MATHEMATICAL MODELING AND COMPUTING, Vol. 11, No. 4, pp. 1128–1134 (2024)

Mathematical Modeling Computing

Incorporating long memory into the modeling of gold prices

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(Received 1 July 2024; Revised 25 November 2024; Accepted 27 November 2024)

Inflation causes many people to move to gold as an option for savings because gold may be used as a hedging tool against currency devaluation and purchasing power erosion. This has contributed to the increased interest in forecasting the prices at the gold market, just like predicting the prices at the stock market, which exhibits uncertain movement, which can be described mathematically with Geometric Brownian Motion (GBM) and Geometric Fractional Brownian Motion (GFBM). This study aims to model Malaysian gold prices using both GBM and GFBM processes and compare the accuracy of these models. Absolute moment and aggregated variance techniques are used to estimate the Hurst exponents to model the prices with GFBM. These models are simulated using the Monte Carlo simulation via the Euler scheme, where the modeled prices will be tested for their accuracy using Mean Average Percentage Error (MAPE). Based on the findings, the MAPE values for both models exhibited significantly low MAPE values, which implies high accuracy in forecasting the gold prices for a long-term period. Nevertheless, the GFBM produces much lower MAPE values than the GBM, thus indicating that the former is more accurate than the latter.

Keywords: *gold price; Hurst exponent; Monte Carlo simulation; long-memory phenomena.*

2010 MSC: 60J65, 60J75, 91G20 **DOI:** 10.23939/mmc2024.04.1128

1. Introduction

Geometric Brownian motion (GBM) is a stochastic process that implies normal distribution and independent stock returns. This approach can be used to replicate short-term stock price movements as well as to model financial processes. According to the Efficient Market Hypothesis (EMH) [1], stock prices exhibit uncertain movement, and its historical stock price is reflected by the current stock price. Due to the unpredictability of stock prices, many studies applied GBM which includes randomness, volatility, and drift, to aid investors in making smart decisions on short-term investments, particularly for forecasting and predicting future prices. For instance, Refs. [2] and [3] used GBM to predict stock prices and indices in Bursa Malaysia. Besides that, Ref. [4] studied GBM to simulate the stock prices and test whether simulated prices matched the real market returns. They adopted a sample of daily stock prices from significant, publicly traded Australian firms that are included in the S&P/ASX 50 index over a 12-month period. Apart from that, Ref. [5] asserted that prices may be represented as a GBM process. The sample data was collected from the Malaysian Rubber Board's official website between December 1 and December 31, 2015, for five distinct grades of Standard Malasian Rubber (SMR) and centrifuged latex rubber. According to the study, simulation prices for the entire month for the two distinct types of rubber were nearly identical to the actual prices. The results demonstrated that the simulated prices are nearly 100 percent accurate, and the MAPE is less than 10 percent, indicating a high degree of accuracy. Then, Ref. [6] conducted a study to forecast stock values using GBM over a 12-month period. They employed data from the Jakarta Composite index for their analysis, which covered daily market close prices from January 2014 to December 2014. As a result of using MAPE to assess the accuracy of simulated pricing, the GBM model was found to have a high accuracy rate, with the value obtained less than 20 percent being the outcome. Prior studies have shown that GBM can be used to anticipate or simulate accurate stock prices.

However, the limitations of GBM in capturing long-range dependence and volatility clustering have motivated the development of more sophisticated models by relaxing the assumptions made by GBM. One known modification is the Geometric Fractional Brownian Motion (GFBM), an extension of GBM, which incorporates a fractional parameter called the Hurst exponent [7] that allows the modeling of long-term dependencies and captures the persistence of volatility observed in financial time series. In recent years, there has been a growing interest in using GFBM to simulate prices in finance and economics, such as the work of [8] demonstrated the modelling of return distribution for the S&P 500 and STOXX Europe 600 indexes with GBM and GFBM, and found that the latter is more suitable than the former.

In [9], GBM and GFBM were used to simulate Malaysia's crude palm oil prices, and to determine whether they exhibit persistent or anti-persistent behaviour across various time periods. The findings demonstrated that both models generated precise price forecasts; where GFBM is more precise than GBM. In [10], several Hurst estimators were used to determine the Hurst exponents in order to be utilized in GFBM to simulate rubber prices. The study found that the smaller the value of the Hurst exponent, more precise simulated rubber prices were generated.

Most studies applied the Monte Carlo Simulation (MCS) to simulate prices, which is a mathematical approach for predicting potential outcomes of an uncertain event [3–6, 8–10]. It is a computational approach that combines random sampling and statistical analysis to model complicated systems in order to analyze past data and forecast a range of future events depending on a choice of action. Reference [11] recommended MCS to obtain numerical results for option valuation problems by simulating the underlying asset prices. This approach is straightforward and is adaptable to various processes governing stock returns. In addition, this technique is distinctive in that the distribution used to generate the underlying stock need not have a closed-form analytic expression, allowing option values to be derived using empirical stock return distributions.

Kijang Emas are gold coins from Bank Negara Malaysia, which serves as an indicator of overall economic health, making it crucial for investors navigating uncertainties. In [12], it stated that the modelling of *Kijang Emas* future price is beneficial for investment purposes in Malaysia; hence to predict its prices accurately, is beneficial to investors. Various forecasting model have been developed to predict the movement of gold price, such as regression method [13], ARIMA [14], and random walk [15]. Hence, the aim of this study is to model the *Kijang Emas* prices using mathematical models, specifically GBM and GFBM, to determine which model accurately describes the movement of the *Kijang Emas* prices. The organization of the remaining of this paper is as follows. Section 2 describes the GBM and GFBM dynamics that are assumed to be followed by the *Kijang Emas* price. Section 3 outlines the data and methodology used to simulate the *Kijang Emas* prices; while Section 4 documents the results and discussion. Section 5 concludes the study.

2. The model

In this section, a brief description of the processes, which are the geometric Brownian motion (GBM) and geometric fractional Brownian motion (GFBM) is presented.

Generally, the Wiener process is defined as:

$$dx = a\,dt + b\,dz, \tag{1}$$

where a and b are constants. In Eq. (1), the adt term implies that x has an expected drift rate of per unit of time. Figure 1 depicts the Wiener process.

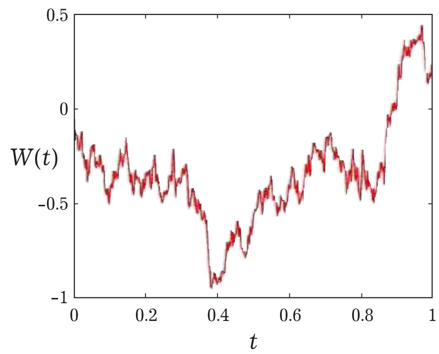


Fig. 1. The Wiener process.

Let S be the *Kijang Emas* price. It is assumed to be governed by GBM and GFBM dynamics, respectively, as follows:

$$dS = \mu\, S\, dt + \sigma\, S\, dW, \tag{2}$$

$$dS = \mu\, S\, dt + \sigma\, S\, dW_H, \tag{3}$$

where dS is the changes in the asset prices, W is the Brownian Motion, μ is the constant drift, σ is the constant volatility, and W_H is an Fractional Brownian Motion with $H \in (0,1)$ is the Hurst exponent.

On that account, by letting $f = \ln S$, and applying Ito's lemma, yields the solution for (2):

$$S(t) = S(0)\, e^{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W(t)}, \tag{4}$$

which is the stochastic differential equation for $\ln S$. Meanwhile, by using the Wick Ito Skorohod integrals, the solution to Eq. (3) for any arbitrary initial value $S(0)$ is obtained as [16]:

$$S_H(t) = S(0)\, e^{\left(\mu t - \frac{1}{2}\sigma^2 t^{2H}\right) + \sigma W_H(t)}. \tag{5}$$

The Hurst exponent, H, measures the long-term memory of a time series, or the degree by which that series deviates from a random walk. The exponent is a real number, which has been explored using numerous estimator approaches such as rescaled range, aggregate variance, and regression residuals [10]. According to [16], depending on the value of H, FBM exhibits different properties, as outlined in Table 1.

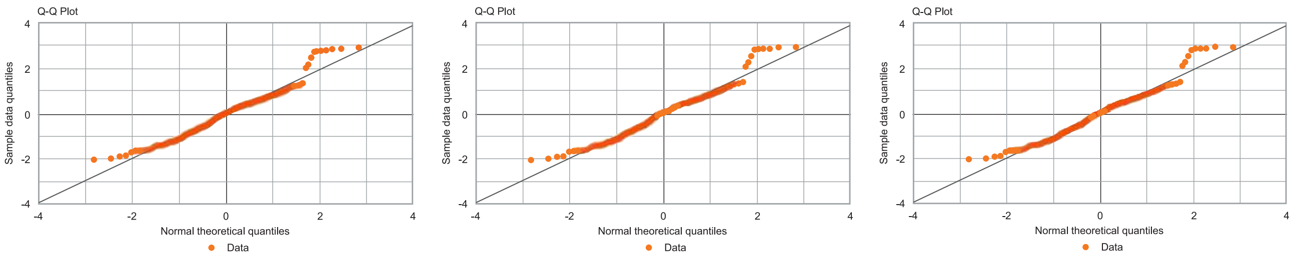
Table 1. Interpretation of the Hurst exponent.

Hurst Exponent	Interpretation
$H < 0.5$	The disjoint increments are negatively correlated which exhibit long memory dependency or anti-persistent.
$H = 0.5$	There is no correlation and the process is a Wiener Process.
$H > 0.5$	The disjoint increments are positively correlated which exhibit short memory dependency or persistent.

3. Data and methodology

In this section, we describe the data used in this study, and the definitions used for parameters estimations.

This study uses daily historical *Kijang Emas* prices from May 2022 to April 2023 for three different types of *Kijang Emas* – 0.25 oz, 0.5 oz, and 1.0 oz, which are retrieved from the official website of Bank Negara Malaysia [17]. Figure 2 plots the Q-Q plot of the *Kijang Emas* prices.



a (0.25 oz) ***b*** (0.50 oz) ***c*** (1.0 oz)

Fig. 2. The Q-Q plot.

In order to simulate the *Kijang Emas* prices with geometric Brownian motion (GBM), we computed the values of mean μ and standard deviation σ obtained as such:

$$\mu = \frac{\sum_{t=1}^{M} \ln \frac{S_t}{S_{t-1}}}{M}, \tag{6}$$

$$\sigma^2 = \frac{\sum_{t=1}^{M} \left[\ln \frac{S_t}{S_{t-1}} - \mu\right]^2}{M-1}. \tag{7}$$

In addition, the Hurst exponent is calibrated using absolute moment and aggregated variance methods. The former is computed as follows [16]:

$$H_{abs} = \frac{\sum_{k=1}^{N/m} \left\| R^{(m)}(k) - \bar{R}_N \right\|}{N/m}, \tag{8}$$

while the latter [18]:

$$H_{agg} = \frac{\sum_{k=1}^{N/m} \left(R^{(m)}(k)\right)^2}{N/m} - \left(\frac{\sum_{k=1}^{N/m} R^{(m)}(k)}{N/m}\right)^2, \tag{9}$$

which divides the logarithmic returns R into N/m blocks with size m, and

$$R^{(m)}(k) = \frac{\sum_{t=(k-1)m}^{km} X(t)}{m}, \tag{10}$$

$$\bar{R}_N = \frac{\sum_{t=1}^{N} X(t)}{N}, \tag{11}$$

for $t = 1, 2, \ldots, N/m$. Following that, we simulate the *Kijang Emas* prices with geometric fractional Brownian motion (GFBM) by using the computed standard deviation $\bar{\sigma}$ and mean $\bar{\mu}$ obtained as such [19]:

$$\bar{\sigma} = \frac{\sigma}{\sqrt{\|\Delta t\|^{2H}}}, \tag{12}$$

$$\bar{\mu} = \frac{\mu}{\Delta t} + \frac{\bar{\sigma}^2}{2}, \tag{13}$$

where $\Delta t = t_N - t_{N-1}$. The fractional Gaussian noise that governs GFBM is chosen to follow the Cholesky decomposition [20].

4. Numerical results

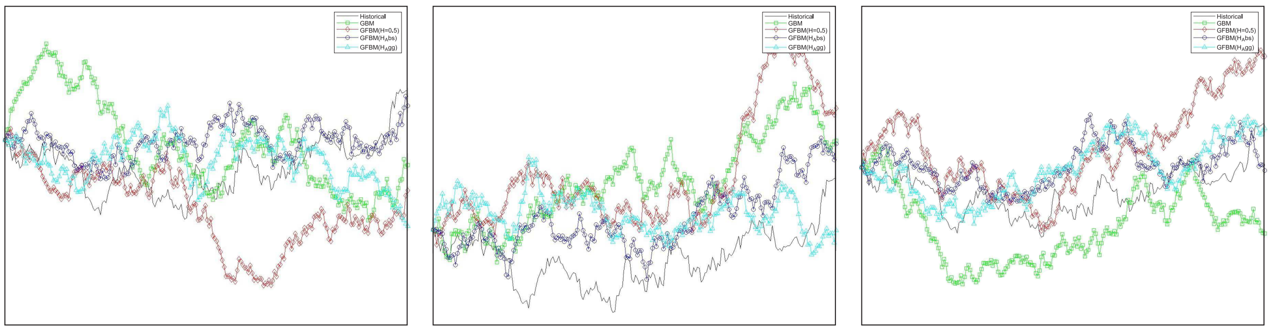
This section documents the illustration of the modeling of *Kijang Emas* using the GBM and GFBM models, as presented previously in Sections 2 and 3. Given the historical *Kijang Emas* prices, we calibrated the mean and standard deviation of the given data set. This is recorded in Table 2.

The mean and standard deviations are obtained by calibrating the historical prices, and are used in the simulation using geometric Brownian motion (GBM) and geometric fractional Brownian motion (GFBM), respectively.

Table 2. Mean and standard deviations.

Type	GBM		GFBM	
	μ	σ	$\bar{\mu}$	$\bar{\sigma}$
0.25 oz	2.4801e-04	0.0083	2.8284e-04	0.0083
0.5 oz	2.4652e-04	0.0083	2.8120e-04	0.0083
1.0 oz	2.4648e-04	0.0083	2.8113e-04	0.0083

We generate the simulated prices for *Kijang Emas* using GBM and GFBM. The estimated values of the Hurst exponents which are obtained using absolute moment and aggregated variance methods are as given in Table 3. Additionally, we simulate the prices usng GFBM at $H = 0.5$ as a control model, since GFBM reduces to GBM when $H = 0.5$.



a (0.25 oz) ***b*** (0.50 oz) ***c*** (1.0 oz)

Fig. 3. *Kijang Emas* price paths simulation for a 12-month period.

Figures 3a, 3b and 3c plot the historical and simulated *Kijang Emas* prices for a 12-month period.

Table 3. Hurst exponents, H.

Type	Absolute Moment	Aggregated Variance
0.25 oz	0.5044	0.4661
0.5 oz	0.4796	0.4560
1.0 oz	0.4796	0.4558

Table 4. Evaluating forecast accuracy using MAPE.

MAPE	Evaluating of forecast accuracy
$\leqslant 10\%$	Highly accurate
11% until 20%	Good accurate
21% until 50%	Reasonable forecast
$\geqslant 51\%$	Inaccurate forecast

It can be seen that the simulated prices follow a similar trend as the historical prices, and display fluctuations. To further evaluate the accuracy of the simulated prices, we computed the MAPE values, and used the judgement scale as given in Table 4 (see [21]).

We computed the MAPE values for both GBM and GFBM as tabulated in Table 5. It shows that both models produced highly accurate simulated prices for *Kijang Emas* since all values fall below the 10% threshold. However, on average, the MAPE value for GBM is 6.4399, while for GFBM, 3.8363 (absolute moment) and 4.3842 (aggregated Variance). The control GFBM has a MAPE value of 7.0252, on average. This implies that GFBM produced a slightly more accurate simulated *Kijang Emas* prices, compared to GBM.

Table 5. MAPE values (%) for a 12-month period.

Type	GBM	GFBM		
		$H = 0.5$	Absolute moment	Aggregated variance
0.25 oz	5.8249	6.0123	3.2702	3.9054
0.5 oz	8.3844	9.3159	4.5340	5.5454
1.0 oz	5.1105	5.7473	3.7047	3.7018

As a result of this computation, both GBM and GFBM are highly accurate models since both models obtained MAPE values less than 10%, but GFBM produced lower MAPE values. Hence, GFBM showed a better accuracy and effectiveness in long term period, which implies the GFBM is more accurate than the GBM.

5. Conclusion

Overall, this study contributes valuable insights into the application of mathematical models for forecasting, specifically, *Kijang Emas* that we considered, with implications for investors to identify the best strategy and minimize risk by making it easier to analyze possible investment outcomes. This helps investors make well-informed decisions, improving their capacity to minimize risk.

In conclusion, this study highlights the effectiveness of mathematical models, particularly GBM and GFBM, in forecasting *Kijang Emas* prices using historical prices. The GFBM model demonstrates higher accuracy over extended time periods compared to the GBM model, as proven by its lower MAPE values. This suggests that incorporating the Hurst exponent in the GFBM model allows for the modeling of long-term dependencies and the persistence of volatility, making it a more suitable choice for accurate forecasting over extended time periods. These findings mark the importance of utilizing mathematical approaches for informed decision-making in volatile financial markets, in which GBM and GFBM provide insightful information by capturing the underlying uncertainties and complexities of financial markets. Hence, the GFBM model outperformed the GBM model in terms of accuracy in forecasting gold prices over the specified time intervals.

Future work may consider incorporating mean-reversion into modeling the *Kijang Emas* prices and using other variance reduction techniques for the Monte Carlo simulation.

Acknowledgement

The present research was originally conducted as part of the author's dissertation work.

[1] Fama E. F. Efficient capital markets: A review of theory and empirical work. The Journal of Finance. **25** (2), 383–417 (1970).

[2] Omar A., Jaffar M. M. Comparative analysis of Geometric Brownian motion model in forecasting FBMHS and FBMKLCI index in Bursa Malaysia. 2011 IEEE Symposium on Business, Engineering and Industrial Applications (ISBEIA). 157–161 (2011).

[3] Abidin S. N. Z., Jaffar M. M. A review on Geometric Brownian Motion in forecasting the share prices in Bursa Malaysia. World Applied Sciences Journal. **17** (Special Issue for Applied Math) , 87–93 (2012).

[4] Reddy K., Clinton V. Simulating stock prices using geometric Brownian motion: Evidence from Australian companies. Australasian Accounting, Business and Finance Journal. **10** (3), 23–47 (2016).

[5] Ibrahim S. N. I. Modeling Rubber Prices as a GBM Process. Indian Journal of Science and Technology. **9** (28), 1–6 (2016).

[6] Agustini W. F., Affianti I. R., Putri E. R. M. Stock price prediction using geometric Brownian motion. Journal of Physics: Conference Series. **974**, 012047 (2018).

[7] Hurst H. E. Long Term Storage Capacity of Reservoirs. Transactions of the American Society of Civil Engineers. **116** (1), (1951).

[8] Brătian V., Acu A.-M. Oprean-Stan C., Dinga E., Ionescu G.-M. Efficient or Fractal Market Hypothesis? A Stock Indexes Modelling Using Geometric Brownian Motion and Geometric Fractional Brownian Motion. Mathematics. **9** (22), 2983 (2021).

[9] Ibrahim S. N. I., Misiran M., Laham M. F. Geometric fractional Brownian motion model for commodity market simulation. Alexandria Engineering Journal. **60** (1), 955–962 (2021).

[10] Balasubramaniam S. S., Ibrahim S. N. I. Simulating Rubber Prices under Geometric Fractional Brownian Motion with Different Hurst Estimators. Malaysian Journal of Fundamental and Applied Sciences. **19** (1), 93–102 (2023).

[11] Boyle P. P. Options: A Monte Carlo approach. Journal of Financial Economics. **4** (3), 323–338 (1977).

[12] Miswan N. H., Ping P. Y., Ahmad M. H. On parameter estimation for Malaysian gold prices modelling and forecasting. International Journal of Mathematical Analysis. **7** (21–24), 1059–1068 (2013).

[13] Ismail Z., Yahya A., Shabri A. Forecasting gold prices using multiple linear regression method. American Journal of Applied Sciences. **6** (8), 1509–1514 (2009).

[14] Areh S. O., Miswan N. H. ARIMA Models For Kijang Emas Price Forecasting: Pre- and Post-COVID Analysis. Malaysian Journal of Mathematical Sciences. **18** (1), 127–140 (2024).

[15] Atsalakis G. S., Valavanis K. P. Surveying stock market forecasting techniques – Part II: Soft computing methods. Expert Systems with Applications. **36** (3), 5932–5941 (2009).

[16] Biagini F., Hu Y., Øksendal B., Zhang T. Stochastic calculus for fractional Brownian motion and applications. Springer Science & Business Media (2008).

[17] Bank Negara Malaysia. 2022 Kijang Emas Prices [Data File]. `https://www.bnm.gov.my/kijang-emas-prices` (2023).

[18] Taqqu M. S., Teverovsky V., Willinger W. Estimators for long-range dependence: an empirical study. Fractals. **3** (4), 785–798 (1995).

[19] Feng Z. Stock-price modeling by the geometric fractional Brownian motion: A view towards the Chinese financial market. Identifier diva2:1257290. Linnaeus University (2018).

[20] Kijima M., Tam C. M. Fractional Brownian motions in financial models and their Monte Carlo simulation. In W. K. Chan (Ed.), Theory and Applications of Monte Carlo Simulations. InTechOpen (2013).

[21] Lawrence K. D., Klimberg R. K., Lawrence S. M. Fundamentals of forecasting using Excel. Industrial Press Inc. (2009).

Включення довгострокової пам'яті в моделювання цін на золото

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Інфляція спонукає багатьох людей перейти до золота як варіанту заощаджень, оскільки золото може використовуватися як інструмент хеджування від девальвації валюти та зниження купівельної спроможності. Це сприяло підвищенню інтересу до прогнозування цін на ринку золота так само, як і до прогнозування ціни на фондовому ринку, що демонструє невизначений рух, який можна описати математично за допомогою геометричного броунівського руху (GBM) і геометричного дробового броунівського руху (GFBM). Це дослідження спрямоване на моделювання цін на золото в Малайзії за допомогою процесів GBM і GFBM та порівняння точностей цих моделей. Використовуються методи абсолютного моменту та агрегованої дисперсії для оцінки показників Херста з метою моделювання цін за допомогою GFBM. Ці моделі симулюються за допомогою метода Монте–Карло за схемою Ейлера, де змодельовані ціни перевірятимуть на точність за допомогою середньої відсоткової похибки (MAPE). Виходячи з отриманих даних, значення MAPE для обох моделей продемонстрували значно нижчі значення MAPE, що свідчить про високу точність прогнозування цін на золото на довгостроковий період. Тим не менш, GFBM дає набагато нижчі значення MAPE, ніж GBM; таким чином, це вказує на те, що перший метод є більш точним, ніж другий.

Ключові слова: *ціни на золото; показник Херста; моделювання методом Монте–Карло; явища тривалої пам'яті.*