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MODELING OF FREE AND FORCED LONGITUDINAL OSCILLATIONS OF A STRAIGHT ROD USING THE DIRECT DISCRETIZATION METHOD

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Abstract. In the study of the dynamics of mechanical systems with distributed parameters, the discretization of partial differential equations by the finite element method, finite difference method, or boundary element method is widely used. This makes it possible to simplify the analysis of dynamic processes by reducing it to solving a system of ordinary differential equations.

The aim of this paper is to mathematically substantiate the method of direct discretization of long-dimensional elastic links as applied to free and forced longitudinal oscillations of a straight rod as a mechanical system with distributed parameters, and to investigate the convergence of the method in determining the natural frequencies and amplitudes of forced oscillations of mechanical systems.

The method of direct discretization is considered in the application to a rod with distributed parameters, which performs longitudinal oscillations. It is shown that by mathematical discretization of the wave equation, the boundary value problem can be reduced to the analysis of oscillatory phenomena in a chain mechanical system of material points. The formulas for determining the masses of the inertial elements of the computational model and the stiffness coefficients of the elastic connections are given.

The convergence of the method is investigated by determining the natural frequencies of oscillations of a rod with free ends, as well as with one end clamped and the other end free, and comparing the approximate results of calculations with the exact results. The effect of the number of degrees of freedom, rod length, and material density on the eigenfrequencies is investigated. It is found that to ensure sufficient accuracy for engineering practice in determining the three or four lowest natural frequencies, computational models with 8–10 degrees of freedom can be used. It is emphasized that the cross-sectional area of the rod does not affect the characteristics of the frequency spectrum of the mechanical system. The zero value of the lowest natural frequency of an elastic rod with free ends can be explained by the fact that this value corresponds not to the oscillatory but to the translational motion of a mechanical system that is not connected to the base.

Forced longitudinal oscillations of an elastic rod, one end of which is clamped at the base and the other end is free, under power and kinematic excitation are considered. The amplitude-frequency characteristics of mechanical systems are constructed on the examples of the dependences of the movement amplitude of one of the material points and the amplitude of the longitudinal force in one of the elastic links of the computational model on the frequency of forced oscillations. Since the dissipation of the mechanical energy of the system during its oscillations is not taken into account, as the frequency of forced oscillations approaches one of the natural frequencies of the system, the movement amplitudes, as well as the amplitudes of internal forces, tend to infinity. The actual values of resonant amplitudes are not of practical interest, since resonant modes of operation are not expected for the vast majority of machine units.

The scientific novelty of the results obtained is that the mathematical substantiation of the method of direct discretization of the rod in its application to its longitudinal oscillations, the creation of computational algorithms and the study of the convergence of the results of determining the natural frequencies and amplitudes of forced oscillations can be considered as a development of the computer methodology for the dynamic analysis of mechanical systems with distributed parameters.

The practical significance and scope of further research of the results of this work are based on the wide possibilities of practical application of the direct discretization method in the study of dynamic processes in rather complex mechanical systems of such technical objects as drilling rigs, cranes, mine hoisting machines, conveyors, rotary kilns, etc. This is facilitated by the simplicity and computational efficiency of the method in solving problems of dynamics of both linear and nonlinear mechanical systems. The method can be applied in the study of both stationary and transient dynamic processes in machine units.

Keywords: mechanical oscillations, systems with distributed parameters, discretization, modal analysis, forced oscillations, power excitation, kinematic excitation, convergence of calculation results.

Introduction

Currently, much attention is paid to the study of the dynamics of machines and engineering structures, as scientific developments in this area form the theoretical basis for optimal design. These studies are mainly aimed at reducing dynamic loads in drive mechanisms and load-bearing structures, reducing the oscillation activity of machines and their components, predicting the service life of parts and critical components, ensuring the stability of operating modes, and improving control systems for machine units. A special place in this research is given to the study of oscillatory phenomena caused by the instability of driving and drag forces, the nonlinearity of elastic and inertial characteristics of links, unbalanced rotating elements, and manufacturing and assembly errors of individual parts and assemblies.

An important stage of research is the equivalent representation of a machine unit by a simplified mechanical system, which is commonly referred to as a computational model or a computational scheme. In scientific and engineering practice, discrete (with a finite number of degrees of freedom), as well as continuum and continuum-discrete (with an infinite number of degrees of freedom) design models of machines and their elements are widely used. A significant advantage of discrete computational models is that their motion is described by ordinary differential equations, the mathematical theory of which is sufficiently developed. This greatly simplifies the analysis of oscillatory processes in machine units. However, for mechanisms and machines that have links with pronounced distributed parameters (long shafts, hoisting ropes, conveyor branches, pipe columns, booms, masts, trusses, etc.), it is advisable to use continuum-discrete computational models, since wave phenomena in large links significantly affect the dynamics of mechanical systems. The motion of such computational models is described by a set of ordinary differential equations and partial differential equations, which greatly complicates the study, especially in the presence of nonlinear factors. Nevertheless, this approach contributes to a significant increase in the accuracy of the calculation of dynamic processes, and in many cases is the only possible solution to the actual problems of the dynamics of machine units.

In the study of mechanical systems with distributed parameters, discretization of partial differential equations by the finite difference method or the finite element method is widely used. This makes it possible to simplify the analysis of dynamic processes by reducing it to solving a system of ordinary differential equations. The transition to a discrete mathematical model must be carried out by ensuring its compliance with the original model. Therefore, considerable attention is paid to this problem in the modern literature.

Problem Statement

The aim of this paper is to mathematically substantiate the method of direct discretization of long-dimensional elastic links as applied to free and forced longitudinal oscillations of a straight rod as a mechanical system with distributed parameters, and to investigate the convergence of the method in determining the natural frequencies and amplitudes of forced oscillations of mechanical systems.

To achieve this goal, the following tasks are considered:

– reduction of the boundary value problem for the wave equation with appropriate boundary and initial conditions to the Cauchy problem for a system of ordinary differential equations;

– to investigate the convergence of the direct discretization method on the examples of determining the natural frequencies of mechanical systems;

– to illustrate the method, to develop mathematical models of forced oscillations of mechanical systems with power and kinematic excitation and to calculate the amplitude-frequency characteristics of these systems.

Review of Information Sources on the Subject of the Article

The design model of a long-dimensional structure that performs longitudinal oscillations, such as a drill pipe string, conveyor line, mine hoist rope, etc., is usually a straight rod of constant cross-section.

The book [1] systematically describes the fundamentals of the theory of oscillations and stability of mechanical systems with a finite number of degrees of freedom, as well as systems with distributed parameters. Considerable attention is paid to the mathematical modeling of longitudinal, torsional, and transverse oscillations of rods. Such important aspects of the problem as oscillation energy dissipation, peculiarities of taking into account external and internal friction, oscillations of multi-span rods, rods with variable parameters along the length, and rods with a curved axis are considered. Most of the problems are formulated and solved in a deterministic formulation, however, the peculiarities of solving problems in a probabilistic formulation are also considered.

Book [2] deals with the dynamics of structures and the problems of designing buildings and structures for seismic effects. Free and forced oscillations of systems with one and many degrees of freedom under various power and kinematic loads are considered. The results of studies of seismic resistance of structures are presented. The most common problems of the dynamics of engineering structures and facilities are considered. Considerable attention is paid to the numerical analysis of oscillatory processes. Among the known methods of discretization of systems with distributed parameters, preference is given to direct discretization, generalized displacements, and finite element methods.

Book [3] describes the theoretical foundations of the finite difference method for solving partial differential equations. This book is aimed at practitioners. Accordingly, it is especially concerned with the construction of finite-difference schemes for differential equations, the formulation and implementation of computational algorithms, verification of implementation results, and analysis of the behavior of physical systems and software for solving various practical problems.

The book [4] covers finite-difference methods for solving parabolic, hyperbolic, and elliptic equations and includes theoretical information on consistency, stability, and convergence of solutions. Attention is paid to the study of stability based on the Lax-Richtmeyer definition and the application of Pade approximations to ordinary differential equations, as well as parabolic and hyperbolic partial differential equations. Iterative methods are considered and improved.

Article [5] discusses the possibility of using numerical approaches, such as finite difference methods and finite element methods, to approximate accurate one-dimensional continuous eigenvalue problems (transverse oscillations of a string, longitudinal or torsional oscillations of a rod, bending of elastic columns). Numerical methods first transform partial differential equations into difference equations, and after performing a continuation procedure or applying a differential approximation method, the difference operators are decomposed into differential operators using Taylor expansion or Pade approximation. Comparisons between the exact discrete eigenvalue problems and the approximated continuous ones show the effectiveness of the continuation procedure.

In [6], a mathematical model and a computer algorithm for analyzing the dynamic processes occurring during the release of a drill pipe string trapped in a well were created. The pipe string is released by means of an impactor or a pulse-wave unit equipped with an electric linear-pulse motor. The drill string with the percussion mechanism, which is installed above the grabbed area, is driven by the drilling rig drive. The string is considered a continuously discrete mechanical system. As a result of the impact of the striker on the body of the impactor, wave processes are formed in the pipe string, which contribute to the release of the trapped drill string. The influence of friction forces on the propagation of longitudinal waves in a drill pipe string is investigated. Practical recommendations for improving the efficiency of oil and gas well drilling are proposed.

The paper [7] considers the problem of determining the resistance forces that impede the drill string extension in a deep curved borehole channel. It is assumed that the geometry of the borehole axis is set discretely at its individual points using the results of geophysical measurements (borehole navigation). A “three-dimensional differential model of a rigid string” is proposed to model the phenomenon of resistance accompanying the operations of raising, lowering, and deepening a well. Based on the theory of curved flexible elastic rods, a system of ordinary differential equations is derived. The transition from the tabular to the analytical description of the geometry of the well trajectory is performed using cubic spline interpolation. The developed approach can be useful for modeling the rotating drillstring pulling, its contact and frictional interaction with the borehole surface, and predicting the situations of blocking the string. Numerous examples illustrate the advantages of the proposed methods.

Work [8] notes that oscillations of rotating parts of a drilling rig often lead to excessive wear of the drill pipe string and premature failure of drilling equipment. In this regard, there is a need to identify the causes of oscillation phenomena and timely make the necessary changes to the design of rig elements to protect equipment and machinery from oscillations. It is believed that the cost of well drilling increases by at least 10 % due to the impact of unforeseen shocks and oscillation of the drill string. The paper analyzes the known results of modeling the drill string oscillation and controlling its oscillation state during well drilling. Models for predicting axial, transverse, and torsional oscillations are proposed. Theoretical and experimental methods for estimating oscillation parameters, as well as ways to reduce their levels, in particular, active and passive control, are discussed.

Drillstring oscillations significantly affect drilling performance, cause bit damage, failure of the bottomhole string, excessive twisting of the locking joints, fatigue loss of the string strength due to torsional loads, etc. [9]. It has been established that the sources of torsional oscillations are the interaction of the wellbore with the drillstring and the interaction of the wellbore with the bit. Real-time analysis of the drill string dynamics is a prerequisite for efficient drilling of oil, gas, and geothermal wells. Paper [9] proposes a mathematical model of torsional oscillations of a drill string based on nonlinear differential equations written separately for the drill pipe string and the bottom hole string. The interaction of the bit with the rock is represented by nonlinear friction forces. The influence of the tool rotation speed and bit load on torsional oscillations was studied. The influence of the stiffness and inertia of the column on its torsional oscillations and the risk of entrapment was determined. The effect of changes in rock strength on the rate of penetration, taking into account the drilling parameters, is also investigated. It is emphasized that the theoretical results confirm the real trends observed in the fields.

Paper [10] considers the effect of drill pipe string oscillations excited by one or more oscillators on reducing the resistance force to the movement of the string. The results show that the excitation of the string oscillations facilitates its movement in the well and better transfer of the load to the bit. The use of oscillations significantly reduces the average coefficient of friction between the drillpipe and the borehole wall. When the energy transferred from the vibrator to the drillstring is constant, oscillations of high amplitude and low frequency are favorable for reducing friction forces. If the other parameters are constant, increasing the amplitude or increasing the number of oscillators improves drilling efficiency and increases the rate of well deepening. The study is important for optimizing the design and safe operation of the drill string. Numerical methods for analyzing dynamic processes were used in the study.

In [11], a model was built using the finite element method to study the spatial oscillations of a drill string with elements having six degrees of freedom. In addition, a comprehensive model of the forces of interaction between the bit and the bottom hole was developed, which serves as a boundary condition of the model, with respect to the oscillations of the lower end of the string in the axial, lateral, and rotational directions. This model takes into account the eccentricity of the drillstring bottom layout, drilling mud damping, bit-rock interaction, and the mechanisms of connection between these factors. The modeling results showed good agreement with field observations and experimental data reported in the literature. The practical significance of the model is illustrated by examples of normal operation of the tool, its oscillations in the sticking-sliding mode and in the vortex oscillation mode.

Paper [12] proposes a strategy for modeling dynamic processes in a drill pipe string, according to which the mathematical model includes a description of the string oscillations as an elastic body of considerable length in a wide frequency band and assumes that the main nonlinear effects are caused by localized areas of strong nonlinearity, for example, the area of interaction between the rock and the drill bit, the area of stabilizer installation, where the distance between the string and the well wall is the smallest, etc. The equations of motion of the string can be written as for a system with a limited number of degrees of freedom, taking into account nonlinear contact laws and subsequent numerical integration in the time domain. Two ways of realizing this approach are presented: with the use of digital filters and with the use of the finite element method to describe dynamic phenomena. Several cases of formulating and solving problems demonstrating various phenomena are considered, in particular, oscillations of the stick-slip type; the occurrence of forward and reverse vortices. Parametric studies are carried out to identify the conditions that lead to oscillations with large amplitudes.

Paper [13] proposes an effective approach to modeling dynamic processes in well drilling machines. Existing methods require a large amount of computation, in particular, the finite element method, or provide low accuracy in the case of complex geometry, for example, in the case of using transfer matrices. In order to take advantage of the advantages of these methods and avoid their disadvantages, a hybrid method is proposed in this paper that combines both of the above modeling approaches due to the unique structural geometry of the drilling system. The proposed method is applied to the modeling of dynamic processes in a well drilling machine, taking into account the dynamic properties of the upper drive, drill string, weighted lower string, and bit-rock interaction.

Paper [14] considers nonlinear lateral oscillations of a drill string in wells with a curved axis. One of the most important features of this work is that the drillstring oscillation in curved wells (including both vertical, inclined, and horizontal well designs) is modeled for real drilling conditions. In the proposed model, the finite element method is used to calculate the oscillations of a drillstring in curved wells. The formulated problem was solved theoretically and verified by reproducing industrial results. The model combines lateral, torsional, and longitudinal oscillations of a drillstring in curved wells, whereas traditionally, attention is paid only to the nonlinear transverse oscillation of the string. For the first time, the effective length of the string for studying lateral oscillation is proposed, and the influence of rotational speed, bit force (WOB), coefficient of friction (COF), and stabilization (STB) on lateral oscillation of a drill string in curved wells is determined.

In recent years, composite drillstring has been developed as a high-tech rotor with complex dynamic behavior to overcome many limitations of drilling operations [15]. In this study, the fully coupled nonlinear oscillation of composite drillstring consisting of orthotropic layers is investigated using the Lagrangian approach and the finite element method.

As follows from the literature review, numerical analysis methods, in particular, the finite difference method and the finite element method, are widely used in the scientific and engineering fields. The practical application of these methods for modeling and analyzing oscillatory processes in mechanical systems with distributed parameters makes it possible to significantly simplify research problems by switching from partial differential equations to ordinary differential or algebraic equations. The increase in the order of the system of equations associated with this transition is overcome by the rapid development of computer technology, as well as mathematical and software of computer-aided design systems. The most important criteria for the quality of computational algorithms include their reliability and ease of implementation. It is in this vein that the purpose and objectives of this study are formulated.

Main material

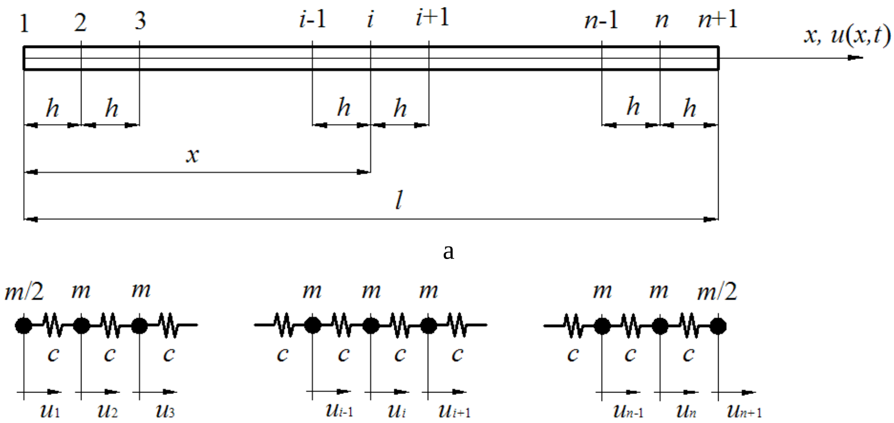
Theoretical justification of the discretization method. Consider the longitudinal oscillations of an elastic rod of length *l* with free ends (Fig. 1, *a*). The displacement of an arbitrary cross-section of the rod in the direction of the *x*-axis is denoted as $u(x, t)$, where *t* is time. The cross-sectional area, the modulus of elasticity of the first kind, and the density of the material are denoted as *A*, *E*, and ρ.

The motion of an elastic rod in the longitudinal direction is described by the wave equation [1],

$$\frac{\partial^2 u}{\partial t^2} - a^2 \frac{\partial^2 u}{\partial x^2} = 0\,, \qquad (1)$$

where *a* is the propagation velocity of the longitudinal strain wave,

$$a = \sqrt{\frac{E}{\rho}}\,. \qquad (2)$$



b

Fig. 1. Unloaded elastic rod with free ends (a)
and its discrete computational model (b)

The boundary conditions at the left and right ends, respectively, are as follows

$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = 0\,; \qquad \left.\frac{\partial u}{\partial x}\right|_{x=l} = 0\,. \qquad (3)$$

The initial conditions for the integration of equation (1) are written in the form

$$u(x,0) = s(x)\,; \qquad \left.\frac{\partial u}{\partial t}\right|_{t=0} = v(x)\,, \qquad (4)$$

where $s(x)$ and $v(x)$ are the functions that determine the initial displacements and initial velocities of the cross-sections of the rod.

In order to avoid the need for an analytical solution of equation (1) when studying dynamic processes in a mechanical system, we will discretize this equation taking into account the boundary (3) and initial (4) conditions.

Let us divide the domain of the function $u(x, t)$ along the x coordinate into n segments of length h (Fig. 1, *a*) and show that the boundary value problem (1)–(4) using the direct discretization method reduces to the analysis of the oscillations of the mechanical system shown in Fig. 1, *b*. The computational model of the rod (Fig. 1, *b*) is a chain mechanical system with $n+1$ material points, the masses of which are shown in the figure, and the stiffness coefficients of elastic connections in the longitudinal direction of the rod are denoted as c, where

$$m = Ah\rho , \qquad c=\frac{EA}{h}, \qquad h=\frac{l}{n}. \tag{5}$$

We denote the coordinates of motion of material points as u_i (i = 0, 1, ..., n+1). Thus, the computational model has n+1 degrees of freedom.

Denoting the increment of the coordinate x as $h/2$, we write down the approximate expressions of the derivatives of the function $u(x, t)$ in terms of the coordinate x in the sections located at a distance h from each other,

$$u'(x-\frac{h}{2},t) \cong \frac{u(x,t)-u(x-h,t)}{h}; \qquad u'(x+\frac{h}{2},t) \cong \frac{u(x+h,t)-u(x,t)}{h}. \tag{6}$$

Then the second derivative of the function $u(x,t)$ at the coordinate x in the middle of the segment under consideration is given by

$$u''(x,t) \cong \frac{u'(x+\frac{h}{2},t)-u'(x-\frac{h}{2},t)}{h}. \tag{7}$$

Substituting expressions (6) into dependence (7), we obtain

$$u''(x,t) = \frac{u(x+h,t)-2u(x,t)+u(x-h,t)}{h^2}. \tag{8}$$

Taking into account (8), let us rewrite the equation of motion (1) in a discrete form:

$$\frac{\partial^2 u(x,t)}{\partial t^2} - a^2\,\frac{u(x+h,t)-2u(x,t)+u(x-h,t)}{h^2} = 0\ . \tag{9}$$

We denote the displacement of the joints of the rod segments as

$$u_{i-1}(t) = u(x-h,t); \qquad u_i(t) = u(x,t); \qquad u_{i+1}(t) = u(x+h,t). \tag{10}$$

Based on equality (9) and notation (10), we write the equation of motion of the discrete model of the rod in the form

$$m\frac{d^2u_i}{dt^2} + c(u_i - u_{i-1}) + c(u_i - u_{i+1}) = 0 \qquad (i = 2,\ 3,\ ...,\ n\ ), \tag{11}$$

where

$$m = \rho Ah; \qquad c = \frac{EA}{h.}. \tag{12}$$

Let's derive the equations of motion of the initial and final material points of the discrete model of the rod. To do this, we extend the domain of the function $u(x, t)$ along the rod axis in both directions by a distance *h*. At the left end of the rod, the boundary condition defined by the first relation (3) is given in the form

$$\frac{u_2 - u_0}{2h} = 0, \tag{13}$$

from

$$u_0 = u_2. \tag{14}$$

Equation (9) for the left end of the rod takes the form

$$m\frac{d^2u_1}{dt^2} + c(u_1 - u_0) + c(u_1 - u_2) = 0. \tag{15}$$

Taking into account (14), we transform equality (15) to the form

$$\frac{m}{2}\frac{d^2u_1}{dt^2} + c(u_1 - u_2) = 0. \tag{16}$$

Similarly to relation (13), we present in a discrete form the boundary condition for the right end of the rod, which is expressed by the second relation (3),

$$\frac{u_{n+2} - u_n}{2h} = 0, \tag{17}$$

from

$$u_{n+2} = u_n. \tag{18}$$

Equation (9) for the right end of the rod takes the form

$$m\frac{d^2u_{n+1}}{dt^2} + c(u_{n+1} - u_n) + c(u_{n+1} - u_{n+2}) = 0. \tag{19}$$

Taking into account (18), we transform equality (19) to the form

$$\frac{m}{2}\frac{d^2u_{n+1}}{dt^2} + c(u_{n+1} - u_n) = 0. \tag{20}$$

Thus, the motion of the rod is described by the system of differential equations (11), (16), (20). In accordance with these equations, the rod, as an elastic body with distributed parameters (Fig. 1, *a*), is reduced by direct discretization to the discrete computational model shown in Fig. 1, *b*. Experience has shown that in practical studies of longitudinal oscillations of long structures with a straight axis, a chain computational model can be successfully used. The masses of the material points of the model, except for the extreme ones, are equal to the masses of the segments into which the rod is divided. The masses of the extreme point bodies are equal to half the mass of the mentioned rod segment. The stiffness coefficients of the elastic elements are equal to the corresponding coefficients of the rod segments. The most important advantage of a discrete computational model is that the dynamic phenomena in it are described by ordinary differential equations, which greatly simplifies calculations.

Study of free oscillations of a rod. To evaluate the convergence of the direct discretization method, we consider the effect of the number of degrees of freedom of the computational model on the accuracy of determining the natural frequencies of a rod with free ends (Fig. 1). The values of the system parameters are as follows: rod length l = 20 m; cross-sectional area $A = 5^{-4}\cdot 10^2$ m^2 ; elastic modulus of the material of the first kind $E = 2.1\cdot 10^{11}$ Pa, material density $\rho = 7.8\cdot 10^3$ kg/m^3. Let us compare the values of natural frequencies found using the direct sampling method with their exact values calculated using the formula [1],

$$f_i = \sqrt{\frac{E}{\rho}} \times \frac{i}{2l} \qquad (i = 0, 1, 2, 3, ...). \tag{21}$$

The results of determining the natural frequencies are shown in Table 1. The first natural frequency of the computational model is zero. This is due to the fact that the mechanical system is not connected to the base and the zero value of the natural frequency corresponds to the translational motion of the system. The second natural frequency of the mechanical system was calculated with satisfactory accuracy using a computational model with four degrees of freedom, with a frequency determination error of less than 5 %. To obtain the third natural frequency with the same error, it is necessary to use a computational model with 7 degrees of freedom. The error in determining the fourth natural frequency using a model with 8 degrees of freedom does not exceed 7.5 %.

Table 1

Influence of the number of degrees of freedom of the calculation model on accuracy determination of natural frequencies of a rod with free ends

Number of steps of freedom n+1	Sequence number and natural frequency value, Hz							
	1	2	3	4	5	6	7	8
2	0	82.581						
3	0	116.79	165.16					
4	0	123.87	214.55	247.74				
5	0	126.41	233.86	305.18	330.33			
6	0	127.60	242.70	334.05	392.70	412.91		
7	0	128.24	247.74	350.36	429.11	478.61	495.49	
8	0	128.63	250.82	360.32	451.95	520.82	563.58	578.07
∞	0	129.72	259.44	389.16	518.88	648.60	778.31	908.03

Thus, to maintain a given level of accuracy, the number of degrees of freedom of the design model must be increased with increasing frequency number. To study the oscillatory processes in the region of the lower three or four natural frequencies of a mechanical system, a computational model with 8 to 10 degrees of freedom can be used.

The effect of the rod length on its natural frequencies is illustrated by the calculation results shown in Table 2. The number of degrees of freedom of the model was assumed to be n+1 = 8. The cross-sectional area of the rod, the elastic modulus of the material of the first kind, and the density of the material were assumed to be, respectively, $A = 5 \cdot 10^{-4}$ m^2, $E = 2.1 \cdot 10^{11}$ Pa, $\rho = 7.8 \cdot 10^3$ kg/m^3. As the results show, a 3-fold increase in the rod length caused a decrease in the values of all natural frequencies by about three times.

Table 2

Influence of the length of a rod with free ends on the value of natural frequencies

l, m	Sequence number and natural frequency value, Hz							
	1	2	3	4	5	6	7	8
10	0	257.27	501.63	720.84	903.91	1042.0	1127.0	1156.1
15	0	171.51	334.42	480.56	602.60	694.43	751.44	770.76
20	0	128.63	250.82	360.42	451.95	520.82	563.58	578.07
25	0	102.91	200.65	288.34	361.56	416.66	450.86	462.46
30	0	85.760	167.21	240.28	301.30	347.22	375.72	385.38

Table 3 illustrates a significant increase in natural frequencies with decreasing rod material density. Calculations were performed at $n+1 = 8$, $l = 20$ m; $A = 5\cdot10^{-4}$ m^2, $E = 2.1\cdot10^{11}$ Pa.

Table 3

Influence of the material density of a rod with free ends on the natural frequencies

ρ, kg/m^3	Sequence number and natural frequency value, Hz							
	1	2	3	4	5	6	7	8
7800	0	128.63	250.82	360.42	451.95	520.82	563.58	578.07
3900	0	181.91	354.71	509.71	639.16	736.56	797.02	817.52
1000	0	359.25	700.49	1007.0	1262.0	1455.0	1574.1	1614.2

It should be noted that the cross-sectional area of the rod does not affect the value of its natural frequencies of longitudinal oscillations, as shown by formula (21).

Similarly, the eigenfrequencies of the longitudinal oscillations of the rod were calculated for the case when one end of the rod is clamped and the other end is free (Fig. 2, *a*). The boundary conditions for solving the wave equation were set as follows

$$u(0,t)=0\,; \qquad \left.\frac{\partial u}{\partial x}\right|_{x=l}=0\,. \tag{22}$$

By discretizing the rod taking into account the relations (22), we obtain a computational model with n degrees of freedom (Fig. 2, *b*).

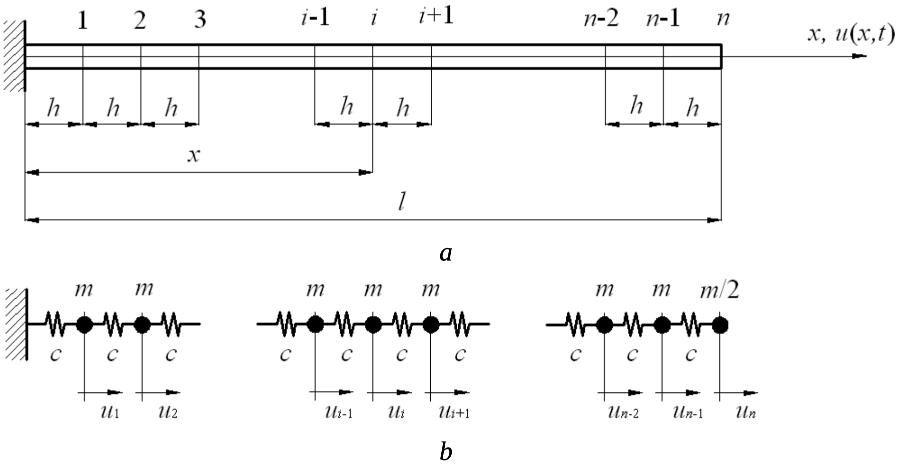


Fig. 2. An unloaded elastic rod with one end clamped,
and the second one is free (a) and its discrete computational model (b)

The values of the rod parameters were taken as follows: $n = 8$, $l = 20$ m, $A = 5\cdot10^{-4}$ m^2, $E = 2.1\cdot10^{11}$ Pa, $\rho = 7.8\cdot10^3$ kg/m^3. The formula for calculating the exact values of the natural frequencies of the longitudinal oscillations of a rod with one clamped end and the other free end is as follows (1)

$$f_i=\sqrt{\frac{E}{\rho}}\cdot\frac{2i-1}{4l} \qquad (i = 1, 3, 5, ...). \tag{23}$$

The results of the calculations given in Tables 4–6 show regularities similar to those found during the study of oscillations of a rod with free ends. These are: good convergence of the results of determining the eigenfrequencies using the direct discretization method with the exact results of calculating the frequency spectra, which manifests itself with an increase in the number of degrees of freedom of the design model; a significant decrease in the eigenfrequencies with an increase in the length of the rod and a decrease in the density of the material; a decrease in the accuracy of determining the eigenfrequency with an increase in its ordinal number at a constant number of degrees of freedom of the design model. The latter regularity necessitates a rational selection of the number of degrees of freedom of the design model, depending on the frequency range of the studied oscillatory phenomena.

Table 4

Influence of the number of degrees of freedom of the calculation model on the accuracy of determination natural frequencies of a rod with one end clamped and the other free

Number of steps of freedom *n*	Sequence number and natural frequency value, Hz							
	1	2	3	4	5	6	7	8
2	63.21	152.59						
3	64.12	175.18	239.30					
4	64.44	183.52	274.66	323.98				
5	64.59	187.46	291.97	407.82	367.90			
6	64.67	189.62	301.63	393.10	457.77	491.25		
7	64.72	190.92	307.55	408.76	489.47	574.44	545.63	
8	64.76	191.78	311.43	419.11	510.69	582.64	632.20	657.47
∞	64.86	194.58	324.30	454.02	583.73	713.45	843.17	972.89

Table 5

The effect of the length of the rod, one end of which is pinched, and the second one is free, by the value of natural frequencies

l, m	Sequence number and natural frequency value, Hz							
	1	2	3	4	5	6	7	8
10	129.51	383.55	622.86	838.23	1021.1	1165.0	1264.3	1315.1
15	86.340	255.70	415.24	558.82	680.92	776.86	842.94	876.63
20	64.755	191.78	311.43	419.11	510.69	582.64	632.20	657.47
25	51.804	153.42	249.14	335.29	408.55	466.11	505.76	525.98
30	43.170	127.85	207.62	279.41	340.46	388.43	421.47	438.31

Table 6

The effect of the material density of the rod, one end of which is pinched, and the second one is free, by the value of natural frequencies

ρ, kg/m^3	Sequence number and natural frequency value, Hz							
	1	2	3	4	5	6	7	8
7800	64.755	191.78	311.43	419.11	510.691	582.64	632.20	657.47
3900	91.578	271.21	440.41	592.72	722.23	823.98	894.07	929.80
1000	180.85	535.60	869.77	1171.0	1426.2	1627.1	1766.0	1836.1

Study of forced oscillations of the rod. Fig. 3, ***a*** shows a straight rod with one clamped end and the other free end in the case of force excitation of oscillations. The boundary conditions for the integration of the wave equation (1) take the form

$$u(0,t) = 0; \qquad EA\left.\frac{\partial u}{\partial x}\right|_{x=l} = F, \tag{24}$$

where $F = F_0 \sin(\omega t)$ is the harmonic power load, and F_0 *is* the load amplitude; ω is the cyclic frequency of power excitation of oscillations.

By discretizing the partial differential equation (1) taking into account the relations (24), we obtain the equation of motion of the model (Fig. 3, *b*), the number of degrees of freedom of which is *n*. The length and cross-sectional area of the rod, the modulus of elasticity of the first kind, and the density of the material are taken, respectively, as l = 20 m, $A = 5{\cdot}10^{-4}$ m^2, $E = 2.1{\cdot}10^{11}$ Pa, $\rho = 7.8{\cdot}10^3$ kg/m^3. The load amplitude is F_0 = 2000 N, and the technical oscillation frequency $f = \omega/2\pi$ is varied in the range from 0 to 700 Hz.

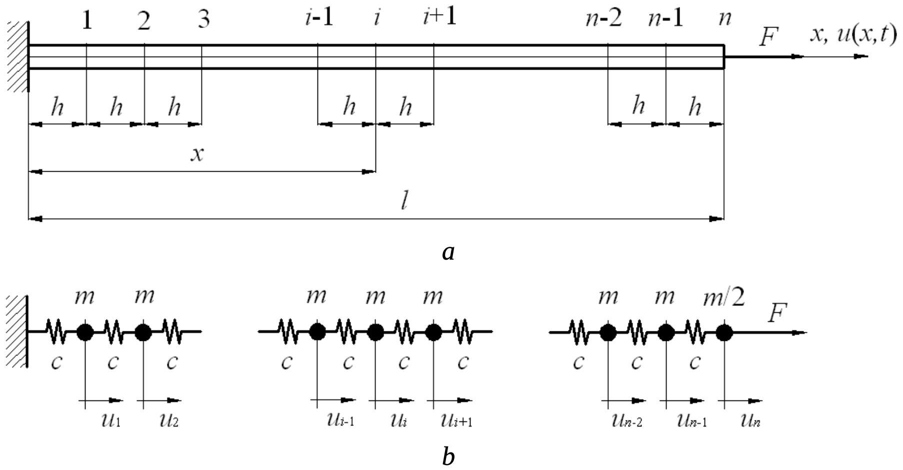


Fig. 3. An elastic rod with one end clamped and the other end free,
loaded with an axial force $F = F_0 \sin(\omega t)$ (a) and its discrete computational model (b)

Based on a comparison of the eigenfrequencies obtained by the direct sampling method with the exact values of these values (Table 4), we can conclude that for engineering practice, the accuracy of the approximate method is quite satisfactory in the frequency range 0 £ *f* £ 450 Hz.

Figs. 4 and 5 show the amplitude-frequency characteristics of the mechanical system in the form of the displacement amplitude of the rightmost (nth) material point $U_8(f)$ and the amplitude of the longitudinal force on the rightmost (nth) section of the rod $N_8(f)$ as functions of the oscillation excitation frequency. As the frequency approaches one of the resonant values, a rapid increase in the displacement amplitude and, consequently, the force amplitude is observed.

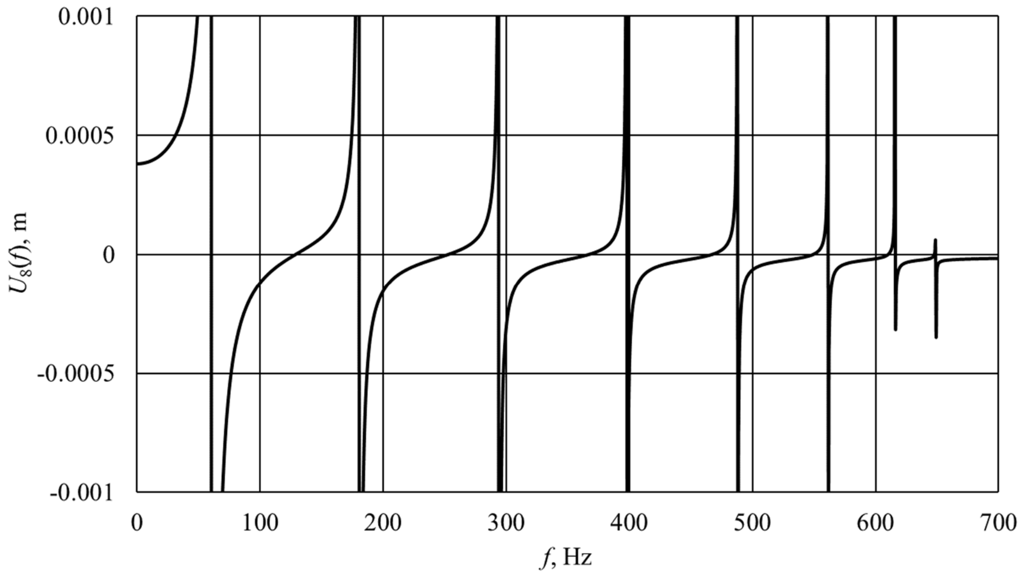


Fig. 4. Dependence of the amplitude of displacements of a material point
with ordinal number *n* from the frequency of power excitation of oscillations

Fig. 6, ***a*** shows a straight rod with one clamped end and the other free in the case of kinematic excitation of oscillations. The boundary conditions for the integration of the wave equation (1) take the form

$$u(0,t) = u_0(t) \quad \left.\frac{\partial u}{\partial x}\right|_{x=l} = 0\,; \tag{25}$$

where $u_0(t) = U_0 \sin(\omega t)$ is the harmonic displacement of the base, and U_0 is the displacement amplitude; ω is the cyclic frequency of the kinematic excitation of the oscillations.

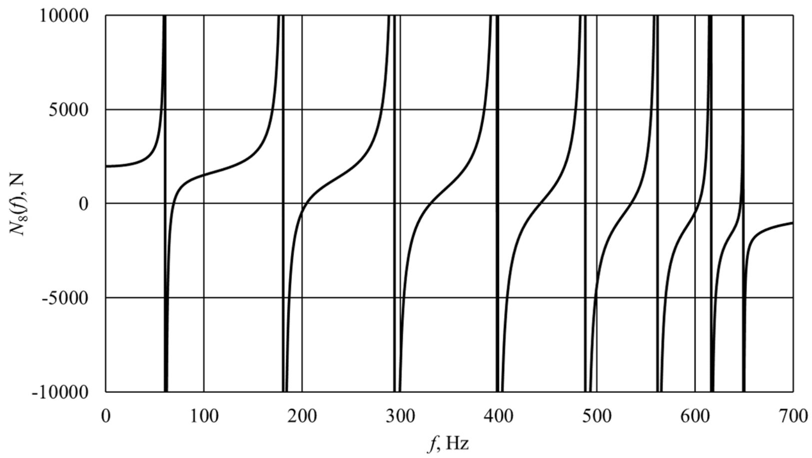


Fig. 5. Dependence of the longitudinal force on the section with the ordinal number ***n*** from the frequency of power excitation of oscillations

By discretizing the partial differential equation (1) taking into account the relations (25), we obtain the equation of motion of the model with ***n*** degrees of freedom (Fig. 4, ***b***). The length and cross-sectional area of the rod, the modulus of elasticity of the first kind, and the density of the material are taken, respectively, as l = 20 m, $A = 5 \cdot 10^{-4}$ m^2, $E = 2.1 \cdot 10^{11}$ Pa, $\rho = 7.8 \cdot 10^3$ kg/m^3. The amplitude of the base movement is set to U_0 = 0.002 m, and the technical oscillation frequency $f = \omega/2\pi$ is varied in the range from 0 to 700 Hz.

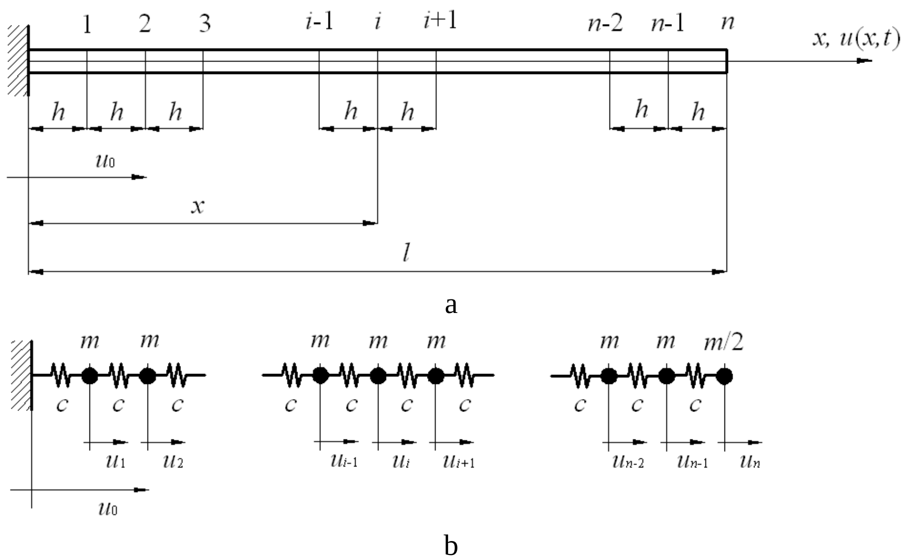


Fig 6. An elastic rod with one end clamped in a movable base and the other end free (a*)* and its discrete computational model (b). The law of displacement of the base $u_0 = U_0 \sin(\omega t)$

As in the calculation of forced oscillations under power excitation, we will assume that for engineering practice, the accuracy of the direct sampling method is quite satisfactory in the frequency range $0 \le f \le 450$ Hz.

Figs. 7 and 8 show the amplitude-frequency characteristics of the mechanical system in the form of the displacement amplitude of the rightmost (nth) material point $U_8(f)$ and the amplitude of the longitudinal force on the leftmost (1st) section of the rod $N_1(f)$ as functions of the oscillation excitation frequency. As the frequency approaches one of the resonant values, a rapid increase in the displacement amplitude and, consequently, the force amplitude is observed.

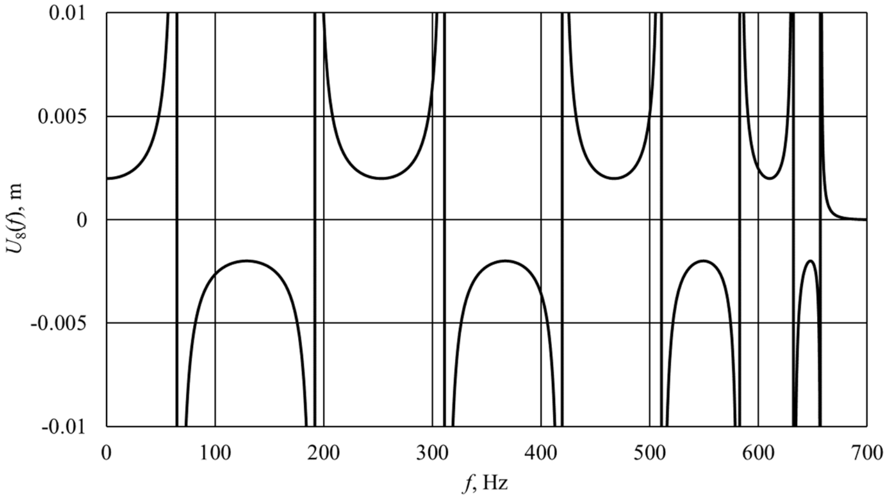


Fig. 7. Dependence of the amplitude of the material point movements on the ordinal number ***n*** from the frequency of kinematic excitation of oscillations

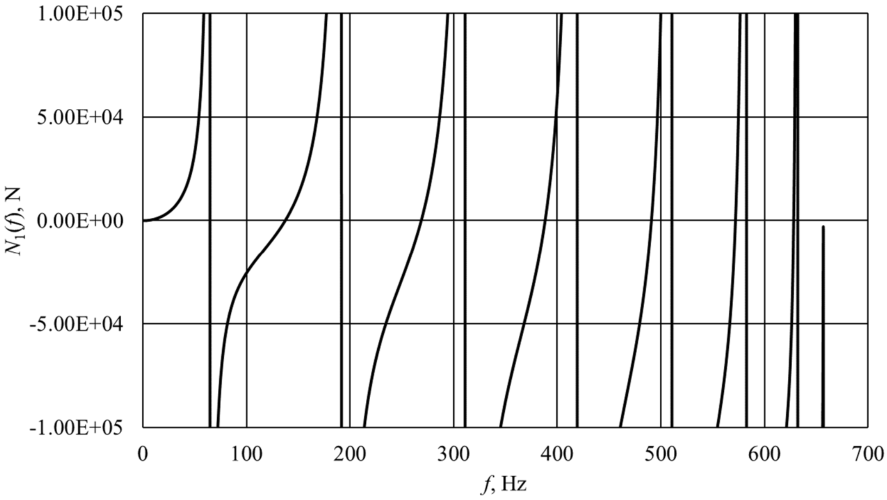


Fig. 8. Dependence of longitudinal force on the first section of the rod on the frequency of kinematic excitation of oscillations

Conclusions

1. In this paper, the method of direct discretization of long-dimensional elastic structures is mathematically substantiated in its application to free and forced longitudinal oscillations of a straight rod as a mechanical system with distributed parameters. It is shown that the boundary value problem for the

wave equation describing the motion of a continuous rod, taking into account the corresponding boundary and initial conditions, is reduced by direct discretization to the Cauchy problem for the system of ordinary differential equations of motion of the computational model with a finite number of degrees of freedom. This computational model is easily obtained by replacing the one-dimensional continuous medium with a chain system of material points.

2. The computational examples illustrate the good convergence of the results of determining the natural frequencies using the direct discretization method with the exact results of calculating the frequency spectra, which manifests itself with an increase in the number of degrees of freedom of the computational model; a significant decrease in the natural frequencies with an increase in the length of the rod and a decrease in the density of the material; a decrease in the accuracy of determining the natural frequency with an increase in its ordinal number at a constant number of degrees of freedom of the computational model. The latter regularity necessitates a rational selection of the number of degrees of freedom of the design model, depending on the frequency range of the studied oscillational phenomena.

3. The use of the direct sampling method in engineering practice greatly facilitates the construction of amplitude-frequency characteristics of a mechanical system, which helps to reduce the oscillation activity of machines, ensure the strength and durability of long structural elements under both power and kinematic excitation of oscillations. This method can be widely used in the study of dynamic processes in rather complex mechanical systems of such technical objects as drilling rigs, cranes, mine hoists, conveyors, rotary kilns, etc. when solving problems of dynamics of both linear and nonlinear mechanical systems.

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