching 55% (Figure 6b). If a point source with predominant power is active in the system, then the values of the averaged concentration significantly increase in the interval $[\bar{\xi_1},\bar{\xi_2}]$ (Figure 7). The longer the duration of the diffusion process, the higher the values of the averaged $\langle c(\tau,\xi)\rangle|_{\xi\in[\bar{\xi_1},\bar{\xi_2}]}$ (Figures 7a and 7b). For example, for $\tau = 0.5$ $\langle c(\tau,\xi)\rangle|_{\xi\in[\xi_1,\xi_2],\omega_{\max}=40}/\langle c(\tau,\xi)\rangle|_{\xi\in[\xi_1,\xi_2],\omega_{\max}=10} \approx 2.1$ (Figure 7b).

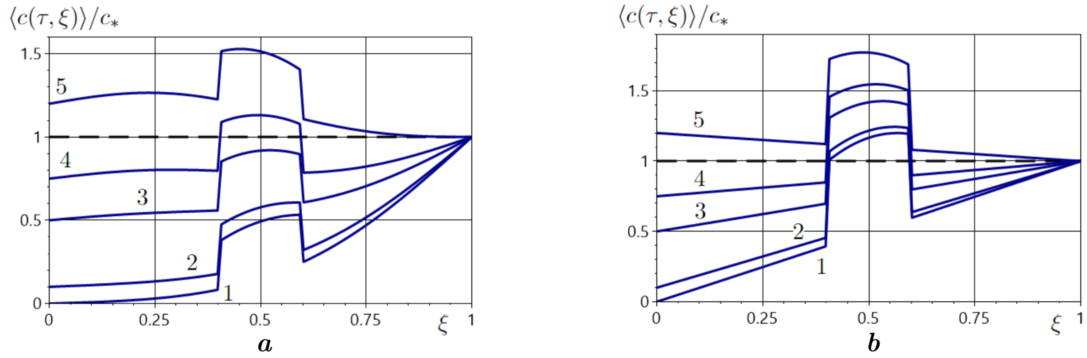

Fig. 5. Graphs of the averaged concentration for different values of concentration at the upper boundary of the layer for small (***a***) and large (***b***) times.

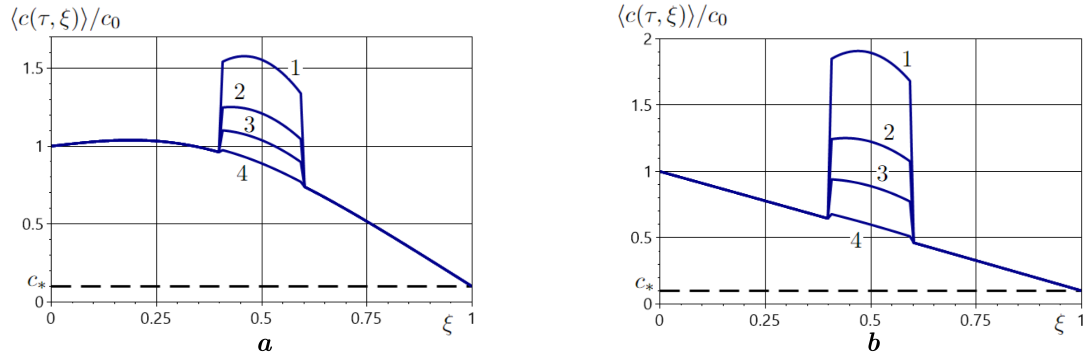

Fig. 6. Graphs of the averaged concentration for a varying number of sources with equal total power for small (***a***) and large (***b***) times.

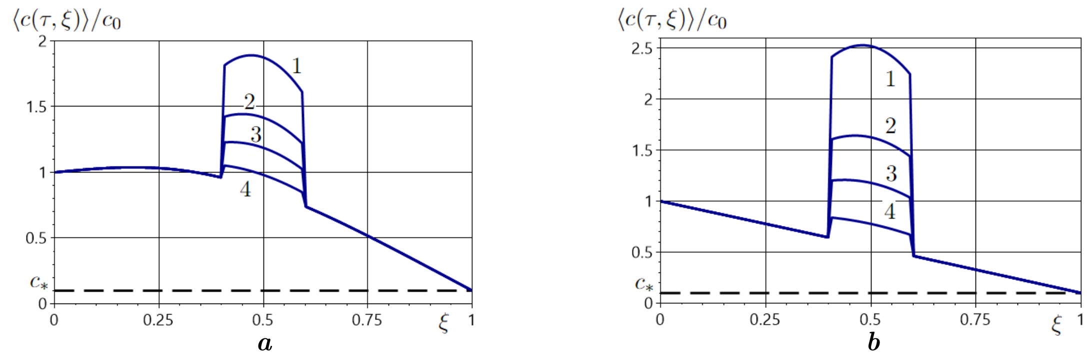

Fig. 7. Graphs of the averaged concentration in the presence of a dominant source of varying power in the system for small (***a***) and large (***b***) times.

For small total powers Ω, their reduction almost does not affect the values of $\langle c(\tau,\xi)\rangle/c_0$. For example, when Ω decreases from 5 to 4, changes in the values of the function $\langle c(\tau,\xi)\rangle/c_0$ occur at most in the third decimal place.

It should be noted for a system of sources where one of them has significantly higher power than then the others, with the changes in the number of sources at a constant total power of sources Ω, the averaged concentration assumes the same values. This is explained by the fact that the result of averaging the concentration over the action interval is affected only by the average power of the active sources Ω/N, which is reflected in the formula (15).

6. Conclusion

Thus, the modeling of the process of impurity diffusion in the layer under the action of a system of random point sources is performed. Mass sources of different power are uniformly distributed in a certain internal interval, which may also coincide with the entire region of the layer. Moreover, statistics of random sources are given. The solution of the initial-boundary value problem is found in the form of the sum of the solution of the homogeneous problem and the convolution of the Green's function with the system of the point sources. Averaging of the obtained solution is performed under uniform distribution on the internal subinterval and in the entire body region. Software modules have been developed for simulating the behavior of the averaged concentration of the system of random point mass sources for different lengths of intervals of the system of point sources and their location in the body region, for different number of sources in the system, at the presence or absence of the source with prevailing power. General laws of behavior of the impurity concentration field are established depending on the parameters of the problem. In particular, it is shown that the presence of a system of point mass sources increases the value of the concentration of the migrating substance, herewith a characteristic increase in the function of the averaged concentration is observed from the surface of the body or in the middle of the layer. It is shown that the presence of an interval of action of the point source system significantly affects the behavior of the averaged concentration of the impurity, in most cases sharply increasing the value of the averaged concentration in this interval. In the case when the interval of the action of point sources coincides with the entire body region, the averaged concentration function is always smooth.

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Математичне моделювання процесу дифузії домішки за заданої статистики системи точкових джерел маси. I

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Запропоновано модель процесу дифузії домішки в шарі, в якому діє система випадкових точкових джерел маси. Джерела маси різної потужності розподілені рівномірно на певному внутрішньому інтервалі тіла. Задана статистика випадкових джерел. Розв'язок крайової задачі побудований як сума розв'язку однорідної задачі і згортки функції Гріна із системою випадкових точкових джерел маси. Розв'язок усереднено як на певному внутрішньому підінтервалі, так і в усій області тіла. Розроблено модулі симуляції для моделювання поведінки функції усередненої концентрації домішкової речовини за дії системи випадкових точкових джерел різної потужності. На цій основі усереднене поле концентрації досліджено в залежності від довжини внутрішнього інтервалу, потужності і кількості джерел в системі, а також значень концентрації на границях шару.

Ключові слова: *математичне моделювання; дифузія; випадкове точкове джерело; функція Гріна; рівномірний розподіл; програмний модуль.*