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MODEL OF SENSITIVITY TO THE INITIAL CONDITIONS OF THE EQUATIONS OF MOTION IN THE ELECTRIC FIELD

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Abstract: A numerical method for calculating steady periodic processes of nonlinear differential equations of motion in an electric field is proposed based on their integration under initial conditions that exclude transient reaction. The calculation of such initial conditions is based on Newton's iterative formula. At each iteration, together with the basic equations of motion, the corresponding equations of the first variation are integrated over the periodicity time interval, which represent a model of sensitivity to the initial conditions, In addition, the number of unknown variational equations has been reduced, which makes it possible to lower their order, and, at the same time, the order of the monodromy matrix. All this contributes to the simplification of the relevant computational algorithms and computer programs. The method takes into account the finite rate of electric field propagation.

Key words: electric field, nonlinear differential equations of motion, periodic solutions, model of sensitivity to initial conditions, reduction of order of variational equations and monodromy matrix.

1. Introduction

The search for periodic solutions to nonlinear differential equations of motion of charged bodies in an electric field still practically remains outside the attention of scientists in this important area of scientific progress.

We suggest that the search for periodic steady-state processes of moving bodies should be based on the most advanced mathematical support, which is a model of sensitivity to initial conditions that excludes a transient reaction and introduces the computational process directly into a periodic state. Moreover, in order to simplify the analysis, such initial conditions were reduced, because there are often situations when they can be reduced. This, as a rule, is achieved by a reasonable rotation of one or another spatial coordinates [1].

2. The Cauchy problem for differential equations

The system of differential equations will be written in vector form

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}\left(\mathbf{x}, t\right), \qquad (1)$$

where $\mathbf{f}\left(x,t\right)$ is the T-periodic, $x = \left(x_1, x_2, ...x_n\right)$ is the vector of subdifferential unknowns.

The uniqueness of the solution is ensured by additional conditions at one or more points of the periodicity interval $[a,b]$. In the case of a single point, these conditions are preferred at the beginning of the interval and are called initial conditions $x_i\big|_{t=a} = x_{ia}$, $i = 1,2,...,n$.

Finding a solution to (1) that satisfies the initial conditions is called the Cauchy problem for differential equations. In the case of two points, these conditions are most often set at the edges of the periodicity interval and are usually called boundary conditions $x_i\big|_{t=a} = x_{ia}$; $x_i\big|_{t=b} = x_{ib}$, $i = 1,2,...,n$. Finding a solution to (1) that satisfies the boundary conditions is called a boundary value problem for differential equations. We will assume that the solution to the problems on $[0,T]$ we set does exist.

We will consider problems for differential equations (Cauchy, boundary value) as nonlinear. Their numerical integration involves temporal discretization of derivatives, as a result of which the computational process is reduced to algebraic operations. The vulnerability of all numerical methods of integrating differential equations is the assessment of the accuracy of the results obtained. The Cauchy problem is particularly unfavorable from this point of view. Due to the connection between the derivative and the value of the function, the errors made in calculating the previous values accumulate, and the rate of their growth can be quite high. Therefore, the transition to the two-point boundary value problem is quite timely!

3. Two-point periodic boundary value problem

As a result of the integration of differential equations over time at $t \to \infty$, the system should reach a T-periodic steady state solution, where T is the period of recurrence of the function over time. The task of finding periodic solutions to nonlinear differential equations is an important stage in the analysis of physical systems, which is known as the analysis of steady processes. From a mathematical point of view, this problem is more difficult than the Cauchy problem, because it imposes an additional condition on the solution, namely: the condition of periodicity

$$\mathbf{x}(t) = \mathbf{x}(t+T). \qquad (2)$$

Expression (2) is often called the periodicity equation, or the target equation.

The simplest, and at the same time the crudest way to achieve a periodic solution is by directly integrating differential equations up to the process stabilization. That is not always realistic! Therefore, it is no coincidence that along with the development of methods for solving the Cauchy problem for differential equations, painstaking work related to the development of methods for the direct solution of a two-point periodic boundary value problem has been and is being conducted [2].

We will not consider the methods of the timeless domain, because they are all pre-computer and devoid of accuracy criteria. Time domain methods are based on the general theory of nonlinear differential equations and are technically named methods of accelerated search for forced periodic states. Their purpose is to achieve a periodic solution at a lower cost than with direct integration of the equations until the process stabilizes.

These methods are divided into three main groups: those based on the construction of sensitivity models to initial conditions, extrapolation and gradient methods. The greatest amount of information about the state of the object under study is provided by methods based on the construction of models of sensitivity to the initial conditions.

4. Model of sensitivity to initial conditions

Expressions (1), (2) constitute a two-point *T*-periodic boundary value problem for nonlinear differential equations. There are such initial conditions which, when integrating (1) over the time interval $[0,T]$, make it possible to enter directly into the periodic solution $\mathbf{x}(0)$, bypassing the transient reaction. We will consider such initial conditions as an argument of the periodicity equation (2), which we will write as

$$\mathbf{f}(\mathbf{x}(0)) = \mathbf{x}(0) - \mathbf{x}(\mathbf{x}(0),T) = 0. \qquad (3)$$

The solution of the nonlinear transcendental equation (3) by the method of simple iteration is identical to the direct integration (1) up to the achievement of a periodic solution, which is unacceptable for us. Therefore, we will apply iterative methods of higher order. The most effective here is

Newton's formula

$$\mathbf{x}(0)^{(s+1)} = \mathbf{x}(0)^{(s)} - \mathbf{f}'\left(\mathbf{x}(0)^{(s)}\right)^{-1} \mathbf{f}\left(\mathbf{x}(0)^{(s)}\right). \qquad (4)$$

The Jacobian matrix $\mathbf{f}'(\mathbf{x}(0))$ is obtained by differentiation with respect to $\mathbf{x}(0)$ of the objective function (3)

$$\mathbf{f}'(\mathbf{x}(0)) = \mathbf{E} - \mathbf{\Phi}(T), \qquad (5)$$

where

$$\mathbf{\Phi}(T) = \left.\frac{\partial \mathbf{x}(\mathbf{x}(0),t)}{\partial \mathbf{x}(0)}\right|_{t=T}. \qquad (6)$$

Matrix (6) is called the fundamental matrix, or the monodromy matrix. In the technical literature, it is called the state transition matrix.

Calculating the monodromy matrix is the main difficulty of the analysis. It can be obtained by numerical differentiation. But we will use a much more efficient method, that leads to variational equations as a result of differentiating (1) by $\mathbf{x}(0)$:

$$\frac{d\mathbf{\Phi}}{dt} = \frac{\partial \mathbf{f}(\mathbf{x},t)}{\partial \mathbf{x}}\mathbf{\Phi}, \qquad (7)$$

where $\mathbf{\Phi} = \partial \mathbf{x} / \partial \mathbf{x}(0)$ is the matrix (6).

On the s-th iteration of Newton's formula (4), linear variational equation (7) is subject to simultaneous integration with nonlinear one (1) on the time interval $[0,T]$. As a result, we find target function (3) and required Jacobi matrix (5), (6), which creates the right-hand side of iterative formula (4), and hence, its desired left-hand side $\mathbf{x}(0)^{(s+1)}$. The iteration process is completed when the specified accuracy of entering the periodic solution is achieved

$$\left|\mathbf{f}\left(\mathbf{x}(0)^{(s)}\right)\right| \le \boldsymbol{\varepsilon}, \qquad (8)$$

where $\boldsymbol{\varepsilon}$ is the vector of specified accuracies.

According to (3), the value of $\mathbf{\Phi}(0)^{(s)}$, including $\mathbf{\Phi}(0)^{(0)}$, is equal to the unit matrix $\mathbf{E}$

$$\mathbf{\Phi}(0)^{(s)} = \mathbf{E}. \qquad (9)$$

Monodromy matrix (6) is, in fact, a matrix of sensitivities to initial conditions. Each of its rows can be considered as the gradient of a certain variable in the space of initial conditions, and each of its columns characterizes the sensitivity of the entire set of variables to the same initial condition. Therefore, differential equation (7) is considered as a model of sensitivity to initial conditions.

The value $\mathbf{x}(0)^{(0)}$ as the zero approximation of Newton's formula and the initial condition for (1) is set arbitrarily. But in those cases when the existence of several periodic states is possible in the solution, the value $\mathbf{x}(0)^{(0)}$ determines the entry of the computational process into the attraction zone of one of them. Therefore, to obtain a set of all possible periodic solutions, it is necessary to vary the values of $\mathbf{x}(0)^{(0)}$.

Example. We use equations of motion (1) of a charged body in an electric field in 2D Euclidean space [1, 3, 4].

$$\frac{d}{dt}\begin{vmatrix} v_x \\ v_y \\ r_x \\ r_y \end{vmatrix} = \begin{vmatrix} -\frac{kQq\,r_x}{mr^3}\left(1+\frac{v^2}{c^2}+2\frac{r_xv_x+r_yv_y}{rc}\right) \\ -\frac{kQq\,r_y}{mr^3}\left(1+\frac{v^2}{c^2}+2\frac{r_xv_x+r_yv_y}{rc}\right) \\ v_x \\ v_y \end{vmatrix}. \quad (10)$$

where *r, v* are the modules of the radius vector **r** directed to the moving body and the vector of the moving body's mutual velocity **v**

$$r=\sqrt{r_x^2+r_y^2}; \quad v=\sqrt{v_x^2+v_y^2}, \quad (11)$$

their projections: r_x, r_y, v_x, v_y; r_x, r_y, v_x, v_y are the charge (field source); q is the charge of the moving body; m is the mass of the moving body; k is the electric constant.

Expressions (10), (11) form a complete system of algebraic-differential equations for the analysis of transient processes in an electric field in 2D Euclidean space and physical time for all possible range of speeds. The uniqueness of the solution is provided by the initial conditions: $r_x(0), r_y(0), v_x(0), v_y(0)$.

If we differentiate (10) by the vector of unknowns $\mathbf{x}=(v_x, v_y, r_x, r_y),$, we obtain the corresponding matrix of coefficients of variational equation (7)

$$\frac{\partial \mathbf{f}(\mathbf{x},t)}{\partial \mathbf{x}} = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ 1 & & & \\ & 1 & & \end{vmatrix}. \quad (12)$$

$$a_{11}=-2b\left(\frac{r_xv_x}{c^2r^3}+\frac{r_x^2}{cr^4}\right) \quad a_{12}=-2b\left(\frac{r_xv_y}{c^2r^3}+\frac{r_xr_y}{cr^4}\right)$$

$$a_{21}=-2b\left(\frac{r_yv_x}{c^2r^3}+\frac{r_xr_y}{cr^4}\right) \quad a_{22}=-2b\left(\frac{r_yv_y}{c^2r^3}+\frac{r_y^2}{cr^4}\right)$$

$$a_{13}=-b\left(1+\frac{v^2}{c^2}\right)\frac{r^2-3r_x^2}{r^5}+$$

$$+\frac{2}{c}\frac{(2r_xv_x+r_yv_y)r^2-4r_x^2(r_xv_x+r_yv_y)}{r^6}$$

$$a_{14}=-b\left(1+\frac{v^2}{c^2}\right)\frac{3r_xr_y}{r^5}+\frac{2}{c}\frac{r_xv_yr^2-4r_y(r_x^2v_x+r_xr_yv_y)}{r^6}$$

$$a_{23}=-b\left(1+\frac{v^2}{c^2}\right)\frac{3r_xr_y}{r^5}+\frac{2}{c}\frac{r_yv_xr^2-4r_x(r_y^2v_y+r_xr_yv_x)}{r^6}$$

$$a_{24}=-b\left(1+\frac{v^2}{c^2}\right)\frac{r^2-3r_y^2}{r^5}+$$

$$+\frac{2}{c}\frac{(r_xv_x+2r_yv_y)r^2-4r_y^2(r_xv_x+r_yv_y)}{r^6} \quad b=\frac{kQq}{m}.$$

The algorithm for finding the initial conditions for entering the periodic state consists in the joint implementation of expressions (4), (5), (7), (8), (10), (11) with a given zero approximation of the initial conditions

$$\mathbf{x}(0)^{(0)}=(v_x(0), v_y(0), r_x(0), r_y(0)). \quad (13)$$

Having obtained the initial conditions $\mathbf{x}(0)$ that exclude the cross-reaction by iterative formula (4), we integrate equation (10) over the time period $[0,T]$ once again. As a result, we obtain the desired periodic state.

The first successful attempt to solve a two-point boundary value problem for differential equations (10) was made in [5]. But the algebraic criteria there had only a specific application, being subject to the needs of quantum ideas.

If the system of differential equations (1) is linear, then the proposed method of sensitivity to initial conditions introduces in the periodic solution in one iteration of formula (4).

5. Reduction of initial conditions

In practical problems of analysis of nonlinear physical systems related to the solution of a two-point periodic boundary value problem for ordinary differential equations, it is very often necessary to determine only a part of the initial conditions, which exclude a transient reaction. Another part of the total number of unknowns may be predetermined or obvious.

In this case, in order to simplify the analysis, we turn to the proposed method of reduction of initial conditions. In multi steady-state systems, sometimes such reduction has to be resorted to subconsciously for the sake of the attraction zone of one or another state.

Let us present the argument of trascendent periodicity equation (3) in block form

$$\mathbf{x}(0) = (\mathbf{x}_1(0), \mathbf{x}_2(0)), \qquad (14)$$

where $\mathbf{x}_1(0)$ is the sub-column of the initial conditions sought; $\mathbf{x}_2(0)$ is the sub-column of known initial conditions.

We represent original differential equation (1) in the form according to (14).

$$\frac{d\mathbf{x}_1}{dt} = \mathbf{f}_1(\mathbf{x}_1, \mathbf{x}_2, t); \ \frac{d\mathbf{x}_2}{dt} = \mathbf{f}_2(\mathbf{x}_1, \mathbf{x}_2, t). \qquad (15)$$

Periodicity equation (3) for the first of them is obvious

$$\mathbf{f}_1(\mathbf{x}_1(0)) = \mathbf{x}_1(0) - \mathbf{x}_1(\mathbf{x}_1(0), T) = 0. \qquad (16)$$

Its solution is also carried out using Newton's iterative formula similar to (4)

$$\mathbf{x}_1(0)^{(s+1)} = \\ = \mathbf{x}_1(0)^{(s)} - \mathbf{f}_1'(\mathbf{x}_1(0)^{(s)})^{-1} \mathbf{f}_{11}(\mathbf{x}_1(0)^{(s)}). \qquad (17)$$

Here, the Jacobi matrix is quite similar to (9)

$$\mathbf{f}_1'(\mathbf{x}(0)) = \mathbf{E} - \mathbf{\Phi}_1(T), \qquad (18)$$

$$\mathbf{\Phi}_1(T) = \left. \frac{\partial \mathbf{x}_1(\mathbf{x}_1(0), t)}{\partial \mathbf{x}_1(0)} \right|_{t=T}. \qquad (19)$$

Variational equation (6) for calculating (18) takes the form

$$\frac{d\mathbf{\Phi}_1}{dt} = \frac{\partial \mathbf{f}_1(\mathbf{x}_1, t)}{\partial \mathbf{x}_1} \mathbf{\Phi}_1. \qquad (20)$$

The reduced initial conditions $\mathbf{x}_1(0)$ are obtained as a result of the compatible implementation of expressions (14)–(20), which is accompanied at each iteration of Newton's formula (17) with the compatible numerical integration over the recurrence time interval (14) $[0, T]$ with a predetermined accuracy (8) provided that $\mathbf{x} = \mathbf{x}_1$.

Example. If in the electron-proton electrical system [1, 3, 5] the origin of the Cartesian coordinates is aligned with the unmovable center of the proton, and the initial conditions that exclude the transient reaction are set at the point of intersection of the orbit with one or another coordinate axis, then the number of unknowns in (12) can be halved. So, if we take the point of intersection of the orbit with the positive *y*-semi-axis, we know that $r_x(0) = 0; v_y(0) = 0$. In this way, reduced vector (12) will be halved

$$\mathbf{x}_1(0)^{(0)} = (v_x(0), r_y(0)). \qquad (21)$$

As a result, the order of matrix (11) will also be halved. The matrix itself, which appears in variational equation (20), will now be searched much easier

$$\frac{\partial \mathbf{f}_1(\mathbf{x}_1, t)}{\partial \mathbf{x}} = \begin{vmatrix} a_{11} & a_{14} \\ & \end{vmatrix}. \qquad (22)$$

Substituting (22) into matrix (18), we obtain the Jacobi matrix, the inverse of which appears in Newton's iterative formula (17)

$$\mathbf{f}_1'(\mathbf{x}_1(0))^{-1} = \frac{1}{1 - a_{11}} \begin{vmatrix} 1 & a_{14} \\ & 1 - a_{11} \end{vmatrix}. \qquad (23)$$

Let us show the mathematical basis for the transition from matrix (12) to reduced matrix (22). In this regard, we will write the right-hand side of equation of motion (10) in vector form

$$\mathbf{f} = (f_1, f_2, v_x, v_y). \qquad (24)$$

Similarly (24), we write the right-hand side of the first equation of motion (14)

$$\mathbf{f}_r = (f_1, v_y). \qquad (25)$$

Then, matrix (22) will acquire its original form

$$\frac{\partial \mathbf{f}_{r1}(\mathbf{x}_1, t)}{\partial \mathbf{x}_1} = \begin{vmatrix} \frac{df_1}{dv_x} & \frac{df_1}{dr_y} \\ \frac{dv_y}{dv_x} & \frac{dv_y}{dr_y} \end{vmatrix}. \qquad (26)$$

It is not difficult to verify that the elements of matrix (26) are nothing other than the elements of original matrix (12), since system of equations (15) is identical to equation (10), as rearranged. This proves the status of matrix (22) as reduced (12).

PS. Based on nonlinear differential equations of motion in an electric field (10) and their analogous equations in a gravitational field (when replacing $k \to G, q \to m$), where G is the gravitational constant; m is the mechanical mass, a number of unsolved physics problems have been successfully solved. In particular:

1. The dynamics of a cosmic black trap (the dynamics of the capture of celestial bodies by a black hole) are simulated [3];

2. The anomaly of solar braking of the flights of the space probes "Pioneer-10" and "Pioneer-11" is substantiated [1,8];

3. The concept of the inapplicability of the laws of classical physics in quantum physics is refuted. Classical models of Hydrogen and Helium have been built [1,5];

4. The problem of precession of elliptical orbits has been solved [4];

5. The equations of electricity and gravity are combined (the theory of electrogravity) [4,5,6];

6. The interaction of electric and gravitational fields is simulated using the example of a cosmic lens. It was shown that the speed of light c is not a constant [4, 7];

7. The dynamics (acceleration-deceleration) of navigation of space probes in the gravitational field of celestial bodies has been studied [9];

8. Faraday's experimental law has been theoretically proven on the basis of electromechanical analogies [10];

9. A heliocentric model of the Moon's motion in the gravitational field of the Sun and the Earth has been constructed [11];

10. In the moving tandem of electron-proton (the classical mathematical model of Hydrogen) in the depths of matter a singularity similar to cosmic black holes has been discovered – at a certain depth, the laws of mechanics (gray holes) collapse, and even deeper – the laws of electricity (white holes).

11. Finally, this work introduces elements of the theory of a two-point boundary value problem into the differential equations of motion of charged bodies in an electric field.

12. Recently, based of the fundamental laws of nature, the physical essence of cosmological dark matter of gravitational and anti-gravitational action has been revealed.

6. Conclusions

1. We hope that the introduction of elements of the theory of a two-point boundary value problem for differential equations into the equation of motion of charged bodies will be useful in the practice of studying periodic movement in an electric field. This primarily concerns orbital, elliptical, circular, etc. motions.

2. The reduction of initial conditions which exclude the transient reaction makes it possible to significantly simplify the analysis by lowering the order of the variational equations

3. On the basis of electromechanical analogies [1, 3, 4], the results of this study can be successfully extended to movement in a gravitational field. It is possible that such an approach will one day be able to reliably predict future cosmic events.

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МОДЕЛЬ ЧУТЛИВОСТИ ДО ПОЧАТКОВИХ УМОВ РІВНЯНЬ РУХУ В ЕЛЕКТРИЧНОМУ ПОЛІ

Чабан Василь

Запропоновано числовий метод розрахунку усталених періодичних процесів нелінійних диференціальних рівнянь руху в електричному полі на підставі їхнього інтегрування за початкових умов, що виключають перехідну реакцію. Обчислення таких початкових умов зійснюється за ітераційною формулою Ньютона. На кожній її ітерації сумісно з основними рівняннями руху на часовому інтервалі періодичности інтегруються відповідні рівняння першої варіації, які представляють модель чутливости до початкових умов. Крім того, здійснено редукцію кількості невідомих варіаційних рівнянь, що дає змогу знизити їхній порядок, а також порядок матриці монодромії. Усе це сприяє спрощенню відповідних обчислювальних алгоритмів і комп'ютерних програм. Метод враховує скінченну щвидкість пропагації електричного поля.

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