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Mathematical modeling of solid municipal waste landfill surface settlement with regard to organic component biodegradation

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Landfilling of municipal solid waste generates a number of problems. This article focuses on one of them – the settling of the landfill surface due to organic residue biodegradation. The task of predicting the settlement of the waste storage surface is proposed to be solved within the framework of the theory of filtration consolidation of porous media. Accordingly, the problem of the dynamics of methane and carbon dioxide emission into the atmosphere was solved on the basis of known experimental data. The application of the proposed approaches made it possible to avoid modeling microbial dynamics and determining the stress-strain state of the waste storage massif, which would significantly complicate the mathematical model of the problem. Numerical solutions of the corresponding boundary value problem in a variable-time domain were found using the finite element method. Numerical experiments show changes in the dynamics of waste storage surface settlements for the case of taking into account biodegradation effects.

Keywords: *biodegradation modeling; greenhouse gas emissions; waste management; mathematical modeling; porous media mechanics.*

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1. Introduction

According to the study [1], the disposal of municipal solid waste (MSW) by creating specialized landfills has become a significant environmental issue worldwide. The problem lies in the dual pollution of the environment with by-products of complex physicochemical and biological transformations in such landfills: the atmosphere with greenhouse gases, and the soil and groundwater with chemical substances. In the work [2], it is noted that in the United States, nearly 268 million tons of MSW were generated in 2017, of which 139.6 million tons (52.1%) were deposited in landfills. In China, according to statistics from 2018, these figures were 663 and 117.06 million tons, respectively.

The authors of the article [3] emphasize that finding a sustainable approach to managing solid household waste is paramount, especially for developing countries. The authors note the following: "Due to the adoption of an inefficient waste management system, society is paying intangible costs such as damage to public health and environment quality. In order to minimize the environmental impacts and health issues associated with waste management practices, a sustainable waste management and disposal strategy is required."

In the work [2], the modeling of interrelated physicochemical and biological processes occurring in MSW landfills is based on the biodegradation of organic residues. This, in turn, leads to the formation of greenhouse gases (primarily carbon dioxide CO_2 and methane CH_4), heat release, formation of liquid solutions of chemical elements, deformation processes, and landfill surface subsidence. The importance of studying processes in waste landfills regarding their subsidence and biodegradation may relate to the stability of landfill slopes, as emphasized in the work [4]. In particular, the authors concluded that ignoring both waste biodegradation and gas generation can lead to an overestimation of the stability index of the landfill slope by approximately 20–50%.

The author's historical overview and analysis of existing approaches to MSW disposal [3] are valuable. Specifically highlighted are open dumps, anaerobic landfills, semi-aerobic landfills, aerated land-

fills, and dioreactor landfills. Open dumps are essentially the most primitive method of waste disposal, characterized by uncontrolled discharge of pollutants into the environment – both into the atmosphere and into soil and groundwater. Unfortunately, as stated by the authors [3], approximately 33% of landfills on Earth are currently organized in this manner, with 90% in low-income countries.

The second method of storage, anaerobic landfills, involves engineering solutions such as geomembranes [5–7] and protective coverings on top, aiming to minimize the environmental impact and human health risks associated with such landfills. However, as noted by the authors of the article [3], monitoring studies of old landfills of this type in Germany show that the potential for greenhouse gas emissions from such landfills persists for at least thirty years, and the danger of emissions of harmful substances into soil and groundwater lasts for centuries. Therefore, the latter three methods of MSW storage described above involve specific engineering solutions to accelerate the biodegradation processes of organic residues. They all involve aeration, with the latter also involving additional moisture and the removal of biodegradation products. As noted by the authors, the "semi-aerobic landfill" method was proposed in the 1970s in Japan and is known as the "Fukuoka method." When applied, the ratio of emitted CO_2 to CH_4 is 4:1, whereas for anaerobic landfills, it is 1:1.

Considering the brief overview regarding MSW landfill issues provided above, this study will focus specifically on two aspects – predicting the magnitude of landfill surface subsidence as a result of organic residue biodegradation processes and predicting the level of greenhouse gas emissions into the atmosphere.

To determine methane emissions from waste disposal sites in Karachi, the authors [3] propose using the formula

$$G_t = 1.868\, C_{\text{org}} \left(0.014T + 0.28\right)\left(1 - e^{-kt}\right),$$

where G_t is the landfill gas production during a specific time, t $\left(\frac{m^3}{\text{tonne fresh waste}}\right)$; C_{org} is total organic carbon in waste $\left(\frac{\text{kg}}{\text{tonne}}\right)$; T is the temperature (35°C); k is the degradation rate constant, $\left(k = \frac{\ln 2}{T_{0.5}}\right)$; and t is the landfill operation time (years). The C_{org} was determined by considering the degradable organic content (DOC) according to the organic fraction of MSW in Karachi. Note that in the specific case considered in the article, DOC = 15.5% or 155 kg/tonne. For the coefficient k, the authors used various values: 0.04; 0.05; 0.095; 0.3 year^{-1}. However, overall, the authors note the presence of scientifically justified formulas where the coefficient k depends on the amount of natural precipitation and the volume of additional liquid. In particular

$$k = 3.2 \times 10^{-5} \times (AP + AL) + 0.01,$$

where AP is the annual precipitation rate in mm and AL is the amount of additional liquid required. The aforementioned studies lay the groundwork for further consideration of variable temperature and variable moisture content of MSW as porous media. In this work, we will not yet consider the non-isothermal nature of the processes, focusing on such research in a separate article. Initial steps in this direction have already been taken in the work [8] to improve the adequacy of the boundary conditions at the interface of the waste repository with the atmosphere. Additionally, we currently consider the waste repository as a porous medium fully saturated with pore fluid. This implies the assumption that generated greenhouse gases are immediately removed beyond the boundaries of the MSW.

A similar formula was used in [9] to estimate the volumes of generated methane:

$$V_{\text{CH}_4} = L_0 \left(1 - \exp(-k_s t)\right),$$

where L_0 is the methane generation potential, k_s is the gas decay rate of the MSWs, and it is different from the decay rate used in the degrees of biodegradation. The corresponding rate of methane generation at time t is expressed as

$$r_{\text{CH}_4} = \begin{cases} \frac{t}{t_c}\tilde{L}_0 k_s & \text{when } t \leqslant t_c, \\ \tilde{L}_0 k_s \exp\left(-k_s(t - t_c)\right) & \text{when } t > t_c, \end{cases}$$

where $\tilde{L}_0$ is the corrected methane generation potential considering the time required to reach the maximum rate, and t_c is the time to reach the maximum methane generation rate. As noted by the

authors [9], the gas generated from MSW consists of methane (CH_4) and carbon dioxide (CO_2), as well as a small amount of other gases. Therefore, the total gas generation rate could be calculated using

$$G = \frac{r_{CH_4}}{C_{CH_4}},$$

where C_{CH_4} represents the percentage of methane in the total gas generation. Depending on the conditions, the methane fraction can range from 30% to 50%. However, in the study [10], methane concentrations ranging from 65% to 82% were obtained depending on the conditions.

Biodegradation processes and methane emissions can be utilized for human needs. However, this requires appropriate technological and engineering solutions. In the work [11], technological solutions were tested to improve and increase biogas production from the organic component of municipal solid waste. For the theoretical estimation of the volume of biogas released, the authors used three model formulas.

Mathematical modeling of complex interconnected processes occurring in landfills is progressing. In the work [12], an overview of approaches to mathematical modeling of interconnected processes in MSW and relevant engineering solutions is provided. In the work [13], a mathematical model of interconnected "consolidation–biodegradation" processes in MSW was constructed and investigated. This involved considering a three-phase porous medium, which significantly complicates the calculations, and biodegradation processes were described by a dynamic system of 13 equations. Similarly, this applies to the work [14].

Landfills of municipal solid waste are also associated with the process of their settlement. In the work [9], a model for predicting the settlement of municipal solid waste landfills is presented, based on biodegradation processes. In the article, settlement calculations are proposed to be carried out based on available field experiments and considerations of porous media mechanics.

The model of biodegradation is based on the relationship between the mass of a certain biodegradable component at time t and its initial mass

$$m_j = m_{0j} \exp(-k_j t),$$

where m_j is the mass of the biodegradable component j at time t, m_{0j} is the initial mass of the biodegradable component j, and k_j is the decay rate of the biodegradable component j.

In the work [15], the first-order decomposition model was used to determine the weight of each organic component during the biodegradation process

$$m_i = m_{0i} \exp(-k_i t),$$

where m_i represents the mass of biodegradable component i at time t, m_{0i} represents the initial mass of biodegradable component i, and k_i represents the decay rate of biodegradable component i. The decay rates of the food, paper, and wood used in the tests were 0.69, 0.24, and 0.05 year^{-1}, respectively. According to the analysis conducted by the authors [1], the approximate composition of municipal solid waste is as follows: 27% paper; 15% wood; 14.3% food waste; 9% heavy metals; 9.6% textile waste; 25.1% plastic.

We will consider the problem of determining surface settlements of a porous medium from the perspective of the theory of its consolidation. Investigating the settlements of a landfill for municipal solid waste, in terms of the correctness of setting up the corresponding boundary problems of mathematical consolidation theory, requires specifying what is called a kinematic boundary condition on the moving boundary.

2. Kinematic boundary condition at the upper moving boundary in the case of biodegradation

In [16], the kinematic boundary condition for vertical settlements of the surface of a porous medium is derived. We will utilize the same methodology as in [16]. However, for the case of considering the biodegradation of organic waste as a component of the solid component of the porous medium, this condition requires clarification. Without loss of generality, let us consider a one-dimensional problem

(Figure 1), where the upper boundary is movable and described by the equation $x = l(t)$, and the lower boundary remains unchanged over time.
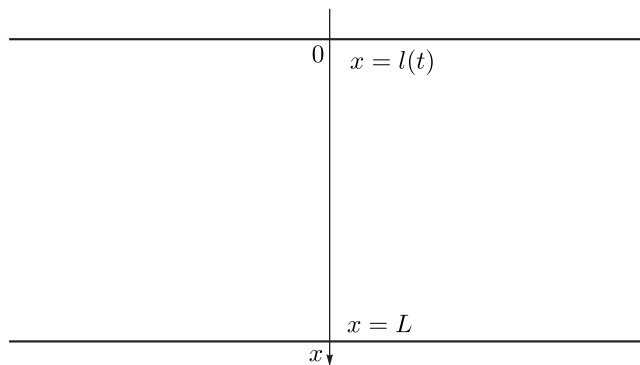
Fig. 1. Layer of porous medium of variable thickness.

To maintain the coherence of the material, we will restate some already known facts. Firstly, as noted in the study itself [16],

$$\frac{dl(t)}{dt} = -\int_{\ell(t)}^{L} \frac{d\varepsilon(z,t)}{dt}\, dz, \tag{1}$$

where integration is carried out over the vertical segment $[l(t), L]$, where $l(t)$ is the upper boundary of the porous medium, L is the lower boundary of the porous medium, and $\varepsilon(x,t)$ is the relative volumetric deformation of the porous medium. Second, it is known that condition (1) is derived from the integral sum

$$\frac{l(t+\Delta t) - l(t)}{\Delta t} = -\sum_{i=1}^{m} \frac{\varepsilon_i(t+\Delta t) - \varepsilon_i(t)}{\Delta t}\Delta z_i,$$

where m is the number of fragments with volumes V_i $(i = \overline{1,m})$ of the porous medium covering the vertical interval $[l(t); L]$. Next, considering the possibility of biodegradation processes, the change in volume V_i over the time interval Δt can occur due to changes in the volume of pores, which is equal to $\Pi_i V_i$, as well as due to changes in the volume of the organic component of the porous medium, which is equal to $W_i V_i$. Then

$$V_i(t) - V_i(t+\Delta t) = \Pi_i(t)V_i(t) - \Pi_i(t+\Delta t)V_i(t+\Delta t) + W_i(t)V_i(t) - W_i(t+\Delta t)V_i(t+\Delta t), \tag{2}$$

where $\Pi_i(t)$ is the averaged porosity of the medium in the selected fragment, and $W_i(t)$ is the relative volumetric content of the organic component in the selected fragment (both quantities are dimensionless, in m^3/m^3). Taking into account that $\Delta V_i = V_i(t) - V_i(t+\Delta t)$, from (2) we obtain

$$\frac{\Delta V_i}{V_i(t)} = \Pi_i(t) - \Pi_i(t+\Delta t) + \Pi_i(t+\Delta t)\frac{\Delta V_i}{V_i(t)} + W_i(t) - W_i(t+\Delta t) + W_i(t+\Delta t)\frac{\Delta V_i}{V_i(t)}.$$

Since by definition the relative volumetric deformation of the porous medium is given by $\frac{\Delta V_i}{V_i(t)} = \Delta\varepsilon_i$, then

$$\Delta\varepsilon_i = \Pi_i(t) - \Pi_i(t+\Delta t) + \Pi_i(t+\Delta t)\Delta\varepsilon_i + W_i(t) - W_i(t+\Delta t) + W_i(t+\Delta t)\Delta\varepsilon_i,$$

$$(1 - \Pi_i(t+\Delta t) - W_i(t+\Delta t))\Delta\varepsilon_i = -\left(\frac{d\Pi_i(t)}{dt} + \frac{dW_i(t)}{dt}\right)\Delta t,$$

where in the above relationship we have neglected infinitely small higher orders (as $\Delta t \to 0$). Taking into account that $\Pi_i = \frac{e_i}{1+e_i}$, where e_i is the averaged void ratio in the selected fragment of the medium, we have

$$\left(1 - \frac{e_i(t+\Delta t)}{1+e_i(t+\Delta t)} - W_i(t+\Delta t)\right)\Delta\varepsilon_i = -\frac{d}{dt}\left(\frac{e_i(t)}{1+e_i(t)}\right)\Delta t - \frac{dW_i(t)}{dt}\Delta t,$$

$$\frac{\Delta\varepsilon_i}{\Delta t} = -\frac{1}{1-(1+e_i(t+\Delta t))W_i(t+\Delta t)}\left(\frac{1+e_i(t+\Delta t)}{1+e_i(t)^2}\frac{de_i(t)}{dt} + (1+e_i(t+\Delta t))\frac{dW_i(t)}{dt}\right),$$

or (as $\Delta t \to 0$)

$$\frac{d\varepsilon_i}{dt} = -\frac{1}{1 - (1 + e_i(t))W_i(t)} \left(\frac{1}{1 + e_i(t)} \frac{de_i(t)}{dt} + (1 + e_i(t)) \frac{dW_i(t)}{dt} \right).$$

Substituting the previously obtained equation into (1), we have

$$\frac{dl(t)}{dt} = \int_{l(t)}^{L} \frac{1}{1 - (1 + e(t,\theta,S))W(z,t)} \left(\frac{1}{1 + e(t,\theta,S)} \frac{de(t,\theta,S)}{dt} + (1 + e(t,\theta,S)) \frac{\partial W(z,t)}{\partial t} \right) dz. \quad (3)$$

Condition (3) is indeed the kinematic boundary one, which describes the change in position of the upper movable boundary of the porous medium while considering biodegradation processes and known dependencies $e = e(t,\theta,S)$. Here $S(x,t) = \{s_j(x,t)\}_{j=1}^n$ is a vector function of physico-chemical influencing factors; θ represents the sum of principal stresses in the porous medium skeleton. By $\frac{de}{dt}$, we mean the total derivative of the function $e = e(t,\theta,S)$. Taking into account that

$$W(x,t) = \frac{1}{\rho_{ws}} m_{ws}(x,t)$$

from condition (3), we obtain:

$$\frac{dl(t)}{dt} = \int_{l(t)}^{L} \frac{1}{1 - (1 + e(t,\theta,S))\frac{m_{ws}(z,t)}{\rho_{ws}}} \left(\frac{1}{1 + e(t,\theta,S)} \frac{de(t,\theta,S)}{dt} + \frac{1 + e(t,\theta,S)}{\rho_{ws}} \cdot \frac{\partial m_{ws}(z,t)}{\partial t} \right) dz. \quad (4)$$

Here, ρ_{ws} is the density of organic waste; $m_{ws}(x,t)$ is the mass of organic waste per unit volume of the porous medium (with the same dimensions as density). It should be noted that condition (4) still needs to be supplemented with a functional dependence $e = e(t,\theta,S)$. This is done in subsequent sections.

3. Modified consolidation equation accounting for biodegradation

The settlement processes of MSW landfill are influenced by two factors – the reduction in the volume of organic components due to biodegradation and the consolidation of the porous medium under the influence of external loads or its own weight. Therefore, we will solve the problem of predicting the settlement of the MSW surface within the framework of the theory of filtration consolidation of porous media. The consolidation equation of MSW, as a porous medium, taking into account the processes of biodegradation, requires modification. For this purpose, we will use the approach proposed in [17] and partially applied and described in the works [18, 19].

It should be noted that, without loss of generality, we will consider a one-dimensional problem of compacting a mass of porous medium with variable thickness (see Figure 1)). Then we obtain the consolidation equation for a saturated porous medium, which is presented by formula (12) in the work [17],

$$\frac{de(t,\theta,S)}{dt} + e(t,\theta,S) \left(\frac{1}{\rho_p} \frac{d\rho_p}{dt} - \frac{1}{\rho_m} \frac{d\rho_m}{dt} \right) = (1 + e) \frac{\partial}{\partial x} \left(k(e) \frac{\partial h}{\partial x} \right), \; x \in \Omega(t) = (l(t); L), \; t > 0. \quad (5)$$

Here, ρ_m represents the total density of the solid component of the porous medium; ρ_p denotes the density of the pore fluid; $k(e)$ is the filtration coefficient, which depends on the void ratio e, which in turn may depend on the vector $S = (s_1, s_2, \ldots, s_n)$ and the sum of principal stresses θ in the porous medium skeleton; h is the excess pressure. Considering the influencing factors in both the consolidation equation (5) and the condition (4), we will account for the change in the void ratio due to stress changes, as well as the biodegradation of the organic waste component. In this work, we neglect the creep of the porous medium skeleton, meaning that in subsequent developments, $e = e(\theta,S)$. Additionally, due to the smallness of the quantities $\frac{1}{\rho_p}$ and $\frac{1}{\rho_m}$, we disregard changes in the densities of the porous medium phases over time. Determining the functional dependence $e = e(\theta,S)$ is a complex task that requires several assumptions. Specifically, we assume that stress changes in the porous medium skeleton and biodegradation processes within it independently affect the variation in the void ratio. Then, to specify $e = e(\theta,S)$, a somewhat generalized compression dependency can be utilized

$$e = -a\theta + E(S)$$

and a relationship that holds true in the one-dimensional case

$$\frac{\partial \theta}{\partial t} = -\gamma \frac{\partial h}{\partial t}.$$

Here, a is the compressibility coefficient of the porous medium, γ is the specific weight of the pore fluid; $E = E(S)$ is a function representing the dependency of the void ratio on the organic component of the waste repository. Then we have

$$\frac{\partial e}{\partial \theta} = -a.$$

Following a similar methodology to derive the kinematic boundary condition (4) in the selected volume V_i of the porous medium over the time interval Δt, we have

$$E_i(t) = \frac{\Pi_i(t)}{T_i(t) + W_i(t)},$$

$$E_i(t + \Delta t) = \frac{\Pi_i(t) + \beta \frac{dW_i}{dt} \Delta t}{T_i(t) + W_i(t) - \frac{dW_i}{dt} \Delta t}.$$

Here, $\Pi_i(t)$ represents the average porosity of the medium in the selected fragment, $W_i(t)$ denotes the relative volumetric content of the organic component in the selected fragment, $T_i(t)$ indicates the relative volumetric content of inorganic particles that are not subject to biodegradation processes (all quantities are in $\mathrm{m}^3/\mathrm{m}^3$); $\beta \in [0,1]$ is the relative coefficient indicating the proportion of voids remaining from biodegraded residues (for example, if $\beta = 1$, it means that voids remain in the place of biodegraded residues and are not occupied by the solid phase of the porous medium). Then, finding the difference $\frac{E_i(t+\Delta t) - E_i(t)}{\Delta t}$ and taking the limit as $\Delta t \to 0$, we obtain

$$\frac{dE_i}{dt} = -\beta(1 + e_i(t))^2 \frac{dW_i}{dt}, \quad i = \overline{1, m}.$$

Then, summarizing the above considerations, we obtain

$$\frac{de(\theta, S)}{dt} = \frac{\partial e}{\partial \theta} \cdot \frac{\partial \theta}{\partial t} + \sum_{i=1}^{n} \frac{\partial e}{\partial s_i} \cdot \frac{\partial s_i}{\partial t} = a\gamma \frac{\partial h}{\partial t} - \beta(1 + e)^2 \sum_{i=1}^{n} \frac{\partial s_i}{\partial t}.$$

To apply the biodegradation relationships for determining $\frac{\partial s_i}{\partial t}$, $i = \overline{1, n}$, we need to transition to the mass $m_i(t)$, $i = \overline{1, n}$. Then, in a unit volume of the porous medium, $m_i(t) = \rho_i s_i(t)$, or

$$\frac{\partial s_i}{\partial t} = \frac{1}{\rho_i} \frac{\partial m_i(t)}{\partial t}, \quad i = \overline{1, n}.$$

Here, ρ_i represents the density of the i-th organic component, $i = \overline{1, n}$. As a result, the kinematic boundary condition (4) and the consolidation equation (5) will take the form:

$$\frac{dl(t)}{dt} = \int_{l(t)}^{L} \frac{1}{1 - (1 + e) \sum_{i=1}^{n} \frac{m_i}{\rho_i}} \left(\frac{a\gamma}{1 + e} \frac{\partial h}{\partial t} + (1 - \beta)(1 + e) \sum_{i=1}^{n} \frac{1}{\rho_i} \frac{\partial m_i}{\partial t} \right) dz;$$

$$\frac{a\gamma}{1 + e} \frac{\partial h}{\partial t} - \beta(1 + e) \sum_{i=1}^{n} \frac{1}{\rho_i} \frac{\partial m_i}{\partial t} = \frac{\partial}{\partial x} \left(k(e) \frac{\partial h}{\partial x} \right), \quad x \in \Omega(t) = (l(t); L), \quad t > 0.$$

4. Mathematical model of porous MSW repository compaction considering the effect of biodegradation

The mathematical model will incorporate the previously modified equation of filtration consolidation, supplemented by the kinematic boundary condition at the upper movable boundary $x = l(t)$ of the porous medium mass. As a result, we have the following boundary problem:

$$\frac{a\gamma}{1 + e} \frac{\partial h}{\partial t} - \beta(1 + e) \sum_{i=1}^{n} \frac{1}{\rho_i} \frac{\partial m_i}{\partial t} = \frac{\partial}{\partial x} \left(k(e) \frac{\partial h}{\partial x} \right), \quad x \in \Omega(t) = (l(t); L), \quad t > 0, \qquad (6)$$

$$h(x,t)\big|_{x=l(t)} = \overline{h}_l(t), \quad t \geqslant 0, \qquad (7)$$

$$u(x,t)\big|_{x=L} = \left(-k(e)\frac{\partial h}{\partial x}\right)\Big|_{x=L} = 0, \quad t \geqslant 0, \tag{8}$$

$$h(x,0) = h_0(x), \quad x \in \overline{\Omega} = [l_0; L], \tag{9}$$

$$\frac{dl(t)}{dt} = \int_{l(t)}^{L} \frac{1}{1 - (1+e)\sum_{i=1}^{n}\frac{m_i}{\rho_i}}\left(\frac{a\gamma}{1+e}\frac{\partial h}{\partial t} + (1-\beta)(1+e)\sum_{i=1}^{n}\frac{1}{\rho_i}\frac{\partial m_i}{\partial t}\right)dz; \tag{10}$$

$$l(t)\big|_{t=0} = l_0 < L. \tag{11}$$

Here: h is excess pressure; k is filtration coefficient of a porous medium depends on the void ratio, which, in turn, is influenced by the pressures and concentration of biowaste; $n, e = \frac{n}{1-n}$ is porosity and the void ratio of the medium; $h_0(x)$, $\overline{h}_l(t)$ are known functions; $l_0 < L$ is specified initial position of the upper movable boundary; a is coefficient of compression of the porous medium; γ is specific weight of the pore fluid; u is filtration velocity, determined according to Darcy's law

$$u = -k(e)\frac{\partial h}{\partial x}.$$

A problem of a similar type for a system of two quasi-linear parabolic equations was studied in [20] within the framework of the qualitative theory of boundary value problems in mathematical physics in domains with moving boundaries. In particular, the existence and uniqueness of a local-in-time solution in the Hölder space were proved.

Similarly to [21], let us introduce the notation $Q_T = \Omega \times (0; T]$.

Definition 1. *A classical solution to the initial-boundary value problem (6)–(11) is a function $h(x,t) \in \Psi_h$ that satisfies for all $(x,t) \in \overline{Q}_T$ the equation (6) and the initial condition (9).*

In the above definition, Ψ_h is the set of functions $\psi_h(x,t)$ which, together with $\frac{\partial(\cdot)}{\partial x}$, are continuous on the closure $\overline{Q}_T$, have bounded continuous partial derivatives $\frac{\partial(\cdot)}{\partial t}$, $\frac{\partial^2(\cdot)}{\partial x^2}$ on Q_T, and satisfy conditions (7) and (8).

5. Finite element method in the problem of waste consolidation

Let H_0 be the space of functions $s(x)$ belonging to the Sobolev space $W_2^1(\Omega)$ on the interval $(l; L)$, with zero values at the ends of the segment $[l; L]$, where the function $h(x,t)$ has first-order boundary conditions accordingly. Let H be the space of functions $v(x,t)$ integrated with their first derivatives over the interval $(l; L)$, $\forall t \in (0, T]$, $T > 0$, satisfying the first-order boundary conditions as $h(x,t)$. Let us take $s(x) \in H_0$. Multiplying equation (6) and the initial condition (9) by $s(x)$, integrating them over the interval $[l; L]$, and considering condition (8), we obtain

$$\int_{l(t)}^{L} \frac{\gamma a}{1+e}\frac{\partial h}{\partial t}s(x)\,dx + \int_{l(t)}^{L} k(e)\frac{\partial h}{\partial x}\frac{ds}{dx}\,dx = \int_{l(t)}^{L}\left(\beta(1+e)\sum_{i=1}^{n}\frac{1}{\rho_i}\frac{\partial m_i}{\partial t}s(x)\right)dx, \tag{12}$$

$$\int_{l_0}^{L} h(x,0)s(x)\,dx = \int_{l_0}^{L} h_0(x)s(x)\,dx. \tag{13}$$

Definition 2. *A function $h(x,t) \in H$, which satisfies the integral equations (12) and (13) for any $s(x) \in H_0$, is called a generalized solution of the boundary value problem (6)–(11).*

An approximate generalized solution of the boundary value problem (6)–(11) is sought in the form

$$h(x,t) = \sum_{i=1}^{N} h_i(t)\,\phi_i(x), \tag{14}$$

where $h_i(t)$, $i = 1, \ldots, N$ are unknown coefficients depending only on time. The functions $\phi_i(x)$, $i = 1, \ldots, N$ are polynomial basis functions with finite support. Then, taking into account the above, from the weak formulation (12), (13) of problem (6)–(11), considering (14), we obtain

$$\mathbf{M_1}\frac{d\mathbf{H}}{dt} + \mathbf{L_1}(\mathbf{H}, \mathbf{S})\,\mathbf{H(t)} = \mathbf{F_1}(\mathbf{S}), \tag{15}$$

$$\widetilde{\mathbf{M}_1}\mathbf{H}^{(0)} = \widetilde{\mathbf{F}}_1, \tag{16}$$

where

$$\widetilde{\mathbf{F}}_1 = \big(\widetilde{f}_i^{(1)}\big)_{i=1}^N, \quad \widetilde{\mathbf{M}}_1 = \big(\widetilde{m}_{ij}^{(1)}\big)_{i,j=1}^N, \quad \widetilde{m}_{ij}^{(1)} = \int_{l_0}^L \phi_i\,\phi_j\,dx, \quad \mathbf{M}_1 = \big(m_{ij}^{(1)}\big)_{i,j=1}^N,$$

$$\mathbf{L}_1 = \big(l_{ij}^{(1)}\big)_{i,j=1}^N, \quad \mathbf{F}_1 = \big(f_i^{(1)}\big)_{i=1}^N,$$

$$\widetilde{f}_i^{(1)} = \int_{l_0}^L h_0\,\phi_i\,dx,$$

$$\mathbf{H} = \big(h_i(t)\big)_{i=1}^N, \quad \mathbf{H}^{(0)} = \big(h_i(0)\big)_{i=1}^N,$$

$$m_{ij}^{(1)} = \int_{l(t)}^L \frac{\gamma a}{1+e}\phi_i\,\phi_j\,dx, \quad l_{ij}^{(1)} = \int_{l(t)}^L k(e)\frac{d\phi_i}{dx}\frac{d\phi_j}{dx}\,dx,$$

$$f_i^{(1)} = \int_{l(t)}^L \left(\beta(1+e)\sum_{j=1}^n \frac{1}{\rho_j}\frac{\partial m_j}{\partial t}\right)\phi_i(x)\,dx.$$

The problem (15), (16) is a Cauchy problem for a system of first-order nonlinear differential equations. Finding its solution also requires the application of appropriate discretization schemes. In terms of practical implementation simplicity, the fully implicit linearized finite difference scheme [22] has proven itself well. For example, for the system (15), it takes the form:

$$\mathbf{M}_1\big(\mathbf{H}^{(j)}\big)\frac{\big(\mathbf{H}^{(j+1)} - \mathbf{H}^{(j)}\big)}{\tau} + \mathbf{L}_1\big(\mathbf{H}^{(j)}, \mathbf{S}^{(j)}\big)\cdot\mathbf{H}^{(j+1)} = \mathbf{F}_1\big(\mathbf{H}^{(j)}, \mathbf{S}^{(j)}\big), \quad j = 0, 1, 2, \ldots, m_\tau - 1.$$

Here, the time interval $[0, T]$ is divided into m_τ equal parts with a step of $\tau = T/m_\tau$; $\mathbf{H}^{(j)}$ represents the approximate solution of the initial value problem (15), (16) at $t = j\tau$. Additionally, the Crank–Nicolson scheme and the predictor-corrector scheme can be applied. Since in the numerical experiments in this work, we do not use the latter two schemes, for more detailed information about them, we refer the reader to, for example, the work [7].

6. Approximation of the kinematic boundary condition

Applying temporal discretization to condition (10) using the implicit finite difference scheme, we obtain

$$\frac{l^{(j+1)} - l^{(j)}}{\tau} = \int_{l^{(j)}}^L \left(\frac{\gamma a}{1+e^{(j+1)}}\frac{h^{(j+1)} - h^{(j)}}{\tau} + (1-\beta)(1+e^{(j+1)})\sum_{i=1}^n \frac{1}{\rho_i}\left(\frac{\partial m_i}{\partial t}\right)^{(j+1)}\right)dz, \ j = 0, 1, 2, \ldots.$$

So, we obtain

$$l^{(j+1)} = l^{(j)} + \int_{l^{(j)}}^L \left(\frac{\gamma a}{1+e^{(j+1)}}\big(h^{(j+1)} - h^{(j)}\big) + \tau(1-\beta)\big(1+e^{(j+1)}\big)\sum_{i=1}^n \frac{1}{\rho_i}\left(\frac{\partial m_i}{\partial t}\right)^{(j+1)}\right)dz, \tag{17}$$

Here, $l^{(j+1)} = l(t_{j+1})$, $j = 0, 1, 2, \ldots$. In equation (17), the position $l^{(j)}$ is known, as well as the values $h^{(j+1)}$, $e^{(j+1)}$, and $h^{(j)}$. The values of pressures, according to the algorithm, at the time level $(j+1)$ are found using the position of the upper boundary $l^{(j)}$. The integral in (17) can be evaluated using numerical integration formulas. The specifics of computing $e^{(j+1)}$ and $\big(\frac{\partial m_i}{\partial t}\big)^{(j+1)}$, $i = 1, \ldots, n$, are provided in the following section.

7. Results of numerical experiments

As a reminder, according to our assumptions:

$$\frac{de}{dt} = a\gamma\frac{\partial h}{\partial t} - \beta(1+e)^2\sum_{i=1}^n \frac{1}{\rho_i}\frac{\partial m_i}{\partial t}.$$

Next, using temporal discretization, we obtain

$$\frac{e^{(j+1)} - e^{(j)}}{\tau} = a\gamma\frac{h^{(j+1)} - h^{(j)}}{\tau} - \beta(1+e^{(j)})^2\sum_{i=1}^n \frac{1}{\rho_i}\left(\frac{\partial m_i}{\partial t}\right)^{(j+1)}, \quad j = 0, 1, 2, \ldots, m_\tau - 1,$$

or

$$e^{(j+1)} = e^{(j)} + a\gamma\big(h^{(j+1)} - h^{(j)}\big) - \tau\beta\big(1 + e^{(j)}\big)^2 \sum_{i=1}^{n} \frac{1}{\rho_i}\left(\frac{\partial m_i}{\partial t}\right)^{(j+1)}, \quad j = 0, 1, 2, \ldots, m_\tau - 1.$$

For the dependence of the filtration coefficient on the void ratio, we will use the Carman–Kozeny formula [19]:

$$k^{(j+1)} = k_0 \frac{(1 + e_0)}{(1 + e^{(j+1)})}\left(\frac{e^{(j+1)}}{e_0}\right)^3,$$

where e_0 is the initial value of the void ratio. For numerical experiments, we will use the following organic components of household waste [1]: paper – initially comprising 27% of the total mass of organic waste w_T per ton of waste (the mass of paper will be denoted by the function $m_p = m_p(t)$, with its initial value denoted as $m_p^{(0)}$), with a density of waste paper $\rho_p = 800$ kg/m^3; wood – initially comprising 15% of the total mass of organic waste w_T per ton of waste (the mass of wood will be denoted by the function $m_w = m_w(t)$, with its initial value denoted as $m_w^{(0)}$), with a density of waste wood $\rho_w = 600$ kg/m^3; food waste – initially comprising 14.3% of the total mass of organic waste w_T per ton of waste (similarly, the mass of food waste is denoted as $m_f = m_f(t)$, with its initial value denoted as $m_f^{(0)}$), with a density of food waste $\rho_f = 400$ kg/m^3. According to the simplest model of biodegradation, we have:

$$m_p(t) = m_p^{(0)} \exp(-k_p t),$$
$$m_w(t) = m_w^{(0)} \exp(-k_w t),$$
$$m_f(t) = m_f^{(0)} \exp(-k_f t),$$

where [9]

$$k_p = 0.00067 \tfrac{1}{\text{day}},$$
$$k_w = 0.00014 \tfrac{1}{\text{day}},$$
$$k_f = 0.00192 \tfrac{1}{\text{day}}.$$

Then we have:

$$\frac{dm_p(t)}{dt} = -k_p m_p^{(0)} \exp(-k_p t),$$
$$\frac{dm_w(t)}{dt} = -k_w m_w^{(0)} \exp(-k_w t),$$
$$\frac{dm_f(t)}{dt} = -k_f m_f^{(0)} \exp(-k_f t).$$

Organic waste is considered uniformly distributed throughout the thickness of the porous medium with an initial total mass value of $w_T^{(0)} = 155$ kg/ton [3] (Table 4). The overall density of household waste is assumed to be $\rho_T = 800$ kg/m^3 [3] (Table 5).

According to [2], Section 3.2.1, the filtration coefficient of waste varies widely. In this study, we used the initial value

$$k^{(0)} = 0.02 \frac{\text{m}}{\text{day}}.$$

According to [3], the total volume of greenhouse gases generated (in cubic meters per ton of waste) as a result of the biodegradation of each component of organic waste will be determined according to the dependencies:

$$G_p = 1.868\, m_p^{(0)}(0.014T + 0.28)\big(1 - e^{-k_p t}\big),$$
$$G_w = 1.868\, m_w^{(0)}(0.014T + 0.28)\big(1 - e^{-k_w t}\big),$$
$$G_f = 1.868\, m_f^{(0)}(0.014T + 0.28)\big(1 - e^{-k_f t}\big).$$

In the same study, it was noted that methane accounts for approximately 64% of the total volume of greenhouse gases generated. Other scientific sources mention different percentage ratios, such as a 1:1

or 1:4 distribution of methane and carbon dioxide. It is worth noting that these ratios depend on the technical operating conditions of the repositories. In turn, technical conditions are determined by the purpose – either minimizing greenhouse gas emissions into the atmosphere or maximizing the methane fraction in biogas generation conditions. The temperature T was assumed to be constant at $25°C$ in the experimental data.

The initial porosity was assumed to be $n_0 = 0.40$, corresponding to the initial value of the void ratio $e_0 = 0.67$. For the model problem, a porous medium layer with a thickness of $l = 10$ m. The spatial step size x was 0.01 m. The time step $\tau = 1$ day. Piecewise quadratic basis functions were used for the finite element method. In the consolidation equation $a = 1 \times 10^{-7} \frac{m^2}{H}$ is the coefficient of compressibility of the porous medium [23], and $\gamma_c = 10^4 \frac{H}{m^2}$ is the specific weight of the pore liquid.

With a density of $\rho_T = 800$ kg/m^3, a volume of thickness $l = 10$ m with a surface area of 1 m^2 will contain 8 tons of waste. Therefore, the total volume of generated greenhouse gases over time t will be

$$G(t) = 8S(G_p + G_w + G_f),$$

where S is the total surface area of the waste repository. Table 1 presents the results of calculating surface subsidence of the waste repository in the absence of excess pressures ($h_0 = 0$ m) and at various values of β. As observed, under these conditions, if the pore space formed as a result of the biodegradation of organic waste is immediately filled with the solid phase ($\beta = 0$), the subsidence over 300 days is approximately 60 centimeters. As β increases, the subsidence logically decreases. It can be assumed that for $\beta \neq 0$, the estimation of subsidence dynamics will be significantly influenced by creep processes, which we have ignored in this study so far. It is noteworthy that without considering biodegradation processes, in the absence of excess pressures $h_0 = 0$ m and neglecting the creep of the solid phase of the household waste repository, subsidence is absent.

Table 2 presents the results of numerical experiments when initially $h_0 = 0$ m, but every 10 days the pressures increase by $\Delta h = 3$ m due to the application of additional loads in the form of freshly deposited waste. Table 3 provides the surface subsidence results for the same problem but without considering the influence of biodegradation processes. As seen from the comparison of data in Tables 2 and 3, for $\beta < 1$ the subsidence when considering biodegradation is always greater than for the case neglecting these processes. Moreover, for $\beta = 0$, the subsidence increases by more than two times compared to the values in Table 3.

Table 1. Settlement of waste storage surface in the absence of excessive pressures considering biodegradation.

β	t, days	subsidence, m
	50	0.121
	100	0.233
	200	0.434
	300	0.606
1/2	50	0.062
	100	0.118
	200	0.220
	300	0.308
3/4	50	0.032
	100	0.060
	200	0.111
	300	0.156

Table 2. Settlement of waste storage surface with discrete increase in excess pressures considering biodegradation.

β	t, days	subsidence, m
0	50	0.193
	100	0.395
	200	0.775
	300	1.127
1/2	50	0.163
	100	0.338
	200	0.668
	300	0.979
3/4	50	0.104
	100	0.222
	200	0.452
	300	0.675

In the two given examples, the volume of greenhouse gas generated at time 300 days will be $G = 172.118$ m^3 with an area of 1 m^2. According to the data from experimental studies, these generation volumes depend on the moisture content and thermal regime of the waste storage as a porous medium. These conditions essentially determine the directions for further research by the authors regarding the problem of predicting greenhouse gas emissions into the atmosphere under conditions of waste biodegradation and consolidation.

Table 3. Settlement of waste storage surface with discrete increase in excess pressures in case of disregarding biodegradation.

t, days	subsidence, m
50	0.072
100	0.162
200	0.342
300	0.522

8. Conclusion

The article presents a mathematical model for predicting surface subsidence of solid waste storage facilities, considering the biodegradation of organic components. The proposed approach is based on the modification of filtration consolidation equations for porous media, accounting for the transformation of organic residues. The model enables the prediction of landfill surface subsidence without the need to model microbial dynamics or the stress-strain state of the waste mass, significantly simplifying calculations. Numerical experiments demonstrate that the filling of pore spaces with solid phases significantly affects the dynamics of subsidence. The results show that biodegradation processes can double the magnitude of subsidence under conditions of complete pore space filling, compared to cases where these processes are not considered. Additionally, the study calculates the volume of greenhouse gases generated. The findings provide a foundation for improving methods of subsidence modeling and assessing the environmental impact of landfills, particularly in forecasting greenhouse gas emissions into the atmosphere.

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Математичне моделювання просідання поверхні сховища твердих побутових відходів з урахуванням біодеградації органічної складової

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Захоронення твердих побутових відходів на звалищах породжує ряд проблем. В статті звернуто увагу на одну з них — це просідання поверхні звалища як результат біодеградації органічних решток. Задачу прогнозування осідання поверхні сховища відходів запропоновано розв'язувати в межах теорії фільтраційної консолідації пористих середовищ. Супровідно вирішено задачу про динаміку емісії метану і вуглекислого газу в атмосферу на основі відомих експериментальних даних. Застосування пропонованих підходів дозволило уникнути моделювання мікробної динаміки та визначення напружено-деформівного стану масиву сховища відходів, що значно ускладнило б математичну модель задачі. Числові розв'язки відповідної крайової задачі в змінній з часом області знайдено методом скінченних елементів. Проведені числові експерименти показують зміни в динаміці просідань поверхні сховища відходів на випадок урахування ефектів біодеградації.

Ключові слова: *моделювання біодеградації; викиди парникових газів; управління відходами; математичне моделювання; механіка пористих середовищ.*