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Stress-strain state of a two-layer orthotropic body under plane deformation

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We consider the problem of determining the stresses and strains of a two-layer body consisting of an orthotropic layer of constant thickness connected to an orthotropic half-space. The surface of the layer is subjected to known external loads, such that the deformation of the body is plane. At infinity, the stresses are zero. The stress-strain state of the body is determined using the method of integral Fourier transforms. The features of solutions of the system of differential equations of the problem are investigated. The solutions of a particular problems are obtained and analyzed.

Keywords: *orthotropic layer; orthotropic half-plane; plane deformation; stress-strain state; stress function; integral Fourier transform.*

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1. Introduction

The widespread use of layered bodies in construction and mechanical engineering has led to an increased interest in the problems of studying the stress-strain state of each layer, the transfer of forces and loads from one layer to another, and the problems of determining contact interactions. The study of the properties of orthotropic materials holds a special place in these tasks [1], as well as structures with orthotropic layers and the task of developing mathematical models describing real materials by orthotropic schemes.

Methods for determining deformations (plane and spatial) for multilayer plates with elastic connections between the layers were considered in [2–4], where the solution was found using the method of elasticity functions with the application of the Fourier integral transform.

The calculations of the plane deformed state of multilayer curved orthotropic plates using the finite element method are described in [5]. The application of the numerical method of spline approximation to find the stress-strain state of rectangular laminated orthotropic plates is described in [6]. The determination of deformations of an orthotropic sandwich layered shell (each layer of which is of finite length) under internal and external pressure is given in [7].

Problems on the interaction of an orthotropic layer with an orthotropic half-space were considered, for example, in [8–13]. Paper [8] presents the problem of incomplete contact between a layer and a half-space solved by the Fredholm integral equation. The problem of determining the stress intensity coefficients for a moving interfacial crack between an orthotropic half-plane and a heterogeneous orthotropic layer is considered in [9]. The analysis of the bending of an orthotropic layer connected to a half-plane under a compressive load is considered in [10, 11]. In [10], the effect of an interfacial crack on the bending under plane deformation is studied. With the help of Fourier transforms, the boundary value problem is reduced to a system of homogeneous singular integral equations of the Cauchy type of the second kind, which is solved numerically to determine the critical deflection loads.

The development of new approaches and extension of existing ones to the problem of determining deformations and stresses in layered structures with consideration of material orthotropy determines the relevance of the research.

This work aims to extend the method of solving elasticity problems for multilayer foundations with isotropic layers, based on the methods of integral Fourier transforms and the method of yielding functions, to the case of foundations with orthotropic layers.

2. Statement of the problem

We consider the problem of calculating stresses and displacements at any point of an orthotropic body consisting of an orthotropic layer of depth h coupled to an orthotropic half-space. The layer is bounded by two parallel planes. An external load P acts on the upper boundary of the layer. At infinity, the stresses tend to zero. The deformation of the layer and the half-space is plane.

We will denote by the index $j=1$ all values related to the layer and $j=2$ those related to the half-space. To build a mathematical model of the problem, we introduce coordinate systems $O_1X_1Y_1$ for the layer and $O_2X_2Y_2$ for the half-space, as shown in Figure 1.

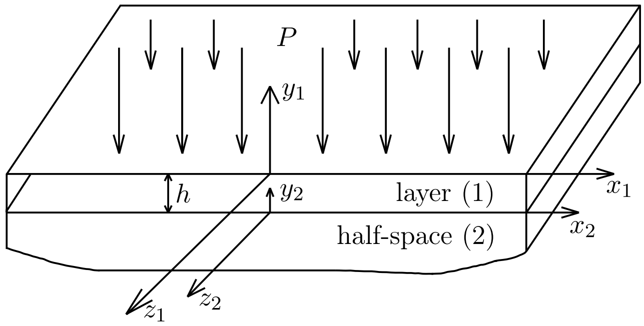


Fig. 1. Statement of the problem.

Geometric areas occupied by the layer and half-space:

$$G_1(x_1,y_1,z_1)\colon \{-\infty<x_1<+\infty, -h\leqslant y_1\leqslant 0, -\infty<z_1<+\infty\},$$
$$G_2(x_2,y_2,z_2)\colon \{-\infty<x_2<+\infty, -\infty<y_2\leqslant 0, -\infty<z_2<+\infty\}.$$

The upper boundary of the layer is $y_1=0$. The materials of the layer and the half-space are characterized by elastic constants $\nu_{ij}^{1,2}$, $E_j^{1,2}$.

The external load $P^1(x,z)$ is such that the deformation of the layer and half-space is plane, the displacements of the body points are parallel to the plane $O_1X_1Y_1$ ($O_2X_2Y_2$):

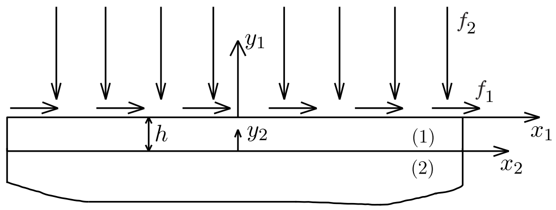


Fig. 2. Illustration of the problem statement.

$$W(x_1,y_1,z_1^*)=u_z(x_1,y_1,z_1^*)=0, \qquad W(x_2,y_2,z_2^*)=u_z(x_2,y_2,z_2^*)=0,$$
$$U(x_1,y_1,z_1^*)=u_x(x_1,y_1,z_1^*)=u_x(x_1,y_1), \qquad U(x_2,y_2,z_2^*)=u_x(x_2,y_2,z_2^*)=u_x(x_2,y_2),$$
$$V(x_1,y_1,z_1^*)=u_y(x_1,y_1,z_1^*)=u_y(x_1,y_1); \qquad V(x_2,y_2,z_2^*)=u_y(x_2,y_2,z_2^*)=u_y(x_2,y_2).$$

Thus, we come to a plane problem of elasticity (Figure 2).

Boundary conditions:

- on the boundary $y_1=0$:

$$\sigma_y^1(x_1,0)=f_1(x_1), \quad \tau_{xy}^1(x_1,0)=f_2(x_1); \tag{1}$$

- common boundary of the layer and the half-plane:

$$\sigma_y^1(x_1,-h)=\sigma_y^2(x_2,0), \quad u_x^1(x_1,-h)=u_x^2(x_2,0)=0; \tag{2}$$

- at infinity:

$$\begin{aligned} &\lim_{x_1^2+y_1^2\to\infty}\sigma_y^1(x_1,y_1)=0, &&\lim_{x_1^2+y_1^2\to\infty}\tau_{xy}^1(x_1,y_1)=0,\\ &\lim_{x_2^2+y_2^2\to\infty}\sigma_y^2(x_2,y_2)=0, &&\lim_{x_2^2+y_2^2\to\infty}\tau_{xy}^2(x_2,y_2)=0. \end{aligned} \tag{3}$$

The materials of the layer and half-space are characterized by the elastic constants ν_{xz}^1, ν_{xy}^1, ν_{yz}^1, ν_{zy}^1, E_x^1, E_y^1 and, respectively, ν_{xz}^2, ν_{xy}^2, ν_{yz}^2, ν_{zy}^2, E_x^2, E_y^2. Thus, it is necessary to find a solution to the system of differential equations of the plane theory of elasticity for an orthotropic material that satisfies the boundary conditions.

To determine the stress-strain state of bodies, we will apply the method of one-dimensional integral Fourier transform [12, 13] to the stress function $\varphi(x,y)$ in the variable x with the transformation parameter ξ:

$$\overline{\varphi}(\xi,y)=\int_{-\infty}^{+\infty}\varphi(x,y)\,e^{i\xi x}\,dx, \quad \varphi(x,y)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\overline{\varphi}(\xi,y)\,e^{-i\xi x}\,d\xi. \tag{4}$$

The first formula defines the direct one-dimensional integral Fourier transform for the function $\varphi(x,y)$, and the second defines the inverse. The function $\overline{\varphi}(\xi,y)$ is called the Fourier transform of the function $\varphi(x,y)$, for which the property [14] is true:

$$\int_{-\infty}^{+\infty} \frac{\partial^k \varphi(x,y)}{\partial x^k}\, e^{i\xi x}\, dx = (-i\xi)^k\, \overline{\varphi}(\xi,y). \tag{5}$$

The solution to the boundary value problem is sought in the space of transformants of the one-dimensional integral transform. In this case, all the basic equations of the problem and boundary conditions are directly transformed by the one-dimensional Fourier integral transform.

We find the solution of the analog of the biharmonic differential equation of a plane problem for an orthotropic material [15,16], to which we apply the Fourier integral transform:

$$A_1 \frac{\partial^4 \overline{\varphi}}{\partial y^4} - 2A_3\xi^2 \frac{\partial^2 \overline{\varphi}}{\partial y^2} + A_2\xi^4\, \overline{\varphi} = 0,$$

where $c_{11} = \frac{1-\nu_{xz}\nu_{zx}}{E_x}$, $c_{22} = \frac{1-\nu_{yz}\nu_{zy}}{E_y}$, $c_{33} = \frac{1}{G_{xy}}$, $c_{12} = c_{21} = \frac{\nu_{xy}+\nu_{xz}\nu_{zy}}{E_y} = \frac{\nu_{yx}+\nu_{zx}\nu_{yz}}{E_x}$, $G_{xy} = \frac{\sqrt{E_xE_y}}{2(1+\sqrt{\nu_{xy}\nu_{yx}})}$ are the elastic constants in Hooke's law, $A_1 = c_{11}$, $A_2 = c_{22}$, $A_3 = \frac{c_{33}-2c_{12}}{2}$, $\overline{\varphi} = \overline{\varphi}(\xi,y)$ is the Fourier transform of the variable x from $\varphi(x,y)$. All values related to the layer are denoted by indices $j=1$, and for the half-plane $j=2$.

Based on the results of [17,18], the transformants of the layer stress function $\overline{\varphi_{21}}(\xi,y)$ and the half-plane stress function $\overline{\varphi_{22}}(\xi,y)$ take the form:

$$\begin{aligned}\overline{\varphi_{21}} &= A_{21}\sinh\left(ry\sqrt{a_1}\right) + B_{21}\sqrt{a_1}y\sinh\left(ry\sqrt{a_1}\right) + C_{21}\cosh\left(ry\sqrt{a_1}\right) + D_{21}\sqrt{a_1}y\cosh\left(ry\sqrt{a_1}\right),\\ \overline{\varphi_{22}} &= A_{22}\sinh\left(ry\sqrt{a_2}\right) + B_{22}\sqrt{a_2}y\sinh\left(ry\sqrt{a_2}\right) + C_{22}\cosh\left(ry\sqrt{a_2}\right) + D_{22}\sqrt{a_2}y\cosh\left(ry\sqrt{a_2}\right),\end{aligned} \tag{6}$$

where $r=|\xi|, \sqrt{a_j} = \sqrt{A_3/A_1}, A_{2j}, B_{2j}, C_{2j}, D_{2j}$ are functions of the parameter ξ, $j=1,2$.

The stress function $\varphi(x,y)$ is chosen to satisfy the conditions exactly:

$$\sigma_x(x,y) = \frac{\partial^2\varphi(x,y)}{\partial y^2}, \quad \sigma_y(x,y) = \frac{\partial^2\varphi(x,y)}{\partial x^2}, \quad \tau_{xy}(x,y) = -\frac{\partial^2\varphi(x,y)}{\partial x\,\partial y}.$$

Taking into account the connection of the stress function $\varphi(x,y)$ with the stresses σ_x, σ_y, τ_{xy} and the Fourier transform property (5), we obtain the corresponding expressions in the space of Fourier transforms:

$$\begin{aligned}\overline{\sigma_x}(\xi,y) &= \int_{-\infty}^{+\infty} \frac{\partial^2\varphi}{\partial y^2}\, e^{i\xi x}\, dx = \frac{\partial^2}{\partial y^2}\int_{-\infty}^{+\infty} \varphi\, e^{i\xi x}\, dx = \frac{\partial^2\overline{\varphi}}{\partial y^2},\\ \overline{\sigma_y}(\xi,y) &= \int_{-\infty}^{+\infty} \frac{\partial^2\varphi}{\partial x^2}\, e^{i\xi x}\, dx = -\xi^2\overline{\varphi},\\ \overline{\tau_{xy}}(\xi,y) &= \int_{-\infty}^{+\infty} -\frac{\partial^2\varphi}{\partial x\,\partial y}\, e^{i\xi x}\, dx = -\frac{\partial}{\partial y}\int_{-\infty}^{+\infty} \frac{\partial\varphi}{\partial x}\, e^{i\xi x}\, dx = i\xi\frac{\partial\overline{\varphi}}{\partial y}.\end{aligned} \tag{7}$$

Substituting the stress functions for the layer $\overline{\varphi_{21}}(\xi,y)$ and the half-plane $\overline{\varphi_{22}}(\xi,y)$ in expressions (7), we obtain:

$$\begin{aligned}\overline{\sigma_x^j}(\xi,y) &= \left(D_{2j}\sqrt{a_j}ry + rC_{2j} + 2B_{2j}\right)a_jr\cosh\left(ry\sqrt{a_j}\right)\\ &\quad + \left(B_{2j}\sqrt{a_j}ry + rA_{2j} + 2D_{2j}\right)a_jr\sinh\left(ry\sqrt{a_j}\right),\\ \overline{\sigma_y^j}(\xi,y) &= \xi^2\cosh\left(ry\sqrt{a_j}\right)\left(D_{2j}\sqrt{a_j}y + C_{2j}\right) - \xi^2\sinh\left(ry\sqrt{a_j}\right)\left(B_{2j}\sqrt{a_j}y + A_{2j}\right),\\ \overline{\tau_{xy}^j}(\xi,y) &= \left(A_{2j}\sqrt{a_j}r + B_{2j}a_jry + D_{2j}\sqrt{a_j}\right)i\xi\cosh\left(ry\sqrt{a_j}\right)\\ &\quad + \left(C_{2j}\sqrt{a_j}r + D_{2j}a_jry + B_{2j}\sqrt{a_j}\right)i\xi\sinh\left(ry\sqrt{a_j}\right),\end{aligned} \tag{8}$$

where the index $j=1$ denotes a layer and $j=2$ denotes a half-space.

Applying the Fourier transform property (5) to the formulas $\varepsilon_x = c_{11}\sigma_x - c_{12}\sigma_y$, $\gamma_{xy} = c_{33}\tau_{xy}$, we obtain the displacement transformants [17]:

$$\overline{u_x}(\xi,y) = \frac{i}{\xi}\left(c_{11}\,\overline{\sigma_x}(\xi,y) - c_{12}\,\overline{\sigma_y}(\xi,y)\right), \quad \overline{u_y}(\xi,y) = \frac{i}{\xi}\left(c_{33}\,\overline{\tau_{xy}}(\xi,y) - \frac{\partial\overline{u_x}}{\partial y}\right). \tag{9}$$

Substituting the transformants $\overline{\varphi_{21}}(\xi,y)$ and $\overline{\varphi_{22}}(\xi,y)$ from formulas (9), we obtain the displacements:

$$\begin{aligned}
\overline{u_x} &= \frac{i\cosh(ry\sqrt{a_j})}{\xi}\left(D_{2j}a_j^{1.5}c_{11}r^2y + D_{2j}\sqrt{a_j}c_{12}\xi^2y + ajrc_{11}(rC_{2j}+2B_{2j}) + C_{2j}c_{12}\xi^2\right) \\
&\quad + \frac{i\sinh(ry\sqrt{a_j})}{\xi}\left(B_{2j}a_j^{1.5}c_{11}r^2y + B_{2j}\sqrt{a_j}c_{12}\xi^2y + ajrc_{11}(rC_{2j}+2B_{2j}) + iA_{2j}c_{12}\xi^2\right), \\
\overline{u_y} &= \big(c_{11}a_j^{1.5}(A_{2j}r+3D_{2j}) - \sqrt{a_j}(c_{33}-c_{12})(A_{2j}r+D_{2j}) \\
&\quad - rya_jB_{2j}(c_{33}-c_{12}-a_jc_{11})\big)\cosh(ry\sqrt{a_j}) \\
&\quad + \sinh(ry\sqrt{a_j})\big(c_{11}a_j^{1.5}(C_{2j}r+3B_{2j}) - \sqrt{a_j}(c_{33}-c_{12})(C_{2j}r+B_{2j}) \\
&\quad - rya_jD_{2j}(c_{33}-c_{12}-a_jc_{11})\big),
\end{aligned} \tag{10}$$

where the index $j=1$ denotes a layer and $j=2$ denotes a half-space.

Boundary conditions (1)–(3) in the transformant space take the form:

$$\overline{\sigma_y^1}(\xi,0)=\overline{f_1}(\xi),\quad \overline{\tau_{xy}^1}(\xi,0)=\overline{f_2}(\xi),\quad \overline{\sigma_y^1}(\xi,-h)=\overline{\sigma_y^2}(\xi,0),\quad \overline{u_x^1}(\xi,-h)=\overline{u_x^2}(\xi,0)=0,$$
$$\lim_{x_1^2+y_1^2\to\infty}\overline{\sigma_y^1}(\xi,y_1)=0,\quad \lim_{x_1^2+y_1^2\to\infty}\overline{\tau_{xy}^1}(\xi,y_1)=0,\quad \lim_{x_2^2+y_2^2\to\infty}\overline{\sigma_y^2}(\xi,y_2)=0,\quad \lim_{x_2^2+y_2^2\to\infty}\overline{\tau_{xy}^2}(\xi,y_2)=0.$$

Let us substitute the stress functions of the layer $\overline{\varphi_{21}}(\xi,y)$ and the half-plane $\overline{\varphi_{22}}(\xi,y)$ into the formulas with the stresses (5) $\overline{\sigma_x}$, $\overline{\sigma_y}$, $\overline{\tau_{xy}}$ and obtain:

$$\begin{aligned}
&\overline{\sigma_y^1}(\xi,0) = -\xi^2C_{21},\quad \overline{\tau_{xy}^1}(\xi,0) = i\xi\sqrt{a_1}(rA_{21}+D_{21}), \\
&\overline{\sigma_y^1}(\xi,-h) = \xi^2\cosh(rh\sqrt{a_1})(D_{21}\sqrt{a_1}h - C_{21}) - \xi^2\sinh(rh\sqrt{a_1})(B_{21}\sqrt{a_1}h - A_{21}), \\
&\overline{\sigma_y^2}(\xi,0) = -\xi^2C_{22},\quad \overline{u_x^2}(\xi,0) = \frac{ia_1rc_{11}(rC_{22}+2B_{22}) + i\xi^2c_{12}C_{22}}{\xi}, \\
&\overline{u_x^1}(\xi,-h) = \frac{i\cosh(rh\sqrt{a_1})}{\xi}\left(\xi^2c_{12}C_{21} + a_1rc_{11}(rC_{21}+2B_{21}) - h\xi^2\sqrt{a_1}(c_{12}D_{21}+a_1c_{11}D_{21})\right) \\
&\qquad + \frac{i\sinh(rh\sqrt{a_1})}{\xi}\left(h\xi^2\sqrt{a_1}(c_{12}B_{21}+a_1c_{11}B_{21}) - \xi^2c_{12}A_{21} - a_1rc_{11}(rA_{21}+2D_{21})\right).
\end{aligned} \tag{11}$$

The formulas for the stress transformants $\overline{\sigma_y^j}(\xi,y)$, $\overline{\tau_{xy}^j}(\xi,y)$ are linear combinations of the functions $\cosh(ry\sqrt{a_j})$, $\sinh(ry\sqrt{a_j})$ and $y\cosh(ry\sqrt{a_j})$, $y\sinh(ry\sqrt{a_j})$ each of which is unboundedly increasing for $r\neq 0$ and $y\to\infty$. Thus, $A_{2j}\sinh(ry\sqrt{a_j}) + C_{2j}\cosh(ry\sqrt{a_j})$ and $B_{2j}\sqrt{a_j}y\sinh(ry\sqrt{a_j}) + D_{2j}\sqrt{a_j}y\cosh(ry\sqrt{a_j})$ tends to 0 at infinity in the cases $A_{2j}=C_{2j}$ and $B_{2j}=D_{2j}$, $j=1,2$.

From the boundary conditions on the upper boundary of the layer, the conditions of contact between the layer and the half-plane, and the conditions at infinity, we obtain a system of linear algebraic equations with respect to the unknown functions A_{21}, B_{21}, C_{21}, D_{21}, A_{22}, B_{22}, C_{22}, D_{22}. The solutions of the system are given by expressions:

$$\begin{aligned}
A_{21} &= C_{21} = -\frac{\overline{f_1}}{\xi^2},\quad B_{21} = D_{21} = \frac{-i(i\overline{f_1}\sqrt{a_1}r + \overline{f_2}\xi)}{\xi^2\sqrt{a_1}}, \\
A_{22} &= C_{22} = \frac{\big(\sinh(ry\sqrt{a_1}) - \cosh(ry\sqrt{a_1})\big)\big(\overline{f_1}\sqrt{a_1}rh - i\overline{f_2}h\xi + \overline{f_1}\big)}{\xi^2}, \\
B_{22} &= D_{22} = \frac{\sinh(ry\sqrt{a_1})\big(i\overline{f_2}rh\xi(\sqrt{a_2}-\sqrt{a_1}) - \overline{f_1}r^2h(\sqrt{a_1}\sqrt{a_2}+a_1) + \overline{f_1}(i\xi - \sqrt{a_2}r)\big)}{\xi^2\sqrt{a_1}} \\
&\quad - \frac{\cosh(ry\sqrt{a_1})\big(i\overline{f_2}rh\xi(\sqrt{a_2}-\sqrt{a_1}) - \overline{f_1}r^2h(\sqrt{a_1}\sqrt{a_2}+a_1) + \overline{f_1}(i\xi - \sqrt{a_2}r)\big)}{\xi^2\sqrt{a_1}}.
\end{aligned} \tag{12}$$

The resulting expressions (12) are substituted into the stress and displacement transforms (8) and (10), and then subjected to the inverse integral Fourier transform from formula (1) to obtain their true values.

3. Numerical results and discussion

Consider an orthotropic layer whose material is characterized by elastic constants $\nu_{xy} = 0.26$, $\nu_{xz} = 0.19$, $\nu_{yz} = 0.41$, $\nu_{zy} = 0.30$, $E_x = 38600\,\text{MPa}$, $E_y = 8270\,\text{MPa}$.

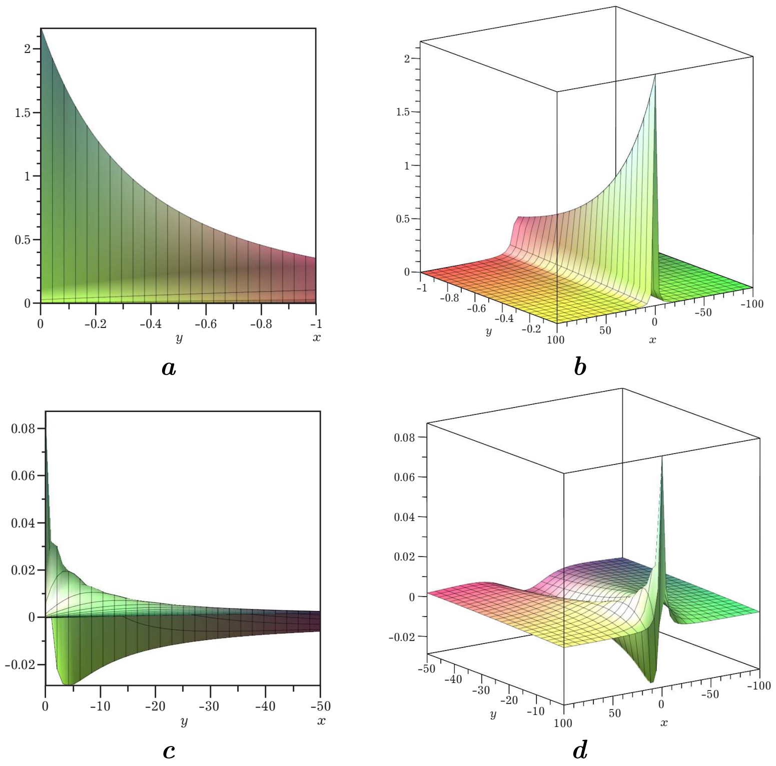


Fig. 3. True values of $\sigma_x(x,y)$ of the layer (***a***,***b***) and half-plane (***c***,***d***).

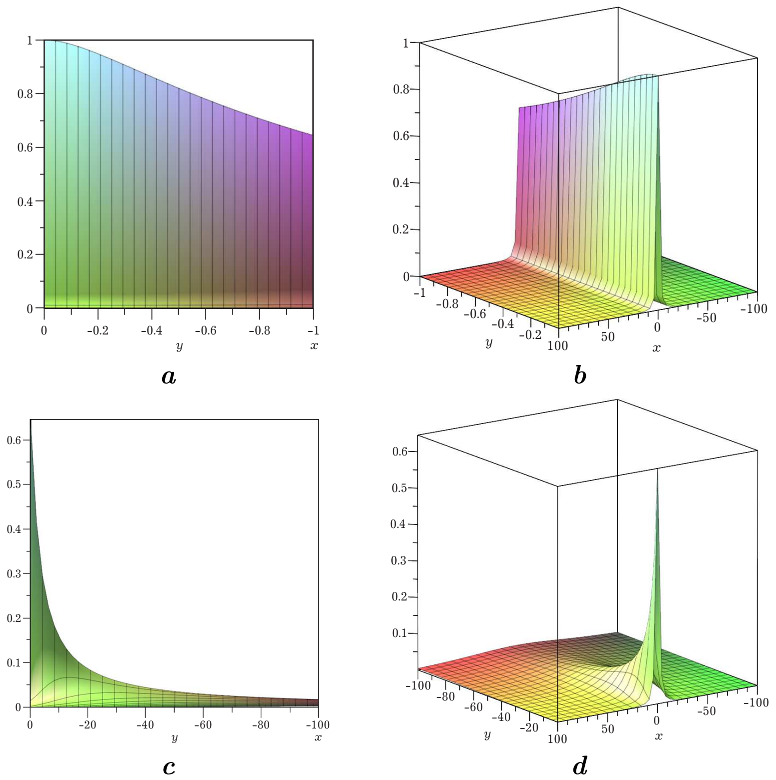


Fig. 4. True values of $\sigma_y(x,y)$ of the layer (***a***,***b***) and half-plane (***c***,***d***).

The orthotropic half-space is characterized by the constants $\nu_{xy} = 0.26$, $\nu_{xz} = 0.235$, $\nu_{yz} = 0.17$, $\nu_{zy} = 0.3$, $E_x = 1730\,\text{MPa}$, $E_y = 3310\,\text{MPa}$.

At the upper boundary of the layer $y_1 = 0$, the load $\sigma_y(x,0) = \frac{1}{x^2+1}$ is set, $\tau_{xy}(x,0) = 0$. Thus, the transformants $\overline{\sigma_y^1}(\xi,0) = \pi e^{-|\xi|}$, $\overline{\tau_{xy}^1}(\xi,0) = 0$.

Applying the inverse transformation to the transformants (5) $\overline{\sigma_x^j}(\xi,y)$, $\overline{\sigma_y^j}(\xi,y)$, $\overline{\tau_{xy}^j}(\xi,y)$, we obtain the true values of the layer and half-plane stresses (Figures 3–5).

The nature of the layer loading determines the symmetry of the stress distribution in the layer and the half-plane, with the maximum modulus stress values observed along the line of action of the maximum normal load.

As can be seen in Figures 3–4, the values of stresses $\sigma_x(x,y)$, $\sigma_y(x,y)$ of the layer and the half-plane take the largest values near $x = 0$ for each y, and at a distance from $x = 0$, the stress values decrease and tend to zero at infinity. This indicates the fulfillment of conditions (3) at infinity, and the fulfillment of conditions (1) at the $y = 0$ boundary. The results also indicate that the condition (2) of contact between the layer and the half-plane (adhesion) is met.

At the points of the layer symmetric with respect to the $x = 0$ plane, we observe the symmetry of the distribution of tangential stresses. These graphical results indicate that the boundary conditions (1) and (3) are met. Comparison of Figures 5a, 5c indicates that condition (2) is met. The results show an increase in the absolute values of τ_{xy} with increasing depth in the layer. In the half-plane, the behavior persists until reaching a maximum value at $y_2 = -3$, followed by a monotonic decrease, and at infinity, it tends to zero.

By applying the inverse Fourier transform (1) to the displacement transforms (7) $\overline{u_x^j}(\xi, y)$, $\overline{u_y^j}(\xi, y)$ we obtain the true values of the layer and half-plane displacements (Figures 6–7).

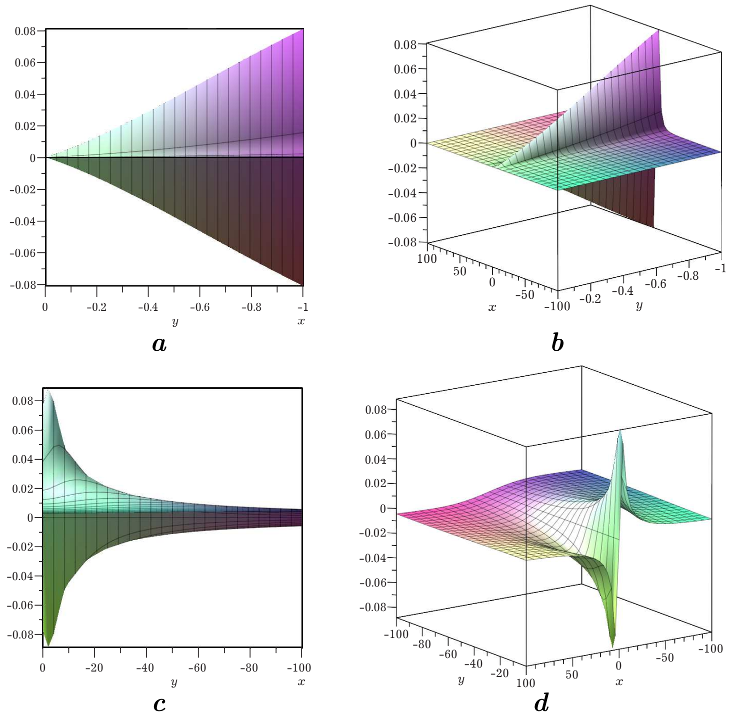


Fig. 5. True values of $\tau_{xy}(x, y)$ of the layer (***a***,***b***) and half-plane (***c***,***d***).

The results obtained for the displacements $u_x(x, y)$, $u_y(x, y)$ are in full agreement with the expected pattern of displacements. A symmetrical nature of deformation is observed. The maximum values of normal displacements $u_y(x, y)$ correspond to the line of action of the maximum normal load. The absolute values of displacements decrease with depth and tend to zero.

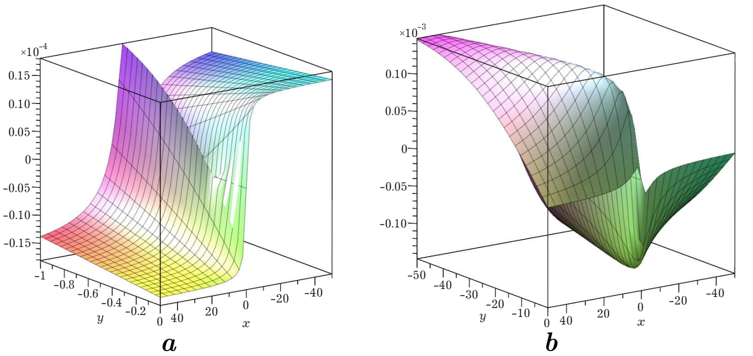


Fig. 6. True values of $u_x(x, y)$ of the layer (***a***) and half-plane (***b***).

To verify the adequacy of the proposed approach, we compared the solution obtained for the problem with isotropic materials with the solution obtained by the finite element method (FEM) using the QFEM finite element package [19]. In Figures 8*a*, 9*a*, 10*a*, 11*a*, the solutions obtained using QFEM are shown, in Figures 8*b*, 9*b*, 10*b*, 11*b* – according to the proposed scheme. The calculations were performed with the following parameters: $\nu^1_{xy} = \nu^1_{xz} = \nu^1_{yz} = \nu^1_{zy} = 0.2$, $E^1_x = E^1_y = 3.86 \times 10^{10}$ Pa. $\nu^2_{xy} = \nu^2_{xz} = \nu^2_{yz} = \nu^2_{zy} = 0.3$, $E^1_x = E^1_y = 3.31 \times 10^9$ Pa. The layer thickness is $h = 1$ m. Parameters of the body for the FEM calculations:

$$G_1 : \{-50 < x_1 < +50, -1 \leqslant y_1 \leqslant 0\},$$

$$G_2 : \{-50 < x_2 < +50, -100 \leqslant y_2 \leqslant 0\}.$$

At the upper boundary of the layer $y_1 = 0$, a load is set $\sigma_y(x, 0) = \frac{1}{x^2+1}$, $\tau_{xy}(x, 0) = 0$. The obtained solutions are in full agreement, which indicates the adequacy of the developed approach.

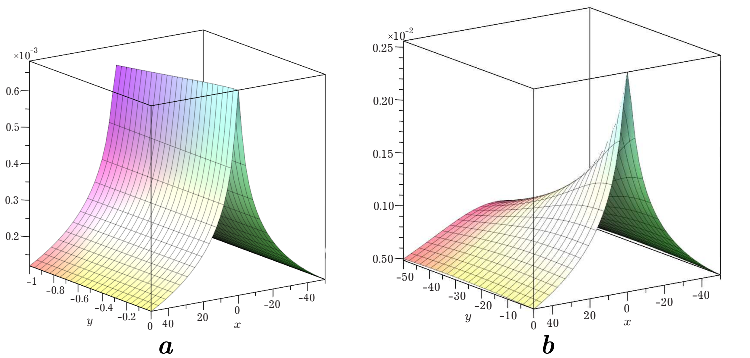
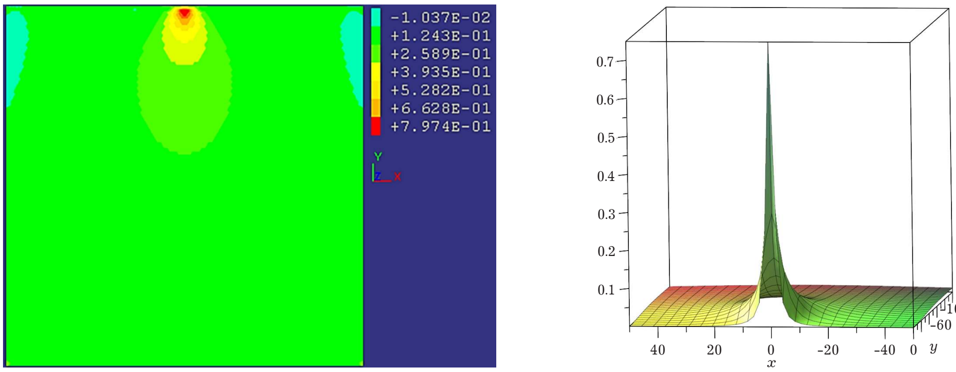


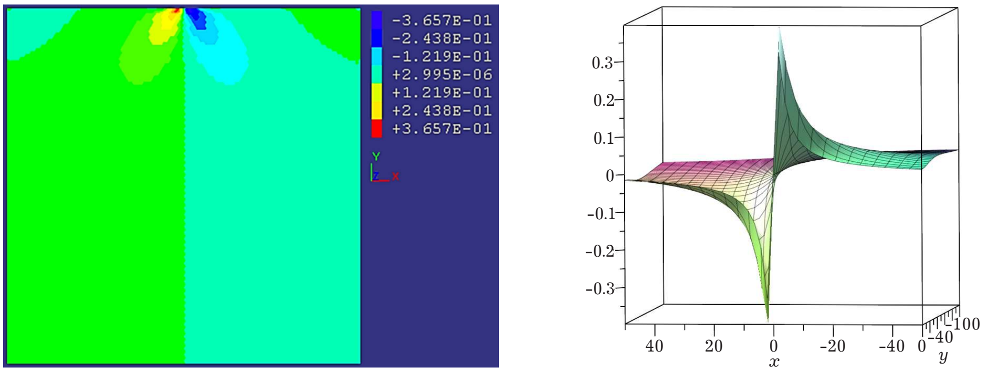
Fig. 7. True values of $u_y(x, y)$ of the layer (***a***) and half-plane (***b***).

In Figures 8–11, we observe the same distribution of the corresponding stresses and displacements. The numerical results obtained by both methods can be considered as coincident.



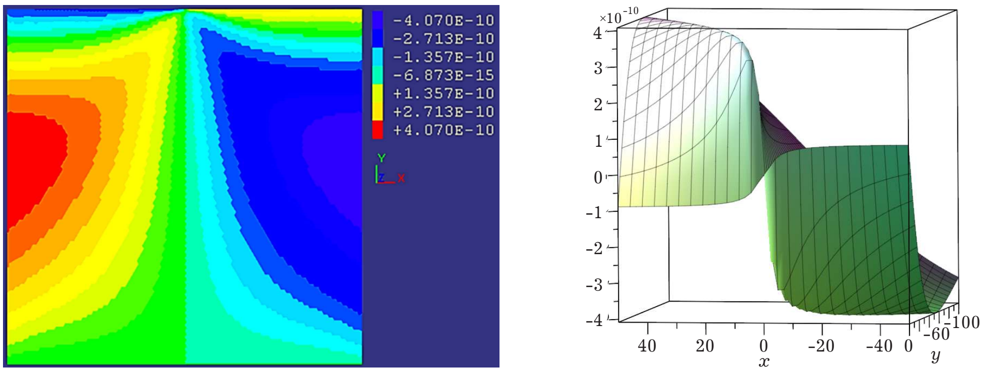
a *b*

Fig. 8. Values of $\sigma_y(x,y)$.



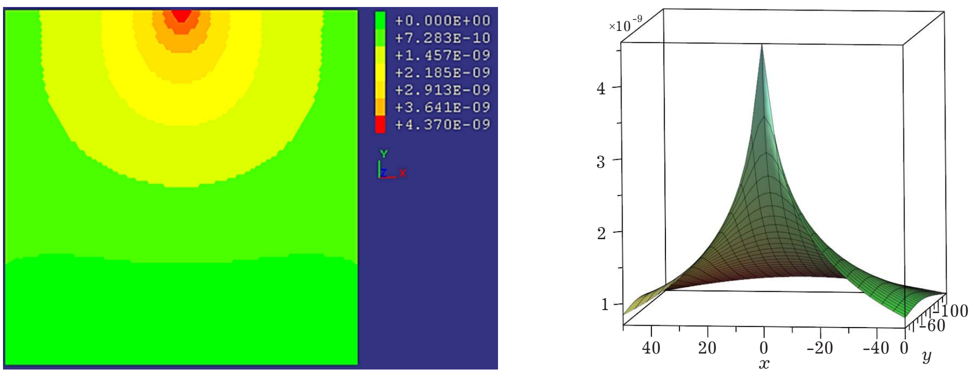
a *b*

Fig. 9. Values of $\tau_{xy}(x,y)$.



a *b*

Fig. 10. Values of $u_x(x,y)$.



a *b*

Fig. 11. Values of $u_y(x,y)$.

The obtained solutions are in full agreement, which indicates the adequacy of the developed approach and the possibility of its application to determine the stress-strain state of layered bodies with sufficient accuracy for technical applications.

4. Conclusions

A plane problem of the theory of elasticity is considered to determine the stress-strain state of an orthotropic layer coupled to an orthotropic half-space subjected to external loads.

The proposed approach to solving the problem is based on the application of the one-dimensional integral Fourier transform method. The search for unknown quantities is carried out in the space of Fourier transforms. A numerical-analytical solution to the problem is obtained, numerical calculations are performed, and their analysis is performed. The results obtained indicate the adequacy of the developed approach.

The next stage of research is to extend the described approach to the class of problems on determining the stress-strain state of multilayer bases with orthotropic layers.

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Напружено-деформований стан двошарового ортотропного тіла за умов плоскої деформації

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Розглядається задача про визначення напружень і деформацій двошарового тіла, що складається з ортотропного шару постійної товщини, зчепленого з ортотропним півпростором. На поверхню шару діють відомі зовнішні навантаження, такі що деформація тіла є плоскою. На нескінченності напруження дорівнюють нулю. Напружено-деформований стан тіла визначається за допомогою методу інтегральних перетворень Фур'є. Досліджено особливості розв'язків системи диференціальних рівнянь задачі. Отримано розв'язки конкретних задач та проведено їх аналіз.

Ключові слова: *ортотропний шар; ортотропна півплощина; плоска деформація; напружено-деформований стан; функція напружень; інтегральне перетворення Фур'є.*