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Numerical stability of the branched continued fraction expansions of the ratios of Horn's confluent hypergeometric functions H_6

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The paper establishes the conditions of numerical stability of a numerical branched continued fraction using a new method of estimating the relative errors of the computing of approximants using a backward recurrence algorithm. Based this, the domain of numerical stability of branched continued fractions, which are expansions of Horn's confluent hypergeometric functions H_6 with real parameters, is constructed. In addition, the behavior of the relative errors of computing the approximants of branched continued fraction using the backward recurrence algorithm and the algorithm of continuants was experimentally investigated. The obtained results illustrate the numerical stability of the backward recurrence algorithm.

Keywords: *branched continued fraction; Horn hypergeometric function; numerical approximation; roundoff error.*

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1. Introduction

Continued fractions and their multidimensional generalizations – branched continued fractions are a powerful mathematical tool for representing and approximating special functions [1–3]. In the analytical theory of branched continued fractions, the branched continued fraction expansions of the hypergeometric functions of Appell [4–8], Lauricella [9], Saran [10, 11], Horn [12–19], Gauss [20], and generalized hypergeometric functions [21, 22] were constructed and studied. Compared to power series, in many cases these expansions converge faster and have wider regions of convergence. Also, the advantage of using continued fractions and branched continued fractions is the property of numerical stability, which consists in the stability of algorithms for computing their approximants to rounding errors and ensures non-accumulation or insignificant accumulation of errors arising in the process of computations [1, 2, 23–25].

Works [26–30] are devoted to the study of the stability of algorithms for computing approximants of continued fractions. The problem of numerical stability of branched continued fractions, the approximants of which are computed by the backward recurrence algorithm, was studied in [1, 31, 32]. Also, in the analytical theory of branched continued fractions, the stability of numerical branched continued fractions to perturbations of their elements is investigated. In works [33–36], estimates of the errors of approximants of branched continued fractions, which arise when their elements are perturbed, were obtained, and sets of resistance to perturbations were constructed. The numerical stability of continued fractions and branched continued fractions, which are expansions of some special functions, was investigated in [8, 23, 37].

This paper investigates the numerical stability of branched continued fractions, which are expansions of ratios of Horn's confluent hypergeometric functions H_6.

Consider the Horn's confluent hypergeometric function H_6 [38], which is defined as a double power series of the form

$$\mathrm{H}_6(a,c;\mathbf{z})=\sum_{m,n=0}^{\infty}\frac{(a)_{2m+n}}{(c)_{m+n}}\frac{z_1^m z_2^n}{m!n!},\quad |z_1|<\frac{1}{4},\quad |z_2|<+\infty,$$

where a, c are complex numbers, $c \notin \{0, -1, -2, \ldots\}$, $(\cdot)_k$ is the Pochhammer symbol, $\mathbf{z} = (z_1, z_2) \in \mathbb{C}^2$.

Let $I_0 = \{1, 2, 3\}$ and for $k \geqslant 1$

$$I_k = \{i(k) = (i_0, i_1, \ldots, i_k)\colon i_0 \in I_0,\ 2 - [(i_{r-1} - 1)/2] \leqslant i_r \leqslant 3 - [(i_{r-1} - 1)/2],\ 1 \leqslant r \leqslant k\}.$$

In the work [15], using a generalization of the classical method of constructing a Gaussian fraction, for each $i_0 \in I_0$, the expansions of the following ratios

$$\frac{\mathrm{H}_6(a, c; \mathbf{z})}{\mathrm{H}_6(a + \delta_{i_0}^1 + \delta_{i_0}^2, c + \delta_{i_0}^2 + \delta_{i_0}^3; \mathbf{z})}$$

into branched continued fractions

$$Q_{i_0} + \sum_{i_1 = 2 - [(i_0 - 1)/2]}^{3 - [(i_0 - 1)/2]} \cfrac{P_{i(1)}(\mathbf{z})}{Q_{i(1)} + \sum\limits_{i_2 = 2 - [(i_1 - 1)/2]}^{3 - [(i_1 - 1)/2]} \cfrac{P_{i(2)}(\mathbf{z})}{Q_{i(2)} + \ddots}}, \tag{1}$$

where $\delta_{i_0}^j$ is the Pochhammer symbol, $[\cdot]$ denotes an integer part, were obtained. The elements of the branched continued fraction (1) are determined by the formulas, for $i(1) \in I_1$,

$$P_{i(1)}(\mathbf{z}) = \begin{cases} -2\dfrac{a+1}{c} z_1,\ i_0 = 1,\ i_1 = 2, \\ -\dfrac{z_2}{c},\ i_0 = 1,\ i_1 = 3, \\ -\dfrac{(2c - a)(a + 1)}{c(c+1)} z_1,\ i_0 = 2,\ i_1 = 2, \\ -\dfrac{c - a}{c(c+1)} z_2,\ i_0 = 2,\ i_1 = 3, \\ \dfrac{a}{2c},\ i_0 = 3,\ i_1 = 1, \\ \dfrac{a}{2c(c+1)} z_2,\ i_0 = 3,\ i_1 = 2, \end{cases} \tag{2}$$

for $i(k+1) \in I_{k+1}$, $k \geqslant 1$,

$$P_{i(k+1)}(\mathbf{z}) = \begin{cases} -\dfrac{2\left(a + k - \sum_{r=0}^{k-1} \delta_{i_r}^3 + 1\right)}{c + k - \sum_{r=0}^{k-1} \delta_{i_r}^1} z_1,\ i_k = 1,\ i_{k+1} = 2, \\ -\dfrac{z_2}{c + k - \sum_{r=0}^{k-1} \delta_{i_r}^1},\ i_k = 1,\ i_{k+1} = 3, \\ -\dfrac{\left(2c - a + k + \sum_{r=0}^{k-1} (\delta_{i_r}^3 - 2\delta_{i_r}^1)\right)\left(a + k - \sum_{r=0}^{k-1} \delta_{i_r}^3 + 1\right)}{\left(c + k - \sum_{r=0}^{k-1} \delta_{i_r}^1\right)\left(c + k - \sum_{r=0}^{k-1} \delta_{i_r}^1 + 1\right)} z_1,\ i_k = 2,\ i_{k+1} = 2, \\ -\dfrac{c - a + \sum_{r=0}^{k-1} (\delta_{i_r}^3 - \delta_{i_r}^1)}{\left(c + k - \sum_{r=0}^{k-1} \delta_{i_r}^1\right)\left(c + k - \sum_{r=0}^{k-1} \delta_{i_r}^1 + 1\right)} z_2,\ i_k = 2,\ i_{k+1} = 3, \\ \dfrac{a + k - \sum_{r=0}^{k-1} \delta_{i_r}^3}{2\left(c + k - \sum_{r=0}^{k-1} \delta_{i_r}^1\right)},\ i_k = 3,\ i_{k+1} = 1, \\ \dfrac{a + k - \sum_{r=0}^{k-1} \delta_{i_r}^3}{2\left(c + k - \sum_{r=0}^{k-1} \delta_{i_r}^1\right)\left(c + k + 1 - \sum_{r=0}^{k-1} \delta_{i_r}^1\right)} z_2,\ i_k = 3,\ i_{k+1} = 2, \end{cases} \tag{3}$$

for $i_0 \in I_0$,

$$Q_{i_0} = 1 - \frac{a}{2c} \delta_{i_0}^3, \tag{4}$$

and, for $i(k) \in I_k$, $k \geqslant 1$,

$$Q_{i(k)} = 1 - \frac{a + k - \sum_{r=0}^{k-1} \delta_{i_r}^3}{2\left(c + k - \sum_{r=0}^{k-1} \delta_{i_r}^1\right)} \delta_{i_k}^3. \tag{5}$$

2. Formulas of relative errors of computing of approximants of branched continued fraction

Let i_0 be an arbitrary index in I_0, $P_{i(k)} = P_{i(k)}(\mathbf{z}_0)$, $i(k) \in I_k$, $k \geqslant 1$, be the values of the partial numerators of the branched continued fraction (1) at an arbitrary fixed point $\mathbf{z}_0$ from its domain of convergence (see [15]). Then

$$f^{(i_0)} = Q_{i_0} + \sum_{i_1=2-[(i_0-1)/2]}^{3-[(i_0-1)/2]} \cfrac{P_{i(1)}}{Q_{i(1)} + \sum\limits_{i_2=2-[(i_1-1)/2]}^{3-[(i_1-1)/2]} \cfrac{P_{i(2)}}{Q_{i(2)} + \ddots}} \tag{6}$$

is the value of the branched continued fraction (1) at the point $\mathbf{z}_0$.

Let n be an arbitrary natural number. To compute the nth approximant

$$f_n^{(i_0)} = Q_{i_0} + \sum_{i_1=2-[(i_0-1)/2]}^{3-[(i_0-1)/2]} \cfrac{P_{i(1)}}{Q_{i(1)} + \sum\limits_{i_2=2-[(i_1-1)/2]}^{3-[(i_1-1)/2]} \cfrac{P_{i(2)}}{Q_{i(2)} + \ddots + \sum\limits_{i_n=2-[(i_{n-1}-1)/2]}^{3-[(i_{n-1}-1)/2]} \cfrac{P_{i(n)}}{Q_{i(n)}}}} \tag{7}$$

of the branched continued fraction (6) we will use the backward recurrence algorithm, which consists in computing the quantities

$$G_{i(k)}^{(n)} = Q_{i(k)} + \sum_{i_{k+1}=2-[(i_k-1)/2]}^{3-[(i_k-1)/2]} \frac{P_{i(k+1)}}{G_{i(k+1)}^{(n)}}, \quad i(k) \in I_k, \quad k = n-1, n-2, \ldots, 0,$$

with initial condition $G_{i(n)}^{(n)} = Q_{i(n)}$, $i(n) \in I_n$. Then $f_n^{(i_0)} = G_{i_0}^{(n)}$.

Let the computation of the nth approximant (7) lead to roundoff errors

$$\varepsilon_{i(k)}^{(P)} = \frac{\hat{P}_{i(k)} - P_{i(k)}}{P_{i(k)}}, \quad P_{i(k)} \neq 0, \quad i(k) \in I_k, \quad 1 \leqslant k \leqslant n,$$

$$\varepsilon_{i(k)}^{(Q)} = \frac{\hat{Q}_{i(k)} - Q_{i(k)}}{Q_{i(k)}}, \quad Q_{i(k)} \neq 0, \quad i(k) \in I_k, \quad 0 \leqslant k \leqslant n,$$

of elements $P_{i(k)}$, $Q_{i(k)}$ respectively, where $\hat{P}_{i(k)} = \mathrm{RN}(P_{i(k)})$, $\hat{Q}_{i(k)} = \mathrm{RN}(Q_{i(k)})$, where $\mathrm{RN}(\cdot)$ is the roundoff function. If $\hat{x} = x = 0$, then we assume that $\varepsilon^{(x)} = 0$.

The value $\hat{f}_n^{(i_0)} = \hat{G}_{i_0}^{(n)}$, where

$$\hat{G}_{i(k)}^{(n)} = \hat{Q}_{i(k)} + \sum_{i_{k+1}=2-[(i_{k-1}-1)/2]}^{3-[(i_{k-1}-1)/2]} \frac{\hat{P}_{i(k+1)}}{\hat{G}_{i(k+1)}^{(n)}}, \quad i(k) \in I_k, \quad k = n-1, n-2, \ldots, 0,$$

with initial condition $\hat{G}_{i(n)}^{(n)} = \hat{P}_{i(n)}$, is the approximate value of the nth approximant $f_n^{(i_0)}$.

Definition 1. *A branched continued fraction* (6) *is called numerical stable if for an arbitrary* $\varepsilon > 0$ *there exists* $\delta_\varepsilon > 0$ *such that for each* $\hat{P}_{i(k)} \in \mathbb{C}$, $i(k) \in I_k$, $1 \leqslant k \leqslant n$, *such that*

$$\left| \frac{\hat{P}_{i(k)} - P_{i(k)}}{P_{i(k)}} \right| < \delta_\varepsilon, \quad i(k) \in I_k, \quad 1 \leqslant k \leqslant n,$$

and each $\hat{Q}_{i(k)} \in \mathbb{C}$, $i(k) \in I_k$, $0 \leqslant k \leqslant n$, *such that*

$$\left| \frac{\hat{Q}_{i(k)} - Q_{i(k)}}{Q_{i(k)}} \right| < \delta_\varepsilon, \quad i(k) \in I_k, \quad 0 \leqslant k \leqslant n,$$

the inequality holds

$$\left|\frac{\hat{f}_n^{(i_0)}-f_n^{(i_0)}}{f_n^{(i_0)}}\right|<\varepsilon,\quad n\geqslant 1.$$

Let

$$\varepsilon_{i(k),n}^{(G)}=\frac{\hat{G}_{i(k)}^{(n)}-G_{i(k)}^{(n)}}{G_{i(k)}^{(n)}},\quad i(k)\in I_k,\quad 0\leqslant k\leqslant n,$$

be the relative errors of the quantities $G_{i(k)}^{(n)}$, $i(k)\in I_k$, $0\leqslant k\leqslant n$. We prove that for the relative errors $\varepsilon_{i(k),n}^{(G)}$, $i(k)\in I_k$, $0\leqslant k\leqslant n$, the formulas hold

$$\varepsilon_{i(n),n}^{(G)}=\varepsilon_{i(n)}^{(Q)},\quad i(n)\in I_n,$$

and, for $i(k)\in I_k$, $0\leqslant k\leqslant n-1$,

$$\varepsilon_{i(k),n}^{(G)}=\left(1-\sum_{i_{k+1}=2-[(i_k-1)/2]}^{3-[(i_k-1)/2]}g_{i(k+1)}^{(n)}\right)\left(1+\varepsilon_{i(k)}^{(Q)}\right)+\sum_{i_{k+1}=2-[(i_k-1)/2]}^{3-[(i_k-1)/2]}\frac{g_{i(k+1)}^{(n)}\left(1+\varepsilon_{i(k+1)}^{(P)}\right)}{1+\varepsilon_{i(k+1),n}^{(G)}}-1,\tag{8}$$

where $\varepsilon_{i(k)}^{(Q)}$, $i(k)\in I_k$, $0\leqslant k\leqslant n$, and $\varepsilon_{i(k)}^{(P)}$, $i(k)\in I_k$, $1\leqslant k\leqslant n$, are the relative errors of the elements of nth approximant (7), and the quantities $g_{i(k)}^{(n)}$, $i(k)\in I_k$, $1\leqslant k\leqslant n$, are defined by

$$g_{i(k)}^{(n)}=\frac{P_{i(k)}}{G_{i(k-1)}^{(n)}G_{i(k)}^{(n)}},\quad i(k)\in I_k,\quad 1\leqslant k\leqslant n.$$

Since for any $i(n)\in I_n$

$$G_{i(n)}^{(n)}=Q_{i(n)},\quad \hat{G}_{i(n)}^{(n)}=\hat{Q}_{i(n)},$$

then $\varepsilon_{i(n),n}^{(G)}=\varepsilon_{i(n)}^{(Q)}$, $i(n)\in I_n$. For any $0\leqslant k\leqslant n-1$ and $i(k)\in I_k$ we have

$$\begin{aligned}\varepsilon_{i(k),n}^{(G)}&=\frac{1}{G_{i(k)}^{(n)}}\left(\hat{Q}_{i(k)}+\sum_{i_{k+1}=2-[(i_k-1)/2]}^{3-[(i_k-1)/2]}\frac{\hat{P}_{i(k+1)}}{\hat{G}_{i(k+1)}^{(n)}}\right)-1\\&=\frac{1}{G_{i(k)}^{(n)}}\left(Q_{i(k)}\left(1+\varepsilon_{i(k)}^{(Q)}\right)+\sum_{i_{k+1}=2-[(i_k-1)/2]}^{3-[(i_k-1)/2]}\frac{P_{i(k+1)}(1+\varepsilon_{i(k+1)}^{(P)})}{G_{i(k+1)}^{(n)}(1+\varepsilon_{i(k+1),n}^{(G)})}\right)-1\\&=\frac{Q_{i(k)}}{G_{i(k)}^{(n)}}\left(1+\varepsilon_{i(k)}^{(Q)}\right)+\sum_{i_{k+1}=2-[(i_k-1)/2]}^{3-[(i_k-1)/2]}\frac{P_{i(k+1)}\left(1+\varepsilon_{i(k+1)}^{(P)}\right)}{G_{i(k)}^{(n)}G_{i(k+1)}^{(n)}\left(1+\varepsilon_{i(k+1),n}^{(G)}\right)}-1.\end{aligned}$$

Taking into account

$$\begin{aligned}\frac{Q_{i(k)}}{G_{i(k)}^{(n)}}&=\frac{1}{G_{i(k)}^{(n)}}\left(G_{i(k)}^{(n)}-\sum_{i_{k+1}=2-[(i_k-1)/2]}^{3-[(i_k-1)/2]}\frac{P_{i(k+1)}}{G_{i(k+1)}^{(n)}}\right)\\&=1-\sum_{i_{k+1}=2-[(i_k-1)/2]}^{3-[(i_k-1)/2]}\frac{P_{i(k+1)}}{G_{i(k)}^{(n)}G_{i(k+1)}^{(n)}}\\&=1-\sum_{i_{k+1}=2-[(i_k-1)/2]}^{3-[(i_k-1)/2]}g_{i(k+1)}^{(n)},\end{aligned}$$

we get (8).

Consistently using the recurrence formulas (8), we obtain the formula for the relative error of the quantities $G_{i(k)}^{(n)}$, $i(k) \in I_k$, $0 \leqslant k \leqslant n$, in the form of a branched continued fraction

$$\begin{aligned}\varepsilon_{i(k),n}^{(G)} = &\left(1 - \sum_{i_{k+1}=2-[(i_k-1)/2]}^{3-[(i_k-1)/2]} g_{i(k+1)}^{(n)}\right)(1+\varepsilon_{i(k)}^{(Q)}) - 1\\ &+ \sum_{i_{k+1}=2-[(i_k-1)/2]}^{3-[(i_k-1)/2]} \frac{g_{i(k+1)}^{(n)}(1+\varepsilon_{i(k+1)}^{(P)})}{\left(1 - \sum\limits_{i_{k+2}=2-[(i_{k+1}-1)/2]}^{3-[(i_{k+1}-1)/2]} g_{i(k+2)}^{(n)}\right)(1+\varepsilon_{i(k+1)}^{(Q)})} + \cdots +\\ &+ \sum_{i_n=2-[(i_{n-1}-1)/2]}^{3-[(i_{n-1}-1)/2]} \frac{g_{i(n)}^{(n)}(1+\varepsilon_{i(n)}^{(P)})}{1+\varepsilon_{i(n)}^{(Q)}},\end{aligned} \tag{9}$$

where $i(k) \in I_k$, $0 \leqslant k \leqslant n-1$, herewith $\varepsilon_{i(n),n}^{(G)} = \varepsilon_{i(n)}^{(Q)}$, $i(n) \in I_n$.

Substituting $k = 0$ into formula (9), we obtain formula

$$\begin{aligned}\varepsilon_{i_0,n}^{(f)} &= \left|\frac{\hat{f}_n^{(i_0)} - f_n^{(i_0)}}{f_n^{(i_0)}}\right|\\ &= \left(1 - \sum_{i_1=2-[(i_0-1)/2]}^{3-[(i_0-1)/2]} g_{i(1)}^{(n)}\right)\left(1+\varepsilon_{i_0}^{(Q)}\right) - 1 \\ &+ \sum_{i_1=2-[(i_0-1)/2]}^{3-[(i_0-1)/2]} \frac{g_{i(1)}^{(n)}\left(1+\varepsilon_{i(1)}^{(P)}\right)}{\left(1 - \sum\limits_{i_2=2-[(i_1-1)/2]}^{3-[(i_1-1)/2]} g_{i(2)}^{(n)}\right)\left(1+\varepsilon_{i(1)}^{(Q)}\right)} + \cdots + \sum_{i_n=2-[(i_{n-1}-1)/2]}^{3-[(i_{n-1}-1)/2]} \frac{g_{i(n)}^{(n)}\left(1+\varepsilon_{i(n)}^{(P)}\right)}{1+\varepsilon_{i(n)}^{(Q)}}\end{aligned} \tag{10}$$

for the relative error of the nth approximant (7).

3. Sufficient conditions for the numerical stability of a numerical branched continued fraction

Let us prove two auxiliary lemmas.

Lemma 1. *Let p and q be the positive real numbers. The modified approximant*

$$T_n(\omega) = q + \underbrace{\frac{-p}{q} + \frac{-p}{q} + \cdots + \frac{-p}{q+\omega}}_{n} \tag{11}$$

of the continued fraction

$$q + \frac{-p}{q} + \frac{-p}{q} + \cdots$$

takes a positive value if

$$4p \leqslant q^2, \tag{12}$$

and one of the conditions

$$\omega > \frac{-q + \sqrt{q^2 - 4p}}{2} \tag{13}$$

or

$$\omega < -q \tag{14}$$

is fulfilled.

Proof. The finite continued fraction (11) is equal to
$$q\Bigg(1+\underbrace{\frac{-p/q^2}{1}\,{+}\,\frac{-p/q^2}{1}\,{+}\cdots{+}\,\frac{-p/q^2}{1+\omega/q}}_{n}\Bigg). \tag{15}$$
If inequality (12) holds, then the partial numerators of the continued fraction (15) can be written in the form
$$p/q^2=d(1-d),$$
where
$$d=\frac{1-\sqrt{1-4p/q^2}}{2}.$$

Consider the periodic continued fraction
$$1+\frac{-d(1-d)}{1}\,{+}\,\frac{-d(1-d)}{1}\,{+}\cdots. \tag{16}$$
Let n be an arbitrary natural number. Let
$$h_n=1+\underbrace{\frac{-d(1-d)}{1}\,{+}\,\frac{-d(1-d)}{1}\,{+}\cdots{+}\,\frac{-d(1-d)}{1}}_{n}$$
be the nth approximant of the continued fraction (16), A_n, B_n be its nth numerator and nth denominator, respectively. It is known (see [3, Theorem 3.2]) that the sequence of approximants $\{h_n\}_{n=1}^{\infty}$ is monotonically decreasing and bounded from below
$$1>h_n>h_{n+1}>\frac{1-\sqrt{1-4d(1-d)}}{2}, \tag{17}$$
the continued fraction (16) converges to the value
$$\frac{1-\sqrt{1-4d(1-d)}}{2},\quad \text{and}\quad B_n=\sum_{i=0}^{n}d^i(1-d)^{n-i},\quad A_n=B_{n+1}.$$
Then
$$T_n(\omega)=q\frac{A_n+\frac{\omega}{q}A_{n-1}}{B_n+\frac{\omega}{q}B_{n-1}}=q\frac{B_{n+1}+\frac{\omega}{q}B_n}{B_n+\frac{\omega}{q}B_{n-1}}.$$
It is clear that $T_n(\omega)>0$, if
$$-\frac{\omega}{q}<\frac{B_{n+1}}{B_n}\quad\text{or}\quad -\frac{\omega}{q}>\frac{B_{n+1}}{B_n}.$$
It follows from inequalities (17) that $T_n(\omega)>0$, if
$$-\frac{\omega}{q}<\frac{1-\sqrt{1-4d(1-d)}}{2}\quad\text{or}\quad -\frac{\omega}{q}>1,$$
that is, inequalities (13) or (14) hold. ■

Lemma 2. *Let*
$$q_0-\frac{-p}{q}\,{+}\,\frac{-p}{q}\,{+}\cdots,\quad p>0,\quad q>0, \tag{18}$$
be a periodic continued fraction whose elements satisfy the inequality (12). Then the sequence of modified approximants
$$S_n(\omega)=q_0-\underbrace{\frac{-p}{q}\,{+}\,\frac{-p}{q}\,{+}\cdots{+}\,\frac{-p}{q+\omega}}_{n},\quad n\geqslant 1, \tag{19}$$
of the continued fraction (18) converges to the value:

1) $q_0+(q-\sqrt{q^2-4p})/2$, if $\omega\neq-(q+\sqrt{q^2-4p})/2$ and $4p<q^2$;
2) $q_0+(q+\sqrt{q^2-4p})/2$, if $\omega=-(q+\sqrt{q^2-4p})/2$ and $4p<q^2$;
3) $q_0+q/2$, if $4p=q^2$.

Proof. Consider the transformations

$$t_0(\omega) = q_0 - \omega, \quad t(\omega) = -p/(q+\omega). \tag{20}$$

The composition

$$t_0 \circ \underbrace{t \circ t \circ \ldots \circ t(\omega)}_{n}$$

of transformations (20) is the nth modified approximant $S_n(\omega)$ of the continuous fraction (18).

Let $4p < q^2$. Then the transformation $t(\omega) = -p/(q+\omega)$ is a fractional-linear mapping of the hyperbolic type with fixed points $\omega_i = (-q + (-1)^{i+1}\sqrt{q^2-4p})/2$, $i \in \{1,2\}$, moreover, ω_1 is an attractive point.

If $\omega \neq \omega_2$, then

$$\lim_{n\to\infty}(q_0 - S_n(\omega)) = \omega_1.$$

Thus

$$\lim_{n\to\infty} S_n(\omega) = q_0 - \omega_1 = q_0 + \frac{q - \sqrt{q^2-4p}}{2}.$$

If $\omega = \omega_2$, then

$$\lim_{n\to\infty}(q_0 - S_n(\omega)) = \omega_2.$$

Thus

$$\lim_{n\to\infty} S_n(\omega) = q_0 - \omega_2 = q_0 + \frac{q + \sqrt{q^2-4p}}{2}.$$

Let $4p = q^2$. In this case $\omega_1 = \omega_2 = -q/2$ and

$$\lim_{n\to\infty}(q_0 - S_n(\omega)) = -q/2.$$

Then

$$\lim_{n\to\infty} S_n(\omega) = q_0 + q/2$$

for arbitrary values ω. ∎

Theorem 1. *Let $i_0 \in I_0$ and let there exist non-negative constants $0 < \alpha < 1$ and $0 < \beta < 1$ such that*

$$|\varepsilon^{(P)}_{i(k)}| \leqslant \alpha,\ i(k) \in I_k,\ 1 \leqslant k \leqslant n,\ n \geqslant 1, \text{ and } |\varepsilon^{(Q)}_{i(k)}| \leqslant \beta,\ i(k) \in I_k,\ 0 \leqslant k \leqslant n,\ n \geqslant 1. \tag{21}$$

A branched continued fraction (6) is numerically stable if there exists a positive constant η such that

$$\sum_{i_k=2-[(i_{k-1}-1)/2]}^{3-[(i_{k-1}-1)/2]} |g^{(n)}_{i(k)}| \leqslant \frac{\eta}{1+\eta}, \quad 1 \leqslant k \leqslant n, \quad n \geqslant 1, \tag{22}$$

moreover, the constants α, β, η satisfy the inequality

$$4\eta(1+\eta)(1+\alpha) \leqslant (1+2\eta)^2(1-\beta)^2. \tag{23}$$

In addition, for the relative error of the nth approximant, the estimate

$$|\varepsilon^{(f)}_{i_0,n}| \leqslant \frac{1+\beta+2\eta\beta - \sqrt{(1+2\eta)^2(1-\beta)^2 - 4\eta(1+\eta)(1+\alpha)}}{2(1+\eta)}, \quad n \geqslant 1, \tag{24}$$

is valid, if $4\eta(1+\eta)(1+\alpha) < (1+2\eta)^2(1-\beta)^2$, and

$$|\varepsilon^{(f)}_{i_0,n}| \leqslant \frac{\beta\sqrt{1+\alpha} + \sqrt{\alpha+\beta(2-\beta)}}{\sqrt{1+\alpha} + \sqrt{\alpha+\beta(2-\beta)}}, \quad n \geqslant 1, \tag{25}$$

if $4\eta(1+\eta)(1+\alpha) = (1+2\eta)^2(1-\beta)^2$.

Proof. Let $i_0 \in I_0$ and n be an arbitrary natural number. Using formulas (8), we show that estimates

$$|\varepsilon^{(G)}_{i(n),n}| \leqslant \beta, \quad i(n) \in I_n, \tag{26}$$

and
$$|\varepsilon^{(G)}_{i(k),n}| \leqslant S_{n-k}(\omega), \quad i(k)\in I_k, \quad 0\leqslant k\leqslant n-1, \tag{27}$$
are valid for relative errors $\varepsilon^{(G)}_{i(k),n}$, $i(k)\in I_k$, $0\leqslant k\leqslant n$, of values $G^{(n)}_{i(k)}$, $i(k)\in I_k$, $0\leqslant k\leqslant n$. Here the approximants $S_{n-k}(\omega)$, $i(k)\in I_k$, $0\leqslant k\leqslant n-1$, are defined by (19), where
$$q_0=\beta-\frac{\eta}{1+\eta}(1-\beta),\quad p=\frac{\eta}{1+\eta}(1+\alpha),\quad q=\frac{1+2\eta}{1+\eta}(1-\beta),\quad \omega=-\frac{\eta}{1+\eta}(1-\beta).$$
Note that if $\eta>0$ and inequality (23) holds, then the elements p, q of the finite continued fraction
$$T_{n-k-1}\left(-\frac{\eta}{1+\eta}(1-\beta)\right)=q+\underbrace{\frac{-p}{q}\,{+}\,\frac{-p}{q}\,{+}\,\cdots\,{+}\,\frac{-p}{q+\omega}}_{n-k-1}$$
satisfy conditions (12) and (13) of Lemma 1. Then
$$T_{n-k-1}\left(-\frac{\eta}{1+\eta}(1-\beta)\right)>0,\quad 0\leqslant k\leqslant n-2.$$
If $k=n$ we have
$$|\varepsilon^{(G)}_{i(n),n}|=|\varepsilon^{(Q)}_{i(n)}|\leqslant\beta,$$
which proves the estimate (26).

We prove the estimate (27) by induction on k. For $k=n-1$ we have
$$\begin{aligned}
|\varepsilon^{(G)}_{i(n-1),n}| &= \left|\left(1-\sum_{i_n=2-[(i_{n-1}-1)/2]}^{3-[(i_{n-1}-1)/2]} g^{(n)}_{i(n)}\right)\left(1+\varepsilon^{(Q)}_{i(n-1)}\right)+\sum_{i_n=2-[(i_{n-1}-1)/2]}^{3-[(i_{n-1}-1)/2]}\frac{g^{(n)}_{i(n)}\left(1+\varepsilon^{(P)}_{i(n)}\right)}{1+\varepsilon^{(G)}_{i(n),n}}-1\right|\\
&\leqslant \left|\varepsilon^{(Q)}_{i(n-1)}+\sum_{i_n=2-[(i_{n-1}-1)/2]}^{3-[(i_{n-1}-1)/2]} g^{(n)}_{i(n)}\left(\frac{1+\varepsilon^{(P)}_{i(n)}}{1+\varepsilon^{(G)}_{i(n),n}}-\left(1+\varepsilon^{(Q)}_{i(n-1)}\right)\right)\right|\\
&\leqslant \left|\varepsilon^{(Q)}_{i(n-1)}\right|+\sum_{i_n=2-[(i_{n-1}-1)/2]}^{3-[(i_{n-1}-1)/2]}\left|g^{(n)}_{i(n)}\right|\left(\left|\frac{1+\varepsilon^{(P)}_{i(n)}}{1+\varepsilon^{(G)}_{i(n),n}}-1\right|+\left|\varepsilon^{(Q)}_{i(n-1)}\right|\right)\\
&\leqslant \beta+\frac{\eta}{1+\eta}\left(\frac{1+\alpha}{1-\beta}-1+\beta\right)\\
&= \beta-\frac{\eta}{1+\eta}(1-\beta)+\frac{\frac{\eta}{1+\eta}(1+\alpha)}{1-\beta}=q_0-\frac{-p}{q+\omega}=S_1(\omega).
\end{aligned}$$
Assume that estimate (27) holds for $k=m+1$, $0\leqslant m\leqslant n-2$, and prove it for $k=m$. We have
$$\begin{aligned}
|\varepsilon^{(G)}_{i(m),n}| &= \left|\left(1+\varepsilon^{(Q)}_{i(m)}\right)\left(1-\sum_{i_{m+1}=2-[(i_m-1)/2]}^{3-[(i_m-1)/2]} g^{(n)}_{i(m+1)}\right)+\sum_{i_{m+1}=2-[(i_m-1)/2]}^{3-[(i_m-1)/2]}\frac{g^{(n)}_{i(m+1)}\left(1+\varepsilon^{(P)}_{i(m+1)}\right)}{1+\varepsilon^{(G)}_{i(m+1),n}}-1\right|\\
&= \left|\varepsilon^{(Q)}_{i(m)}+\sum_{i_{m+1}=2-[(i_m-1)/2]}^{3-[(i_m-1)/2]} g^{(n)}_{i(m+1)}\left(\frac{1+\varepsilon^{(P)}_{i(m+1)}}{1+\varepsilon^{(G)}_{i(m+1),n}}-\left(1+\varepsilon^{(Q)}_{i(m)}\right)\right)\right|\\
&\leqslant \left|\varepsilon^{(Q)}_{i(m)}\right|+\sum_{i_{m+1}=2-[(i_m-1)/2]}^{3-[(i_m-1)/2/]}\left|g^{(n)}_{i(m+1)}\right|\left(\left|\frac{1+\varepsilon^{(P)}_{i(m+1)}}{1+\varepsilon^{(G)}_{i(m+1),n}}-1\right|+\left|\varepsilon^{(Q)}_{i(m)}\right|\right)\\
&\leqslant \beta+\frac{\eta}{1+\eta}\left(\frac{1+\alpha}{1-\left|\varepsilon^{(G)}_{i(m+1),n}\right|}-1+\beta\right)=\beta-\frac{\eta}{1+\eta}(1-\beta)+\frac{\frac{\eta}{1+\eta}(1+\alpha)}{1-\left|\varepsilon^{(G)}_{i(m+1),n}\right|}\\
&\leqslant \beta-\frac{\eta}{1+\eta}(1-\beta)-\frac{-\frac{\eta}{1+\eta}(1+\alpha)}{\frac{1+2\eta}{1+\eta}(1-\beta)}\,{+}\,\underbrace{\frac{-\frac{\eta}{1+\eta}(1+\alpha)}{\frac{1+2\eta}{1+\eta}(1-\beta)}\,{+}\,\cdots\,{+}\,\frac{-\frac{\eta}{1+\eta}(1+\alpha)}{1-\beta}}_{n-m-1}
\end{aligned}$$

$$= q_0 - \underbrace{\frac{-p}{q} \, \frac{}{+} \, \frac{-p}{q} \, \frac{}{+} \, \frac{\cdots}{} \, \frac{}{+} \, \frac{-p}{q+\omega}}_{n-m} = S_{n-m}(\omega).$$

Substituting $k = 0$ into formula (27), we obtain formula

$$\left|\varepsilon_{i_0,n}^{(f)}\right| \leqslant S_n(\omega),$$

for the relative error of the nth approximant (6), where $S_n(\omega)$ is defined by (19).

For the difference between approximants $S_{n+1}(\omega)$ and $S_n(\omega)$, the formula

$$S_{n+1}(\omega) - S_n(\omega) = \frac{p^n(\omega^2 + q\omega + p)}{T_n(\omega)\prod_{k=0}^{n-1} T_k^2(\omega)}$$

is valid. If $\eta > 0$ and the inequality (23) holds, then conditions (12) and (13) of Lemma 1 are fulfilled. Then

$$\frac{p^n(\omega^2 + q\omega + p)}{T_n(\omega)\prod_{k=0}^{n-1} T_k^2(\omega)} > 0$$

and sequence $\{S_n(\omega)\}_{n=1}^{\infty}$ increases monotonically. Therefore, for the relative error of the nth approximant of the branched continued fraction (6), the estimate holds

$$\left|\varepsilon_{i_0,n}^{(f)}\right| \leqslant S,$$

where

$$S = \lim_{n\to\infty} S_n(\omega).$$

Let $(1+2\eta)^2(1-\beta)^2 - 4\eta(1+\eta)(1+\alpha) > 0$. Since

$$\omega \neq -\frac{q + \sqrt{q^2 - 4p}}{2},$$

then according to Lemma 2

$$S = q_0 + \frac{q - \sqrt{q^2 - 4p}}{2} = \frac{1 + \beta + 2\eta\beta - \sqrt{(1+2\eta)^2(1-\beta)^2 - 4\eta(1+\eta)(1+\alpha)}}{2(1+\eta)}.$$

This proves the validity of the estimate (24).

Consider the function

$$\varphi(\alpha, \beta) = \frac{1 + \beta + 2\eta\beta - \sqrt{(1+2\eta)^2(1-\beta)^2 - 4\eta(1+\eta)(1+\alpha)}}{2(1+\eta)}.$$

Since

$$\lim_{\substack{\alpha\to+0\\ \beta\to+0}} \varphi(\alpha, \beta) = 0,$$

then for any $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that for all $\alpha > 0$ and $\beta > 0$ such that $\sqrt{\alpha^2 + \beta^2} < \delta_\varepsilon$, the inequality $\varphi(\alpha, \beta) < \varepsilon$ is valid. If $\alpha < \delta_\varepsilon/\sqrt{2}$, then for each $\hat{P}_{i(k)} \in \mathbb{C}$, $i(k) \in I_k$, $1 \leqslant k \leqslant n$, such that

$$\left|\frac{\hat{P}_{i(k)} - P_{i(k)}}{P_{i(k)}}\right| \leqslant \alpha < \frac{\delta_\varepsilon}{\sqrt{2}}, \quad i(k) \in I_k, \quad 1 \leqslant k \leqslant n,$$

and each $\hat{Q}_{i(k)} \in \mathbb{C}$, $i(k) \in I_k$, $0 \leqslant k \leqslant n$, such that

$$\left|\frac{\hat{Q}_{i(k)} - Q_{i(k)}}{Q_{i(k)}}\right| \leqslant \alpha < \frac{\delta_\varepsilon}{\sqrt{2}}, \quad i(k) \in I_k, \quad 0 \leqslant k \leqslant n,$$

the inequality

$$|\varepsilon_{i_0,n}^{(f)}| \leqslant \varphi(\alpha, \beta) < \varepsilon,$$

that proves the numerical stability of the branched continued fraction (6).

Assume that $(1+2\eta)^2(1-\beta)^2 - 4\eta(1+\eta)(1+\alpha) = 0$. Then

$$\eta = \frac{\sqrt{(1+\alpha)(\alpha+\beta(2-\beta))} - \alpha - \beta(2-\beta)}{2(\alpha+\beta(2-\beta))}$$

and according to Lemma 2

$$\lim_{n\to\infty} S_n(\omega) = q_0 + \frac{q}{2} = \beta + \frac{1-\beta}{2(1+\eta)} = \frac{\beta\sqrt{1+\alpha} + \sqrt{\alpha+\beta(2-\beta)}}{\sqrt{1+\alpha} + \sqrt{\alpha+\beta(2-\beta)}}.$$

Thus, for the relative error of the computation of the nth approximant of the branched continued fraction (6), the estimate (25) is valid and, as in the previous case, we obtain its numerical stability. ■

4. The set of numerical stability of branched continued fraction expansions of ratios of Horn's confluent hypergeometric functions $\mathbf{H}_6$

Definition 2. *A functional branched continued fraction (1) is called numerically stable at the point* $\mathbf{z} = \mathbf{z}_0 \in \Omega$ *if the numerical branched continued fraction (6) is numerically stable.*

Definition 3. *A set* Ω *is called a set of numerical stability of the branched continued fraction* (1) *if it is numerically stable at every point* $\mathbf{z} \in \Omega$.

The following theorem is true.

Theorem 2. *Let* $i_0 \in I_0$ *and* a, c *be the real constants such that*

$$a \geqslant 0, \quad c \geqslant a + 1 + \delta_{i_0}^1. \tag{28}$$

A set

$$\Omega = \left\{\mathbf{z} \in \mathbb{R}^2\colon -L_1 \leqslant z_1 \leqslant 0, -L_2 \leqslant z_2 \leqslant 0\right\},$$

where L_1, L_2 *are positive constants such that*

$$2L_2 < c + 1, \tag{29}$$

is a set of numerical stability of the branched continued fraction (1), if the relative errors of the computation of its elements satisfy the conditions (21) and the inequality (23) holds, where

$$\eta = \max\left\{2L_1 + \frac{2(c+1)L_2}{c(c+1-L_2)}, \frac{c+1+L_2}{c+1-2L_2}\right\}.$$

In addition, for the relative error of its n*th approximants, the estimates (24) and (25) are valid.*

Proof. Let $i_0 \in I_0$, n be an arbitrary natural number and $\mathbf{z}$ be an arbitrary fixed point in Ω. If the conditions (28), (29) are valid, then for the quantities

$$G_{i(k)}^{(n)}, \quad i(k) \in I_k, \quad 1 \leqslant k \leqslant n, \quad \text{and} \quad \sum_{i_{k+1}=2-[(i_k-1)/2]}^{3-[(i_k-1)/2]} \left|g_{i(k+1)}^{(n)}(\mathbf{z})\right|, \quad i(k) \in I_k, \quad 1 \leqslant k \leqslant n-1,$$

the estimates

$$G_{i(k)}^{(n)}(\mathbf{z}) \geqslant 1, \quad i(k) \in I_k, \quad 1 \leqslant k \leqslant n, \quad i_k \neq 3,$$

$$G_{i(k)}^{(n)}(\mathbf{z}) \geqslant \frac{c+1-L_2}{2(c+1)} > 0, \quad i(k) \in I_k, \quad 1 \leqslant k \leqslant n, \quad i_k = 3,$$

$$\sum_{i_{k+1}=2-[(i_k-1)/2]}^{3-[(i_k-1)/2]} \left|g_{i(k+1)}^{(n)}(\mathbf{z})\right| \leqslant \frac{\eta}{1+\eta}, \quad i(k) \in I_k, \quad 1 \leqslant k \leqslant n-1,$$

are valid (see, [15, Theorem 3.2]).

We prove that

$$\sum_{i_1=2-[(i_0-1)/2]}^{3-[(i_0-1)/2]} \left|g_{i(1)}^{(n)}(\mathbf{z})\right| < \frac{\eta}{1+\eta}, \quad i_0 \in I_0. \tag{30}$$

Note that the inequality (30) is equivalent to

$$\sum_{i_1=2-[(i_0-1)/2]}^{3-[(i_0-1)/2]} \frac{|P_{i(1)}(\mathbf{z})|}{|G^{(n)}_{i(1)}(\mathbf{z})|} \leqslant \eta\left(|G^{(n)}_{i_0}(\mathbf{z})| - \sum_{i_1=2-[(i_0-1)/2]}^{3-[(i_0-1)/2]} \frac{|P_{i(1)}(\mathbf{z})|}{|G^{(n)}_{i(1)}(\mathbf{z})|}\right).$$

If $i_0 = 1$, then

$$|G^{(n)}_1(\mathbf{z})| = G^{(n)}_1(\mathbf{z}) = Q_1 + \sum_{i_1=2}^{3} \frac{P_{1,i_1}(\mathbf{z})}{G^{(n)}_{1,i_1}(\mathbf{z})} = 1 + \frac{2(a+1)|z_1|}{c\,G^{(n)}_{1,2}(\mathbf{z})} + \frac{|z_2|}{c\,G^{(n)}_{1,3}(\mathbf{z})} \geqslant 1,$$

$$|G^{(n)}_1(\mathbf{z})| - \sum_{i_1=2}^{3} \frac{|P_{1,i_1}(\mathbf{z})|}{|G^{(n)}_{1,i_1}(\mathbf{z})|} = Q_1 = 1,$$

and

$$\sum_{i_1=2}^{3} \frac{|P_{1,i_1}(\mathbf{z})|}{G^{(n)}_{1,i_1}(\mathbf{z})} = \frac{2(a+1)|z_1|}{c\,G^{(n)}_{1,2}(\mathbf{z})} + \frac{|z_2|}{c\,G^{(n)}_{1,3}(\mathbf{z})} \leqslant \frac{2(a+1)L_1}{c} + \frac{2(c+1)L_2}{c(c+1-L_2)}$$
$$< 2L_1 + \frac{2(c+1)L_2}{c(c+1-L_2)} \leqslant \eta.$$

Thus,

$$\sum_{i_1=2}^{3} |g^{(n)}_{1,i_1}(\mathbf{z})| < \frac{\eta}{1+\eta}.$$

If $i_0 = 2$, then

$$|G^{(n)}_2(\mathbf{z})| = G^{(n)}_2(\mathbf{z}) = Q_2 + \sum_{i_1=2}^{3} \frac{P_{2,i_1}(\mathbf{z})}{G^{(n)}_{2,i_1}(\mathbf{z})} = 1 + \frac{(2c-a)(a+1)|z_1|}{c(c+1)G^{(n)}_{2,2}(\mathbf{z})} + \frac{(c-a)|z_2|}{c(c+1)G^{(n)}_{2,3}(\mathbf{z})} \geqslant 1,$$

$$|G^{(n)}_2(\mathbf{z})| - \sum_{i_1=2}^{3} \frac{|P_{2,i_1}(\mathbf{z})|}{G^{(n)}_{2,i_1}(\mathbf{z})} = Q_2 = 1,$$

and

$$\sum_{i_1=2}^{3} \frac{|P_{2,i_1}(\mathbf{z})|}{G^{(n)}_{2,i_1}(\mathbf{z})} = \frac{(2c-a)(a+1)|z_1|}{c(c+1)G^{(n)}_{2,2}(\mathbf{z})} + \frac{(c-a)|z_2|}{c(c+1)G^{(n)}_{2,3}(\mathbf{z})}$$
$$\leqslant \frac{(2c-a)(a+1)L_1}{c(c+1)} + \frac{2(c-a)L_2}{c(c+1-L_2)} < 2L_1 + \frac{2(c+1)L_2}{c(c+1-L_2)} \leqslant \eta.$$

Thus,

$$\sum_{i_1=2}^{3} |g^{(n)}_{2,i_1}(\mathbf{z})| < \frac{\eta}{1+\eta}.$$

If $i_0 = 3$, then

$$|G^{(n)}_3(\mathbf{z})| = G^{(n)}_3(\mathbf{z}) = Q_3 + \sum_{i_1=1}^{2} \frac{P_{3,i_1}(\mathbf{z})}{G^{(n)}_{3,i_1}(\mathbf{z})} = 1 - \frac{a}{2c} + \frac{a}{2c\,G^{(n)}_{3,1}(\mathbf{z})} - \frac{a|z_2|}{2c(c+1)G^{(n)}_{3,2}(\mathbf{z})}$$
$$\geqslant 1 - \frac{a}{2c} - \frac{a|z_2|}{2c(c+1)G^{(n)}_{3,2}(\mathbf{z})} \geqslant \frac{1}{2} - \frac{aL_2}{2c(c+1)} \geqslant \frac{c+1-L_2}{2(c+1)},$$

$$|G^{(n)}_3(\mathbf{z})| - \sum_{i_1=1}^{2} \frac{|P_{3,i_1}(\mathbf{z})|}{G^{(n)}_{3,i_1}(\mathbf{z})} = 1 - \frac{a}{2c} - \frac{a|z_2|}{c(c+1)G^{(n)}_{3,2}(\mathbf{z})} \geqslant 1 - \frac{a}{2c} - \frac{aL_2}{c(c+1)} > \frac{c+1-2L_2}{2(c+1)},$$

and

$$\sum_{i_1=1}^{2} \frac{|P_{3,i_1}(\mathbf{z})|}{G^{(n)}_{3,i_1}(\mathbf{z})} = \frac{a}{2c\,G^{(n)}_{3,1}(\mathbf{z})} + \frac{a|z_2|}{2c(c+1)G^{(n)}_{3,2}(\mathbf{z})} \leqslant \frac{a}{2c} + \frac{aL_2}{2c(c+1)} < \frac{c+1+L_2}{2(c+1)}$$

$$= \frac{c+1+L_2}{c+1-2L_2}\,\frac{c+1-2L_2}{2(c+1)} \leqslant \frac{c+1-2L_2}{2(c+1)}\eta.$$

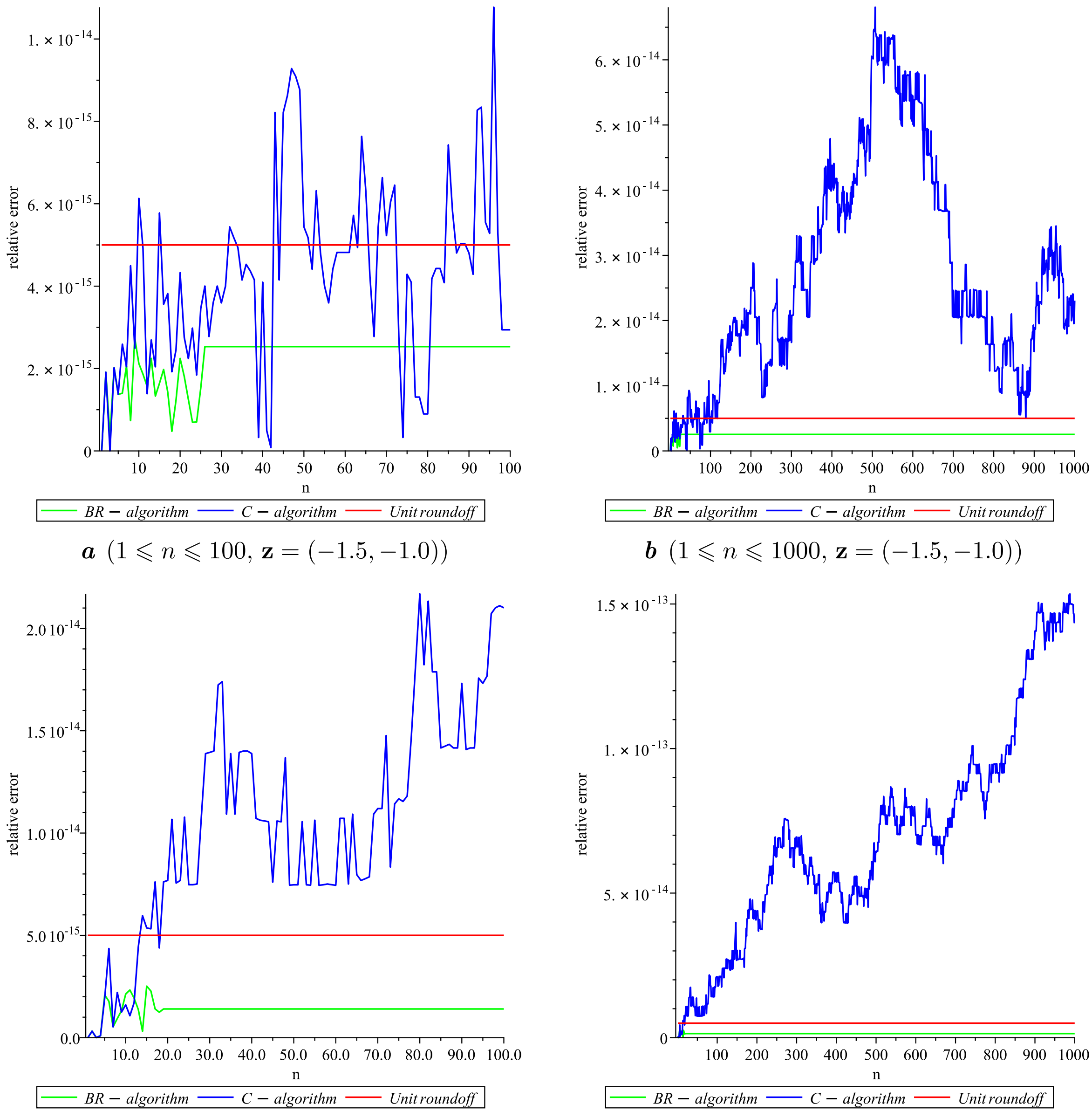
Thus,

$$\sum_{i_1=1}^{2} \left|g_{3,i_1}^{(n)}(\mathbf{z})\right| < \frac{\eta}{1+\eta}.$$

Finally, according to the conditions (21)–(23) of Theorem 1, for each $i_0 \in I_0$, the set Ω is the set of numerical stability of the branched continued fraction (1), in addition, the estimates (24) and (25) are valid for the relative error of its nth approximants. ∎

5. Numerical experiments

In the analytical theory of branched continued fractions, there are two algorithms for computing approximants: the backward recurrence algorithm and the algorithm of continuants [1,39]. For $i_0 = 1$, let us investigate the errors of computing the approximants of the branched continued fraction (1) using



a $(1 \leqslant n \leqslant 100,\ \mathbf{z} = (-1.5, -1.0))$ ***b*** $(1 \leqslant n \leqslant 1000,\ \mathbf{z} = (-1.5, -1.0))$

c $(1 \leqslant n \leqslant 100,\ \mathbf{z} = (-10.0, -2.0))$ ***d*** $(1 \leqslant n \leqslant 1000,\ \mathbf{z} = (-10.0, -2.0))$

Fig. 1. Relative errors of the computing of nth approximants of (1).

these algorithms. For computations, we will use floating-point arithmetic (IEEE 754-2019 standard) with $t = 15$ accuracy and rounding mode to the nearest [25]. Figures $1a$ and $1b$ show the computation errors of the first 100 and 1000 approximants, respectively, at the point $\mathbf{z} = (-1.5, -1.0)$ for the parameters $a = 1$ and $c = 3$. Similarly, Figures $1c$ and $1d$ show the computation errors of the first 100 and 1000 approximants, respectively, at the point $\mathbf{z} = (-10.0, -2.0)$ for the parameters $a = 1$ and $c = 3$.

Numerical experiments show that the backward recurrence algorithm is stability to the accumulation of errors, and the maximum value of the relative error of the approximant computation does not exceed the rounding unit $u = 0.5 \cdot 10^{1-t}$, which ensures high accuracy of computations. Instead, the errors of approximant computations using the algorithm of continuants tend to accumulate and exceed the value of the rounding unit.

6. Conclusions

The formulas for the relative errors of calculating the approximant of the branched continued fractions using the backward recurrence algorithm have been established. The obtained formulas are shown in the form of branched continued fractions (10). Using the methods of the analytical theory of continued fractions, estimates of these errors in the form of periodic continued fractions were obtained and the conditions for numerical stability of a branched continued fractions were established. The obtained results are used to study of the stability of the branched continued fraction expansions of the ratios of Horn's confluent hypergeometric functions H_6 with real parameters in the region of convergence. Further studies on the numerical stability of these expansions with complex parameters of the Horn's confluent hypergeometric function H_6 are relevant.

[1] Bodnar D. I. Branched Continued Fractions. Naukova Dumka (1986).

[2] Cuyt A., Petersen V. B., Verdonk B., Waadeland H., Jones W. B. Handbook of Continued Fractions for Special Functions. Springer (2008).

[3] Jones W. B., Thron W. J. Continued Fractions: Analytic Theory and Applications. Addison-Wesley Pub. Co. (1980).

[4] Antonova T., Cesarano C., Dmytryshyn R., Sharyn S. An approximation to Appell's hypergeometric function F_2 by branched continued fraction. Dolomites Research Notes on Approximation. **17** (1), 22–31 (2024).

[5] Bodnar D. I., Manzii O. S. Expansion of the ratio of Appel hypergeometric functions F_3 into a branching continued fraction and its limit behavior. Journal of Mathematical Sciences. **107**, 3550–3554 (2001).

[6] Dmytryshyn R. On the analytic continuation of Appell's hypergeometric function F_2 to some symmetric domains in the space $\mathbb{C}^2$. Symmetry. **16** (11), 1480 (2024).

[7] Hladun V. R., Hoyenko N. P., Manziy O. S., Ventyk L. On convergence of function $F_4(1, 2; 2, 2; z_1, z_2)$ expansion into a branched continued fraction. Mathematical Modeling and Computing. **9** (3), 767–778 (2022).

[8] Hoyenko N., Hladun V., Manzij O. On the infinite remains of the Nórlund branched continued fraction for Appell hypergeometric functions. Carpathian Mathematical Publications. **6** (1), 11–25 (2014).

[9] Bodnar D. I., Hoyenko N. P. Approximation of the ratio of Lauricella functions by a branched continued fraction. Matematychni Studii. **20** (2), 210–214 (2003).

[10] Antonova T., Dmytryshyn R., Goran V. On the analytic continuation of Lauricella–Saran hypergeometric function $F_K(a_1, a_2, b_1, b_2; a_1, b_2, c_3; \mathbf{z})$. Mathematics. **11** (21), 4487 (2023).

[11] Dmytryshyn R., Goran V. On the analytic extension of Lauricella–Saran's hypergeometric function F_K to symmetric domains. Symmetry. **16** (2), 220 (2024).

[12] Antonova T., Dmytryshyn R., Kravtsiv V. Branched continued fraction expansions of Horn's hypergeometric function H_3 ratios. Mathematics. **9** (2), 148 (2021).

[13] Antonova T., Dmytryshyn R., Kril P., Sharyn S. Representation of some ratios of Horn's hypergeometric functions H_7 by continued fractions. Axioms. **12** (8), 738 (2023).

[14] Antonova T., Dmytryshyn R., Lutsiv I.-A., Sharyn S. On some branched continued fraction expansions for Horn's hypergeometric function $H_4(a,b;c,d;z_1,z_2)$ ratios. Axioms. **12** (3), 299 (2023).

[15] Antonova T., Dmytryshyn R., Sharyn S. Branched continued fraction representations of ratios of Horn's confluent function H_6. Constructive Mathematical Analysis. **6** (1), 22–37 (2023).

[16] Antonova T. M. On convergence of branched continued fraction expansions of Horn's hypergeometric function H_3 ratios. Carpathian Mathematical Publications. **13** (3), 642–650 (2021).

[17] Dmytryshyn R., Lutsiv I.-A., Bodnar O. On the domains of convergence of the branched continued fraction expansion of ratio $H_4(a,d+1;c,d;\mathbf{z})/H_4(a,d+2;c,d+1;\mathbf{z})$. Researches in Mathematics. **31** (2), 19–26 (2023).

[18] Dmytryshyn R., Lutsiv I.-A., Dmytryshyn M. On the analytic extension of the Horn's hypergeometric function H_4. Carpathian Mathematical Publications. **16** (1), 32–39 (2024).

[19] Hladun V., Rusyn R., Dmytryshyn M. On the analytic extension of three ratios of Horn's confluent hypergeometric function H_7. Researches in Mathematics. **32** (1), 60–70 (2024).

[20] Manziy O., Hladun V., Ventyk L. The algorithms of constructing the continued fractions for any rations of the hypergeometric Gaussian functions. Mathematical Modeling and Computing. **4** (1), 48–58 (2017).

[21] Antonova T., Dmytryshyn R., Sharyn S. Generalized hypergeometric function ${}_3F_2$ ratios and branched continued fraction expansions. Axioms. **10** (4), 310 (2021).

[22] Dmytryshyn R., Oleksyn V. On analytical extension of generalized hypergeometric function ${}_3F_2$. Axioms. **13** (11), 759 (2024).

[23] Backeljauw F., Becuwe S., Cuyt A. Validated evaluation of special mathematical functions. Intelligent Computer Mathematics. 206–216 (2008).

[24] Higham N. J. Accuracy and Stability of Numerical Algorithms. SIAM (2002).

[25] Muller J.-M., Brisebarre N., De Dinechin F., Jeannerod C.-P., Lefevre V., Melquiond G., Torres S. Handbook of floating-point arithmetic. Birkhäuser (2018).

[26] Blanch G. Numerical evaluation of continued fractions. SIAM Review. **6** (4), 383–421 (1964).

[27] Cuyt A., Van der Cruyssen P. Rounding error analysis for forward continued fraction algorithms. Computers & Mathematics with Applications. **11**, 541–564 (1985).

[28] Gautschi W. Computational aspects of three-term recurrence relations. SIAM Review. **9**, 24–82 (1967).

[29] Jones W. B., Thron W. J. Numerical stability in evaluating continued fractions. Mathematics of Computation. **28**, 795–810 (1974).

[30] Macon N., Baskervill M. On the generation of errors in the digital evaluation of continued fractions. Journal of the ACM. **3** (3), 199–202 (1956).

[31] Bodnar D., Hladun V. Sufficient conditions of stability of branched continued fractions with positive elements. Mat. Metody Fiz.-Mekh. Polya. **45**, 22–27 (2002), (in Ukrainian).

[32] Hladun V., Kravtsiv V., Dmytryshyn M., Rusyn R. On numerical stability of continued fractions. Matematychni Studii. **62**, 168–183 (2024).

[33] Bodnar D. I., Hladun V. R. On the stability of branched continued fractions with complex elements under perturbations. Matematychni Studii. **25**, 207–212 (2006), (in Ukrainian).

[34] Hladun V. R., Bodnar D. I., Rusyn R. S. Convergence sets and relative stability to perturbations of a branched continued fraction with positive elements. Carpathian Mathematical Publications. **16** (1), 16–31 (2024).

[35] Hladun V., Bodnar D. Some domains of relative stability under perturbations of branched continued fractions with complex elements. Bukovinian Mathematical Journal. **288**, 18–27 (2006), (in Ukrainian).

[36] Hladun V. R. Some sets of relative stability under perturbations of branched continued fractions with complex elements and a variable number of branches. Journal of Mathematical Sciences. **215**, 11–25 (2016).

[37] Dmytryshyn R., Cesarano C., Lutsiv I.-A., Dmytryshyn M. Numerical stability of the branched continued fraction expansion of Horn's hypergeometric function H_4. Matematychni Studii. **61** (1), 51–60 (2024).

[38] Horn J. Hypergeometrische Funktionen zweier Veränderlichen. Mathematische Annalen. **105**, 381–407 (1931).

[39] Manziy O., Hladun V., Seredynsky V. Continuants algorithm for evaluation approximants of branched continued fraction. Physico-mathematical modelling and informational technologies. **37**, 7–11 (2023), (in Ukrainian).

Обчислювальна стійкість гіллястих ланцюгових дробових розвинень відношень вироджених гіпергеометричних функцій Горна H_6

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В роботі, використовуючи новий метод оцінки відносних похибок обчислення апроксимант за допомогою оберненого рекурентного алгоритму, встановлено умови обчислювальної стійкості числового гіллястого ланцюгового дробу. Використовуючи їх, побудовано область обчислювальної стійкості гіллястих ланцюгових дробів, які є розвиненнями вироджених гіпергеометричних функцій Горна H_6 з дійсними параметрами. Окрім того, експериментально досліджено поведінку відносних похибок обчислення апроксимант гіллястого ланцюгового дробу за допомогою оберненого рекурентного алгоритму та алгоритму континуант. Отримані результати ілюструють стійкість оберненого рекурентного алгоритму.

Ключові слова: *гіллястий ланцюговий дріб; гіпергеометрична функція Горна; чисельна апроксимація; похибка заокруглення.*