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Analytical images of the coordinates time dependence of Keplerian motion

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In order to calculate the time dependence of polar coordinates of relative motion in the problem of two bodies in the analytical form it was proposed variants of iterative algorithms with fast convergence, which are based on the usage of approximating functions. It was shown that an independent determining the time dependence of radial coordinate in the elliptical motion, as well as at large distance from the pericenter in the case of the hyperbolic motion yields good convergence using the method of ordinary successive iterations.

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1. Introduction

As it was known from the celestial mechanics, the relative motion of the system of two point gravitating bodies with masses m_1 and m_2 occurs by Keplerian orbits [1, 2]

$$\rho = p\,\{1 + e \cdot \cos v\}^{-1}, \tag{1}$$

where ρ and v are polar coordinates, and the focal parameter p and eccentricity e are determined by the masses of bodies and the motion integrals – angular momentum $\mathbf{l}$ and energy $\mathcal{E}$:

$$p = l^2\mu^{-2}K^{-1}, \quad e = \left\{1 + 2\mathcal{E}l^2\mu^{-3}K^{-2}\right\}^{1/2}. \tag{2}$$

Here $K = G(m_1 + m_2)$, G is the gravitational constant, $\mu = m_1 m_2 (m_1 + m_2)^{-1}$ is the reduced mass. Using the definition of angular momentum

$$\mathbf{l} = \mu\,\rho^2\,\frac{dv}{dt}\,\mathbf{n}, \tag{3}$$

where $\mathbf{n}$ is the unit normal vector to the plane of the orbit, using equation (1) and the first of equations (2), we obtain equation for the time dependence of true anomaly $v(t)$ in the form

$$p^{3/2}K^{-1/2}\int_0^{v(t)} [1 + e \cdot \cos v]^{-2} dv = t. \tag{4}$$

Here used the initial condition $v(0) = 0$, therefore equation (4) determines the time of motion from the pericenter to the point of orbit with the given value of anomaly $v(t)$.

The integral in equation (4) is expressed in elementary functions, but it has a different images depending on the eccentricity value. The trivial case $e = 0$ corresponds to the uniform motion by circular orbit of the radius p with the angular velocity $\omega_p = K^{1/2}/p^{3/2}$, $v(t) = \omega_p t$. At the nonzero eccentricities, the integral in equation (4) is calculated using the substitution $x = \tan(v/2)$. In the case $e = 1$ (at $\mathcal{E} = 0$) the motion occurs by the parabolic orbit, and equation (4) takes the form

$$\tan\frac{v}{2} + \frac{1}{3}\left[\tan\frac{v}{2}\right]^3 = \frac{2K^{1/2}}{p^{3/2}}\,t \equiv 2\omega_p\,t. \tag{5}$$

This equation is known as the Barker's equation [3]. At the changing time in the region $-\infty < t < \infty$, true anomaly changes in the interval $-\pi < v < \pi$.

At $0 < e < 1$, it is realized the motion by the elliptical orbit, and equation (4) takes the form

$$\arctan\left\{\left(\frac{1-e}{1+e}\right)^{1/2}\tan\frac{v}{2}\right\} = \frac{t_*}{2} + \frac{e}{2}\,(1-e^2)^{1/2}\frac{\sin v}{1+e\cdot\cos v}, \tag{6}$$

where

$$t_* = 2\pi t/T = t\,K^{1/2}(1-e^2)^{3/2}\,p^{-3/2} \tag{7}$$

is the dimensionless time, and T is the orbital motion period. At $e > 1$, equation (4) has such an image in elementary functions

$$\ln\frac{(e+1)^{1/2}-(e-1)^{1/2}\tan\frac{v}{2}}{(e+1)^{1/2}+(e-1)\tan\frac{v}{2}} = t_{\text{H}} - e(e^2-1)^{1/2}\frac{\sin v}{1+e\cdot\cos v}, \tag{8}$$

where

$$t_{\text{H}} = t\,K^{1/2}\frac{(e^2-1)^{3/2}}{p^{3/2}} \equiv t_p(e^2-1)^{3/2}. \tag{9}$$

Solving equation (5) relative to $\tan(v/2)$ by the Cardano's formulae [3], we find the exact expression for true anomaly,

$$v(t) = 2\arctan\left\{\left[\left(1+s^2(t)\right)^{1/2}\right) + s(t)\right]^{1/3} - \left[\left(1+s^2(t)\right)^{1/2} - s(t)\right]^{1/3}\right\}, \tag{10}$$

where $s(t) = 3K^{1/2}p^{-3/2}t$. It is shown from relation (10) that $v(-t) = -v(t)$, and function $v(t)$ has the following asymptotics

$$v(t) \to \begin{cases} 4\omega_p t + \ldots \quad \text{at} \quad |\omega_p t| \ll \frac{\pi}{2}, \\ 2\arctan(6\omega_p t) + \ldots \quad \text{at} \quad |\omega_p t| \gg \frac{\pi}{2}. \end{cases} \tag{11}$$

2. The elliptical motion

Using the substitution

$$\left(\frac{1-e}{1+e}\right)^{1/2}\tan\frac{v}{2} = \tan\frac{E}{2}, \tag{12}$$

equation (6) is reduced to the Kepler's equation

$$E - e\sin E = t_*, \tag{13}$$

and the auxiliary function E is an eccentric anomaly. As it is shown from equation (13), the difference $E - t_*$ is the periodic function with period 2π. Many works are devoted to finding approximate solutions of this equation. The most famous is the Lagrange iterative method [4], in which the function $E^{(0)}(t_*) = t_*$ is chosen as zero approximation, which corresponds to the uniform motion along a circular orbit, and the term $e\cdot\sin E$ is considered as perturbation. In such way, we obtain the analytical solution in the form of infinity expansion by the powers of eccentricity

$$E(t_*) = t_* + \sum_{k=1}^{\infty}\frac{e^k}{k!}\cdot\frac{d^{k-1}}{dt_*^{k-1}}\{\sin(t_*)\}^k. \tag{14}$$

As it was shown by Laplace, this expansion coincides only in the region $0 < e < \bar{e} = 0.66274\ldots$ [4].

Application of Fourier series leads to the solution in the form [4]

$$E(t_*) = t_* + \sum_{k=1}^{\infty} I_k(ke)\sin(kt_*), \tag{15}$$

where

$$I_k(ke) = \frac{1}{\pi}\int_0^{\pi}\cos\{k(z - e\cdot\sin z)\}\,dz \tag{16}$$

are the Bessel functions of first kind with an integer index value [3]. The necessity to numerically calculate a large number of these functions makes this approach cumbersome and irrational.

Usage the theory of complex variable function allows us to obtain the solutions of equation (13) in the form of quadratures depending on e and t [5,6]. For example, in the interval $\pi/2 - e \leqslant t_* < \pi$, the solution is represented by one quadrature,

$$E = t_* - (t_* - \pi)\exp\left\{\frac{1}{\pi}\int_0^1 \arg\Omega_0^+(\eta|e,t_*)\,\frac{d\eta}{\eta}\right\}, \tag{17}$$

where

$$\begin{aligned}
&\arg\Omega_0^+(\eta|e,t_*) = \arctan\Psi(\eta|e,t_*),\\
&\Psi(\eta|e,t_*) = \left\{-2\eta\,C(\eta)\left[e+\frac{\pi}{2}\,\eta\right]\right\}\left\{\left[e+\frac{\pi}{2}\,\eta\right]^2 - \eta^2[C^2(\eta)+(\pi-t_*)^2]\right\}^{-1};\\
&C(\eta) = \ln[f(\eta)+\eta^{-1}], \quad f(\eta) = (\eta^{-2}-1)^{-1/2}.
\end{aligned} \tag{18}$$

In the region $0 \leqslant t_* \leqslant \pi/2 - e$, the solution is represented by three cumbersome quadratures. This approach can be considered as a proof of existence of exact analytical solution of the Kepler's equation, but it is too cumbersome for practical use. The work [5] has not interested astronomers and there are no references to it in the literature.

To obtain the approximate analytical solutions of equation (13) using the iterative method with fast convergence, in the work [7] it was proposed the approach of approximating functions. With this purpose, equation (13) is presented in the equivalent form

$$E - e\,f(E) - t_* = e\,\{\sin E - f(E)\}. \tag{19}$$

The auxiliary function $f(E)$ is chosen in the form of a low-degree polynomial, so that the zero approximation equation

$$E^{(0)} - e\,f(E) - t_* = 0 \tag{20}$$

has the analytical solution, and the right-hand side of equation (19) would be a small perturbation, which is taken into account using the successive iterations.

In connection with the periodicity of the function $(E - t_*)$, it is enough to consider equation (13) in the region $0 \leqslant (t_*, E) \leqslant 2\pi$. Moreover, from equation (13) it follows that in the interval $\pi \leqslant t_* \leqslant 2\pi$

$$E(t_*) = 2\pi - E(2\pi - t_*), \tag{21}$$

therefore, it is enough to find the solution in the region $0 \leqslant t_* \leqslant \pi$. With a sufficiently large value of eccentricity, we can to distinguish two regions in which the behavior of the function $E(t_*)$ has a different character: the region of fast change of $E(t_*)$

$$\text{I.} \qquad 0 \leqslant E(t_*) \leqslant \frac{\pi}{2}; \quad 0 \leqslant t_* < \bar{t}_*(e), \tag{22}$$

and the region of slow change of $E(t_*)$

$$\text{II.} \qquad \frac{\pi}{2} \leqslant E(t_*) \leqslant \pi; \quad \bar{t}_*(e) \leqslant t_* \leqslant \pi; \quad \bar{t}_*(e) = \frac{\pi}{2} - e. \tag{23}$$

It is appropriate to make substitutions

$$E = \frac{\pi}{2} - F_1 \tag{24}$$

in the region I,

$$E = \frac{\pi}{2} + F_2 \tag{25}$$

in the region II. Equation for the functions F_1 and F_2 are written in the form

$$\begin{aligned}
F_1 + e\cdot\cos F_1 &= \frac{\pi}{2} - t_*,\\
F_2 - e\cdot\cos F_2 &= t_* - \frac{\pi}{2}.
\end{aligned} \tag{26}$$

In the role of the function $f(F)$ described above, it is used a polynomial of fourth degree

$$f_4(F) = 1 + a_2F^2 + a_3F^3 + a_4F^4, \tag{27}$$

which at

$$a_2 = -0.503491, \quad a_3 = 0.011168, \quad a_4 = 0.032752 \tag{28}$$

is well approximate $\cos F$ on the interval $0 \leqslant F \leqslant \pi/2$, and the deviation does not exceed $3 \cdot 10^{-4}$ even in the vicinity $F = \pi/2$. We will consider in more detail equation for the function F_2, presented it in the form

$$F_2(t_*) - e\, f_4(F_2(t_*)) = t_* - \frac{\pi}{2} + e\big\{\cos F_2(t) - f_4(F_2(t_*))\big\}. \tag{29}$$

The zero approximate for the function $F_2(t_*)$ is the root of algebraic equation of the fourth degree

$$F_2^{(0)}(t_*) - e\, f_4\big(F_2^{(0)}(t_*)\big) - t_* + \frac{\pi}{2} = 0. \tag{30}$$

The refined solution of equation (29) (in the first iteration)

$$F_2^{(1)}(t_*) = F_2^{(0)}\big(t_*^{(1)}\big), \tag{31}$$

where

$$t_*^{(1)} \equiv t_* + e\left\{\cos F_2^{(0)}(t_*) - f_4\big(F^{(0)}(t_*)\big)\right\}, \tag{32}$$

and etc. According to the Cardano's formulae [3], the solution satisfying the condition $0 \leqslant F_2^{(0)}(t_*) \leqslant \pi/2$, and determined by expression

$$F_2^{(0)}(t_*) = c - \left\{c^2 - \frac{1}{2}\,u + \left[\frac{u^2}{4} - \alpha_0\right]^{1/2}\right\}^{1/2}. \tag{33}$$

Here, the following notations is used

$$\begin{aligned}
&c = -\frac{1}{2}\left\{\frac{1}{2}\alpha_3 - \left(\frac{1}{4}\alpha_3^2 + u - \alpha_2\right)^{1/2}\right\};\\
&u = \left\{g + (g^2+q^3)^{1/2}\right\}^{1/3} + \left\{g - (g^2+q^3)^{1/2}\right\}^{1/3} - \frac{1}{3}\,b_2;\\
&q = \frac{1}{3}b_1 - \frac{1}{9}b_2^2; \quad g = \frac{1}{6}\{b_1 b_2 - 3b_0\} - \frac{1}{27}b_2^3;\\
&b_0 = 4\alpha_0\alpha_2 - \alpha_1^2 - \alpha_0\alpha_3^2; \quad b_1 = \alpha_1\alpha_3 - 4\alpha_0; \quad b_2 = -\alpha_2;\\
&\alpha_0 = \frac{1}{a_4}\left\{1 + \frac{1}{e}\left(t_* - \frac{\pi}{2}\right)\right\}; \quad \alpha_1 = -\frac{1}{e\,a_4}; \quad \alpha_2 = \frac{a_2}{a_4}; \quad \alpha_3 = \frac{a_3}{a_4}.
\end{aligned} \tag{34}$$

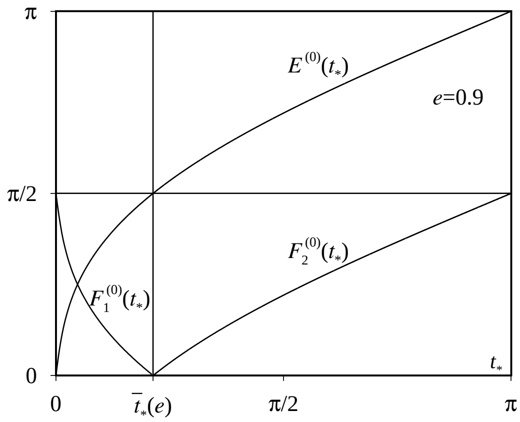

Fig. 1. The time dependence of functions $F_1^{(0)}(t_*)$ and $F_2^{(0)}(t_*)$ and eccentric anomaly $E^{(0)}(t_*)$ at $e = 0.9$.

In the region I, the solution of zero approximation is

$$F_1^{(0)}(t_*) = -c + \left\{c^2 - \frac{1}{2}u + \left[\frac{u^2}{4} - \alpha_0\right]^{1/2}\right\}^{1/2}, \tag{35}$$

if in the notations (34) we make the replacement $\alpha_1 \to -\alpha_1$, $\alpha_3 \to -\alpha_3$.

The time dependence of functions $F_1^{(0)}(t_*)$ and $F_2^{(0)}(t_*)$ at $e = 0.9$ and eccentric anomaly in this approximation are shown in Figure 1. The maximal deviation $E^{(0)}(t_*)$ from the equation solution (13) found numerically does not exceed $5 \cdot 10^{-4}$.

3. The hyperbolic motion

Using the substitution

$$\tan\frac{v}{2} = \left(\frac{e+1}{e-1}\right)^{1/2} \tanh\frac{H}{2}, \tag{36}$$

equation (8) reduces to the form

$$e \cdot \sinh H - H = t_{\text{H}}, \tag{37}$$

where the auxiliary function H is the analog of eccentric anomaly. Herewith $-\infty < t_{\text{H}} < \infty$, and $-\infty < H < \infty$. However, $H(-t_{\text{H}}) = -H(t_{\text{H}})$, then it is enough to find the solution of equation (37) in the region $0 \leqslant t_{\text{H}} < \infty$. Unfortunately, the function $\sinh H$ is not possible to approximate a polynomial of low degree in a sufficiently wide region of change H, because $\sinh H$ increases exponentially at $H \gtrsim 2.5$. However, approximation is possible for the region near the pericenter, which has the greatest practical interest. For this region, we will rewrite equation (37) in the form

$$e\, f_3(H) - H - t_{\text{H}} = e\,\{f_3(H) - \sinh H\}, \tag{38}$$

and be choosing the approximation function in the form

$$f_3(H) = H + a\, H^3 \tag{39}$$

at $a = 0.188479$. The solution of equation (38), we search by iteration method using the root of equation in the role of zero approximation

$$e\, f_3\big(H^{(0)}\big) - H^{(0)} - t_{\text{H}} = 0, \tag{40}$$

which determined by the Cardano's formula,

$$\begin{aligned} &H^{(0)}(t_{\text{H}}) = \big\{(g^2+q^3)^{1/2} + g\big\}^{1/3} - \big\{(g^2+q^3)^{1/2} - g\big\}^{1/3}, \\ &g = \frac{t_{\text{H}}}{2e\,a}, \quad q = \frac{e-1}{3e\,a}. \end{aligned} \tag{41}$$

Refinement of this solution can be performed by the method of iterations of equation (38),

$$\begin{aligned} &H^{(1)}(t_{\text{H}}) = H^{(0)}\big(t_{\text{H}}^{(1)}\big), \\ &t_{\text{H}}^{(1)} = t_{\text{H}} + e\,\big\{f_3(H^{(0)}(t_{\text{H}})) - \sinh H^{(0)}(t_{\text{H}})\big\}; \\ &H^{(2)}(t_{\text{H}}) = H^{(0)}\big(t_{\text{H}}^{(2)}\big), \\ &t_{\text{H}}^{(2)} = t_{\text{H}} + e\,\big\{f_3\big(H^{(0)}(t_{\text{H}}^{(1)})\big) - \sinh H^{(0)}\big(t_{\text{H}}^{(1)}\big)\big\}, \end{aligned} \tag{42}$$

and etc. The iterative algorithm based on equation (37) has even better convergence,

$$\begin{aligned} H^{(1)}(t_{\text{H}}) &= \operatorname{arsinh}\left\{\frac{1}{e}\big[t_{\text{H}} + H^{(0)}(t_{\text{H}})\big]\right\}, \\ H^{(2)}(t_{\text{H}}) &= \operatorname{arsinh}\left\{\frac{1}{e}\big[t_{\text{H}} + H^{(1)}(t_{\text{H}})\big]\right\}, \end{aligned} \tag{43}$$

and etc. In Table 1, it is shown the convergence of iterative process of equation (43) for the case $e = 1.4$. In the second column of Table, the solution of equation (37), found by the numerical method, is given.

Table 1. Dependence of the function $H(t_{\text{H}})$ on time in different approximations.

t_{H}	$H^{(\text{num})}(t_{\text{H}})$	$H^{(0)}(t_{\text{H}})$	$H^{(1)}(t_{\text{H}})$	$H^{(2)}(t_{\text{H}})$
0.5	0.86210	0.84789	0.85481	0.85836
1.0	1.25444	1.24056	1.24919	1.25246
1.5	1.50824	1.50432	1.50706	1.50789
2.0	1.69869	1.70874	1.70123	1.69933
3.0	1.98161	2.02469	1.98991	1.98321

4. An alternative approach for the description of Keplerian motion

The described above methods of calculating of the coordinate time dependence of material point in Keplerian motion based on the determination of time dependence of angular variable – true anomaly $v(t)$. If the function $v(t)$ is known, then the time dependence of radial coordinate $\rho(t)$ is determined with help of orbit equation. Such an algorithm is generally accepted. But at the same time, the physical reason is veiled for the existence of different types of motion – elliptical, parabolic, hyperbolic or one-dimensional radial. Because of that, it is useful to derive the relation that corresponds to the time dependence of radial coordinate, directly from equations of relative motion.

Using the energy integral of relative motion $\mathcal{E}$, which is written in the polar coordinates,

$$\mathcal{E} = \frac{\mu}{2}\left(\dot{\rho}^2 + \rho^2\dot{v}^2\right) - \frac{\mu K}{\rho}, \tag{44}$$

and the integral of angular momentum (3), we obtain the relation

$$\mathcal{E} = \frac{\mu}{2}\,\dot{\rho}^2 + V_{\text{eff}}(\rho), \tag{45}$$

where

$$V_{\text{eff}}(\rho) = -\frac{\mu K}{\rho} + \frac{l^2}{2\mu\rho^2} \equiv \frac{\mu K}{p}\left\{-\frac{1}{\xi} + \frac{1}{2\xi^2}\right\} \tag{46}$$

is the effective potential energy of relative motion, and $\xi = \rho/p$ is the dimensionless radial coordinate.

From relations (44)–(46), we find equation for the time dependence of radial coordinate

$$\frac{d\xi}{dt} = \pm\left(\frac{2K}{p^3}\right)^{1/2}\left\{\tilde{\mathcal{E}} + \frac{1}{\xi} - \frac{1}{2\xi^2}\right\}^{1/2}, \tag{47}$$

where $\tilde{\mathcal{E}} = \mathcal{E}(\mu K)^{-1}p$ is the dimensionless energy of relative motion. The condition

$$\tilde{\mathcal{E}} + \frac{1}{\xi} - \frac{1}{2\xi^2} = 0 \tag{48}$$

determines the turning points that have restricted the region of motion relative to the radial variable. The minimum of potential of equation (46) will be found at $\xi = 1/2$ that corresponds to a circular motion with radius $R = p$, and $\tilde{\mathcal{E}} = -1/2$. At the elliptical motion ($\mathcal{E} = -\mu K(2a)^{-1}$, where a is the semimajor axis of ellipse)

$$\tilde{\mathcal{E}} = -\frac{p}{2a} = \frac{e^2-1}{2} < 0, \tag{49}$$

and the radial coordinate varies in the region

$$\xi_1 \leqslant \xi \leqslant \xi_2, \quad \xi_1 = \frac{1}{1+e}, \quad \xi_2 = \frac{1}{1-e}. \tag{50}$$

At the parabolic motion

$$\frac{1}{2} \leqslant \xi < \infty; \quad \tilde{\mathcal{E}} = 0, \tag{51}$$

and at the hyperbolic ($e > 1$)

$$\tilde{\mathcal{E}} = \frac{e^2-1}{2} > 0, \quad \frac{1}{e+1} \leqslant \xi < \infty. \tag{52}$$

Assuming that the material point is at the pericenter at the moment $t = 0$, from equation (47), we find the time dependence of radial coordinate, which is written in the quadratures,

$$\left(\frac{K}{p^3}\right)^{1/2} t = \int_{\xi_1}^{\xi} \frac{x\,dx}{\left[(e^2-1)\,x^2 + 2x - 1\right]^{1/2}}, \tag{53}$$

that the analog of equation (4).

In the case of elliptical motion ($0 < e < 1$), equation (53) reduces to the transcendental equation

$$2\arctan\left\{\left(\frac{1-e}{1+e}\right)^{1/2}\frac{\left[(e^2-1)\,\xi^2+2\xi-1\right]^{1/2}}{1+\xi\,(e-1)}\right\} - (1-e^2)\left[(e^2-1)\,\xi^2+2\xi-1\right]^{1/2} = t_*. \tag{54}$$

To analyze and find the solutions of equation (54), it is convenient to pass from the eccentricity to the dimensionless turning points $\xi_1 = (1+e)^{-1}$, $\xi_2 = (1-e)^{-1}$. Taking into account that

$$\begin{aligned}&\left[(e^2-1)\,\xi^2+2\xi-1\right]^{1/2} = (\xi_1\xi_2)^{-1/2}[(\xi-\xi_1)(\xi_2-\xi)]^{1/2},\\&1+\xi\ (e-1) = 1-\xi/\xi_2,\end{aligned} \tag{55}$$

we rewrite equation (54) in the form, that is convenient to find the approximate analytical solution, namely

$$\frac{\xi-\xi_1}{\xi_2-\xi} = \tan^2\left\{\frac{1}{2}\left[t_* + (\xi_1\xi_2)^{-1}(\xi-\xi_1)^{1/2}(\xi_2-\xi)^{1/2}\right]\right\}. \tag{56}$$

Let us note that equation (54) is equivalent to equation (6), which is easy to see if we pass from the variable $\xi = \xi(t)$ to true anomaly $v(t)$, using the orbit equation. But equation (54) is convenient for finding the approximate analytical solutions. One of the simplest variants of this approach is the iterative method.

Neglecting the second term of the square brackets in equation (56), we find the zero approximation

$$\xi^{(0)}(t_*) = \xi_1\cos^2\left(\frac{1}{2}t_*\right) + \xi_2\sin^2\left(\frac{1}{2}\ t_*\right), \tag{57}$$

which is an exact in the vicinity of points $\xi\ (t_*) = \xi_1$, $\xi\ (t_*) = \xi_2$. In the first iteration

$$\begin{aligned}&\xi^{(1)}(t_*) = \xi^{(0)}\big(t_*^{(1)}\big),\\&t_*^{(1)} = t_* + \frac{\xi_1\xi_2}{\big(\xi^{(0)}(t_*)-\xi_1\big)^{1/2}\big(\xi_2-\xi^{(0)}(t_*)\big)^{1/2}}.\end{aligned} \tag{58}$$

In general $n \geqslant 2$

$$\begin{aligned}&\xi^{(n)}(t_*) = \xi^{(0)}\big(t_*^{(n)}\big),\\&t_*^{(n)} = t_* + \frac{\xi_1\xi_2}{\big(\xi^{(0)}(t_*^{(n-1)})-\xi_1\big)^{1/2}\big(\xi_2-\xi^{(0)}(t_*^{(n-1)})\big)^{1/2}}.\end{aligned} \tag{59}$$

The speed of convergence of the iterative process illustrates in Figure 2. Curve 1 in this Figure corresponds to the approximation (57), curve 2 to the approximation (58), curve 3 to the approximation (59) at $n = 3$. The relative deviation of curve 3 from the numerical solution of equation (56) does not exceed 0.03%.

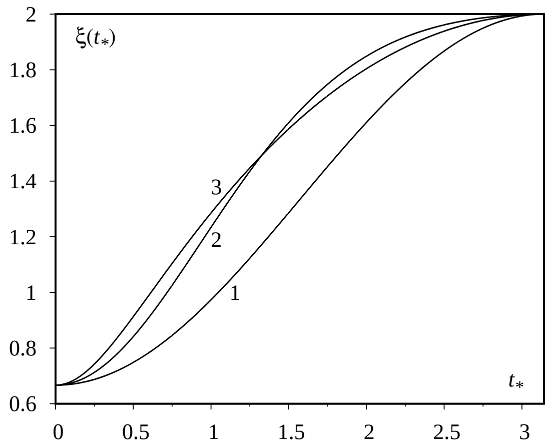

Fig. 2. Dependence of the function $\xi(t_*)$ on the variable t_* in different approximation at $e = 0.5$. Curve 1 corresponds to the approximation (57), curve 2 corresponds to the approximation (58) and curve 3 corresponds to the approximation (59) at $n = 3$.

If we use the orbit equation and pass from the variable $\xi(t_*)$ to true anomaly $v(t)$, then we see that

$$(\xi_1\xi_2)^{-1}(\xi(t_*)-\xi_1)^{1/2}(\xi_2-\xi(t_*))^{1/2} = (1-e^2)^{1/2}\sin v(t)\{1+e\cdot\cos v(t)\}^{-1} = e\cdot\sin E(t_*), \tag{60}$$

where $E(t_*)$ is eccentric anomaly. Using the analytical solution of the Kepler's equation in the form (33) and (35), the solution of equation (56) we can rewrite in the form

$$\begin{aligned}\xi(t_*) &= \xi_1\cos^2\left[\frac{1}{2}(t_*+e\cdot\sin E(t_*))\right] + \xi_2\sin^2\left[\frac{1}{2}(t_*+e\cdot\sin E(t_*))\right]\\&= (1-e^2)^{-1}\{1-e\cdot\cos[t_*+e\cdot\sin E(t_*)]\} = (1-e^2)^{-1}\{1-e\cdot\cos E(t_*)\}.\end{aligned} \tag{61}$$

It follows from here that equation (56) is equivalent to the Kepler's equation, and in the role of zero approximation of equation (56) we can use expression (61) with the eccentric anomaly in the form of the equation of solution (20).

In the parabolic motion ($e = 1$), equation (53) takes the form of cubic algebraic equation

$$\frac{1}{3}\Big(1+\xi\Big)^2\Big(2\xi-1\Big) = \left[\frac{K}{p^3}\right]\cdot t^2. \tag{62}$$

Its solution is rewritten in the form

$$\xi(t)=\frac{1}{2}\left\{-1+\left[1+2s^2(t)+2s(t)(1+s^2(t))^{1/2}\right]^{1/3}+\left[1+2s^2(t)-2s(t)(1+s^2(t))^{1/2}\right]^{1/3}\right\} \tag{63}$$

and has the asymptotics

$$\xi(t)\to\begin{cases}2^{-1}+\dots & \text{at} \quad t\to 0,\\ 2^{-1/3}\ s^{2/3}(t)+\dots=2^{-1/3}3^{2/3}K^{1/3}p^{-1}t^{2/3}+\dots & \text{at} \quad t\to\infty.\end{cases} \tag{64}$$

As a result of integration over the variable x, in the case of hyperbolic motion, equation (53) takes the form

$$D-\operatorname{arsinh}\left(\tfrac{D}{e}\right)=t_{\text{H}}. \tag{65}$$

Here, the auxiliary function is introduced

$$D(t_{\text{H}})=\left\{\left[1+\xi(t_{\text{H}})(e^2-1)\right]^2-e^2\right\}^{1/2}, \tag{66}$$

from which it follows that the solution of equation (53) is rewritten in the form

$$\xi(t_{\text{H}})=(e^2-1)^{-1}\left\{\left[D^2(t_{\text{H}})+e^2\right]^{1/2}-1\right\}. \tag{67}$$

From equation (65), we obtain the asymptotics

$$D(t_{\text{H}})\to\begin{cases}t_{\text{H}}(1+e^{-1})+\dots & \text{at} \quad t_{\text{H}}\ll e,\\ t_{\text{H}}+\ln(2e^{-1}t_{\text{H}})+\dots & \text{at} \quad t_{\text{H}}\gg e.\end{cases} \tag{68}$$

It determines the asymptotics of the equation solution (53) at $e>1$ in the form

$$\xi(t_{\text{H}})\to\begin{cases}\xi_1+2^{-1}t_{\text{H}}^2e^{-3}\ (e+1)(e-1)^{-1}+\dots & \text{at} \quad t_{\text{H}}\ll e,\\ \xi_1(e-1)^{-1}[t_{\text{H}}+\ln\,(2t_{\text{H}}e^{-1})]+\dots & \text{at} \quad t_{\text{H}}\gg e.\end{cases} \tag{69}$$

As with finding solutions of the Kepler's equation (13), in the case of hyperbolic motion, it is also appropriate to consider the solution of equation (65) in two regions: in the region of fast change of the function $D(t_{\text{H}})$ (in the pericenter vicinity, at small values of t_{H}), and in the region far from the pericenter. In the first region, we will use the method of approximating functions. The function

$$\operatorname{arsinh}\left(\tfrac{D}{e}\right)\equiv\ln\left[\tfrac{D}{e}+\left[1+\left(\tfrac{D}{e}\right)^2\right]^{1/2}\right] \tag{70}$$

in equation (65), we approximate by the polynomial of a third degree

$$\begin{gathered}f_3\left(\tfrac{D}{e}\right)=\tfrac{D}{e}+a_2\left(\tfrac{D}{e}\right)^2+a_3\left(\tfrac{D}{e}\right)^3,\\ a_2=-0.124789, \quad a_3=-0.00596395,\end{gathered} \tag{71}$$

and equation (65), we represent in the equivalent form

$$t_{\text{H}}-D+f_3\left(\tfrac{D}{e}\right)=f_3\left(\tfrac{D}{e}\right)-\operatorname{arsinh}\left(\tfrac{D}{e}\right). \tag{72}$$

In the role of zero approximation, we will choose the root of equation of third degree

$$f_3\left(\tfrac{D}{e}\right)-D+t_{\text{H}}=0. \tag{73}$$

By the Cardano's formulae we find that equation (73) has three real roots, two of them are negative. The third root is a monotonically increasing function of time, that has the zero asymptotics at $t_{\text{H}}\to 0$ and corresponds to the physical content of the problem

$$D^{(0)}(t_{\text{H}})=e\left\{2|q|^{1/2}\cos\left[3^{-1}\arctan\left([|q|^3-r^2]^{1/2}r^{-1}\right)\right]-\frac{1}{3}b_2\right\}. \tag{74}$$

Here are used the notations

$$\begin{gathered}q=\frac{b_1}{3}-\frac{b_2^2}{9}, \quad r=\frac{1}{6}\left\{b_1b_2-3b_0-\frac{b_2^3}{27}\right\},\\ b_1=\frac{1-e}{a_3}, \quad b_2=\frac{a_2}{a_3}, \quad b_0=\frac{(e^2-1)^{3/2}}{a_3}t_p.\end{gathered} \tag{75}$$

To specify the found solution, we use equation (65). By the analogy with formulae (43) in the case of the hyperbolic motion

$$\begin{aligned} D^{(1)}(t_{\text{H}}) &= t_{\text{H}} + \operatorname{arsinh}\left(\frac{D^{(0)}(t_{\text{H}})}{e}\right), \\ D^{(n)}(t_{\text{H}}) &= t_{\text{H}} + \operatorname{arsinh}\left(\frac{D^{(n-1)}(t_{\text{H}})}{e}\right) \end{aligned} \tag{76}$$

at $n \geqslant 2$. In Table 2 is illustrated the convergence of iterative process (76) for the case $e = 1.3$.

Table 2. Dependence of the function $D(t_{\text{H}})$ on time in different approximations.

t_{H}	$D^{(\text{num})}(t_{\text{H}})$	$D^{(0)}(t_{\text{H}})$	$D^{(1)}(t_{\text{H}})$	$D^{(3)}(t_{\text{H}})$
0.5	0.98724	0.94571	0.96161	0.97755
1.0	1.61681	1.60773	1.61243	1.61579
1.5	2.12821	2.14193	2.13370	2.12909
2.0	2.58466	2.59917	2.58967	2.58526
3.0	3.41202	3.36914	3.40021	3.41113

In the region that is far from the pericenter, it is possible to use the ordinary iterations of equation (65),

$$\begin{aligned} D^{(0)}(t_{\text{H}}) &= t_{\text{H}}, \\ D^{(1)}(t_{\text{H}}) &= t_{\text{H}} + \operatorname{arsinh}\left(\frac{t_{\text{H}}}{e}\right), \\ D^{(n)}(t_{\text{H}}) &= t_{\text{H}} + \operatorname{arsinh}\left(\frac{D^{(n-1)}(t_{\text{H}})}{e}\right). \end{aligned} \tag{77}$$

In Figure 3, the convergence of such iterative algorithm is illustrated at $e = 1.3$.

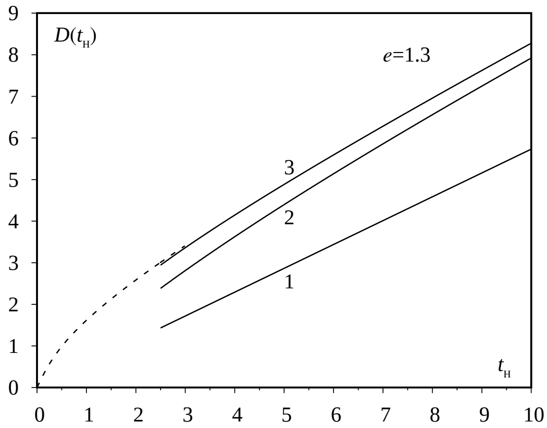

Fig. 3. Dependence of the function $D(t_{\text{H}})$ on time t_{H} in the region far from the pericenter in different approximation at $e = 1.3$. Curve 1 corresponds to the zero approximation of $D^{(0)}(t_{\text{H}})$, curve 2 – $D^{(1)}(t_{\text{H}})$, curve 3 – $D^{(4)}(t_{\text{H}})$. The dashed curve is the exact solution of equation (65).

5. Conclusions

The determination of time dependence of true anomaly in the two-bodies problem is the traditional approach in the theory of Keplerian motion. We have proposed a fast-converging algorithm for finding solutions of the Kepler's equation (13) (or (26)) and its analogue (37). In the role of zero approximations are used solutions of algebraic equations (30) or (40), obtained by replacing $\cos F$ or $\sinh H$ by approximating polynomials $f_4(F)$ or $f_3(H)$. In such approach, the role of small parameter plays the differences $\cos F - f_4(F)$ or $\sinh H - f_3(H)$, which insures the fast convergence of iterations regardless of the eccentricity.

The alternative approach consists of independently determining the time dependence of the radial coordinate. In the case of elliptical motion, this leads to equation (56), which equivalent to the Kepler's equation. As it is shown in Figure 2, the approximate solution of this equation can be obtained with help of ordinary iterations, using in the role of zero approximation expression (57). In the case of hyperbolic motion, the time dependence of radial variable is determined by equations (65) and (67). To find the solution of equation (65) near the pericenter, it is convenient to use the method of approximating functions, the convergence of which is illustrated in Table 2. In the region far from the pericenter, it is enough to use the method of ordinary iterations, which can be seen in Figure 3.

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Аналітичні зображення часової залежності координат кеплерівського руху

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З метою розрахунку часової залежності полярних координат відносного руху у задачі двох тіл в аналітичній формі запропоновано варіанти ітераційних алгоритмів зі швидкою збіжністю, що грунтуються на використанні апроксимуючих функцій. Крім того, показано, що при незалежному визначенні часової залежності радіальної координати в еліптичному русі, а також на великих віддалях від перицентру у випадку гіперболічного руху добру збіжність дає метод звичайних послідовних ітерацій.

Ключові слова: *задача двох тіл; часова залежність координат; аналітичні методи.*