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Inverse problems of determining an unknown coefficient depending on time for a parabolic equation with involution and anti-periodicity conditions

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Inverse problems of determining an unknown depending on time coefficient for a parabolic equation with involution and anti-periodicity conditions. The solution of the investigated problem with an unknown coefficient in the equation was constructed using the method of separation of variables. The properties of the induced spectral problem for the second-order differential equation with involution are studied. The dependence of the spectrum and its multiplicity and the structure of the system of root functions and partial solutions to the problem on the involutive part of this equation was studied. The conditions for the existence and uniqueness of the solution to the inverse problem have been established. To determine the required coefficient, Voltaire's integral equation of the second kind was found and solved.

Keywords: *inverse problem; heat conduction equation; method of separation of variables; nonlocal conditions; involution; Riesz basis.*

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1. Introduction

Problems of determining the coefficients or the right-hand side of a differential equation simultaneously with its solution are called inverse problems of mathematical physics. Such problems appear, for example, in the simulation of hyperthermia, thrombosis and sclerosis of vessels, optical tomography.

Inverse heat conduction problems arise in various branches of applied heat engineering. In particular, the problem of modeling the thermo-diffusion process is described in the paper [1]. The authors analyzed a problem that describes a mathematical model of the process of heat diffusion in a closed metal rod, the insulation of which is slightly permeable. Therefore, the temperature at the point of the rod on one side of the insulation affects the diffusion process in the rod on the other side of the insulation. The authors proposed to consider the following heat conduction equation with involution for modeling the process:

$$\frac{\partial u(x,t)}{\partial t} = \alpha \frac{\partial^2 u(x,t)}{\partial x^2} + \beta \frac{\partial^2 u(-x,t)}{\partial x^2}. \tag{1}$$

In the paper [2] for equation

$$\frac{\partial u(x,t)}{\partial t} - \frac{\partial^2 u(x,t)}{\partial x^2} + \frac{\partial^2 u(-x,t)}{\partial x^2} = f(x), \quad (x,t) \in \Omega, \quad \Omega = \{-\pi < x < \pi, 0 < t < T\} \tag{2}$$

inverse problems of determining a pair of unknown functions $\{u(x,t),\ f(x)\}$ with boundary conditions

$$\begin{gathered} u(-\pi,t) = u(\pi,t) = 0, \\ \frac{\partial u(-\pi,t)}{\partial x} = \frac{\partial u(\pi,t)}{\partial x} = 0, \\ u(-\pi,t) - u(\pi,t) = 0, \quad \frac{\partial u(-\pi,t)}{\partial x} - \frac{\partial u(\pi,t)}{\partial x} = 0, \\ u(-\pi,t) + u(\pi,t) = 0, \quad \frac{\partial u(-\pi,t)}{\partial x} + \frac{\partial u(\pi,t)}{\partial x} = 0 \end{gathered} \tag{3}$$

are investigated.

In the paper [3] for equation (1), the inverse problem with nonlocal conditions, which are weak perturbations of conditions (3):

$$\frac{\partial u(-\pi,t)}{\partial x}-\frac{\partial u(\pi,t)}{\partial x}-\alpha u(\pi,t)=0,\quad u(-\pi,t)-u(\pi,t)=0$$

was considered.

In [4] for equation (2) the inverse problem of finding $\{u(x,t),f(x)\}$ with the initial condition

$$u(x,0)=\varphi(x),$$

condition of redefinition

$$u(x,E)=\psi(x)$$

and Ionkin–type conditions

$$\frac{\partial u(-\pi,t)}{\partial x}+\alpha\frac{\partial u(\pi,t)}{\partial x}=0,\quad u(-\pi,t)-u(\pi,t)=0,$$

are studied.

The inverse problem of mathematical biology is considered in [5], namely, the problem of finding a time-dependent source function for a population model with nonlocal boundary conditions of the population density.

So, in $\Omega=\{0<x<1,0<t<T\}$ for equation

$$\frac{\partial u(x,t)}{\partial t}=\frac{\partial^2 u(x,t)}{\partial x^2}+r(t)u(x,t)+f(x,t) \tag{4}$$

the inverse problem of finding $\{u(x,t),\, r(t)\}$ with the initial condition

$$u(x,0)=\varphi(x),$$

condition of redefinition

$$\int_0^1 u(x,t)\,dt=E(t),$$

and perturbed antiperiodicity conditions

$$\frac{\partial u(0,t)}{\partial x}+\frac{\partial u(1,t)}{\partial x}=0,\quad u(0,t)+bu(1,t)=0,$$

was considered.

In [6–8], inverse problems of determining $\{u(x,t),r(t)\}$ with nonlocal boundary conditions used in models of population age description were analyzed.

Mixed and boundary value problems for equations with partial derivatives, which contain involution, were studied in [4, 9–13]. For ordinary differential operators with involution boundary value problems were studied in the papers [14–18].

2. Basic designations and results

Let

$$W_2^2(-1,1):=\left\{y\in L_2(-1,1)\colon y^{(m)}\in C[-1,1],y^{(2)}\in L_2(-1,1),m=0,1\right\},$$

$$(y;u)_{W_2^2(-1,1)}:=\sum_{k=0}^{2}\left(y^{(k)};u^{(k)}\right)_{L_2(-1,1)},\quad \|y\|^2_{W_2^2(-1,1)}:=(y;y)_{W_2^2(-1;1)}.$$

E be identical transformation in the space $L_2(-1,1)$, $I\colon L_2(-1,1)\to L_2(-1,1)$, $Iy(x)\equiv y(-x)$ be involution operator in $L_2(-1,1)$, $p_j:=\frac{1}{2}(E+(-1)^jI)$ are orthoprojectors of space $L_2(-1,1)$ and

$$L_{j,2}(-1,1):=\{y\in L_2(-1,1)\colon y=p_jy\},\quad j=0,1.$$

Definition 1. *The system of elements $\{e_m\}_{m=1}^{\infty}\subset H$ is called closed (complete) in the space H if the linear shell of this system is dense everywhere in H. That is, an arbitrary element of the space H can be approximated by a linear combination of the elements of this system with any precision according to the norm of the space H.*

Definition 2. *The system of elements* $\{e_m\}_{m=1}^{\infty} \subset H$ *is called total in* H, *if only the zero element* 0 *of the space* H *is orthogonal to all elements of this system.*

Definition 3. *The system of elements* $\{g_s\}_{s=1}^{\infty} \subset H$ *is called biorthogonal in* H *to the system of elements* $\{e_m\}_{m=1}^{\infty} \subset H$, *if* $(g_s; e_m)_H = \delta_{s,m}$, $s, m \in \mathbb{N}$.

Definition 4. *The system of elements* $\{e_m\}_{m=1}^{\infty} \subset H$ *is called the Riesz basis of the space* H, *if there exists a bounded operator* $A\colon H \to H$ *with its inverse such that the system* $\{Ae_m\}_{m=1}^{\infty}$ *is orthonormal basis in* H.

Definition 5. *Let the operator* $A\colon H \to H$ *has an eigenvalue* $\lambda \in \mathbb{C}$. *Arbitrary solution of the equations* $(A - \lambda E)^2\nu = 0$, $(A - \lambda E)\nu \neq 0$ *will be called the root function of this operator, which corresponds to the eigenvalue* $\lambda \in \mathbb{C}$ *[19].*

Let us consider the heat conduction equation with involution in the region $D_T = \{-1<x<1, 0<t\leqslant T\}$:

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} + \alpha_1(1+\gamma x)\left(\frac{\partial^2 u(x,t)}{\partial x^2} - \frac{\partial^2 u(-x,t)}{\partial x^2}\right) + \alpha_2\left(\frac{\partial u(x,t)}{\partial x} + \frac{\partial u(-x,t)}{\partial x}\right) - r(t)u(x,t) + f(x,t), \quad (x,t) \in D_T, \tag{5}$$

with boundary conditions

$$\begin{cases} \beta_1 u(-1,t) + \beta_2 u(1,t) = 0, \\ \dfrac{\partial u(-1,t)}{\partial x} + \dfrac{\partial u(1,t)}{\partial x} = 0, \end{cases} \quad 0 \leqslant t \leqslant T, \quad \beta_1, \beta_2 \in \mathbb{R}, \quad \beta_1 \neq \beta_2, \tag{6}$$

initial condition

$$u(x,0) = \eta(x), \quad -1 \leqslant x \leqslant 1 \tag{7}$$

and redefinition condition

$$\int_{-1}^{1} x\, u(x,t)\, dx = E(t). \tag{8}$$

Definition 6. *A pair of functions* $\{r(t), u(x,t)\}$ *from the set* $C[-1,1] + \left(C^{2,1}(D_T) \cap C^{1,0}(\overline{D_T})\right)$ *will be called a classical solution of the inverse problem (5)–(8).*

Let $\boldsymbol{L}\colon L_2(-1,1) \to L_2(-1,1)$ be an operator of the problem

$$-\nu''(x) + \alpha_1(1+\gamma x)\big(\nu''(x) - \nu''(-x)\big) + \alpha_2\big(\nu'(x) + \nu'(-x)\big) = f(x), \quad \alpha_1, \alpha_2 \in \mathbb{R}, \quad -1 < x < 1, \tag{9}$$

$$\begin{cases} \ell_1\nu := \beta_1\nu(-1) + \beta_2\nu(1) = 0, \\ \ell_2\nu := \nu'(-1) + \nu'(1) = 0, \end{cases} \quad \beta_1 \neq \beta_2. \tag{10}$$

$$D(\boldsymbol{L}) = \left\{\nu \in W_2^2(-1,1)\colon \ell_1\nu = \ell_2\nu = 0\right\}.$$

Theorem 1. ***A.*** *For any* $\beta_1, \beta_2 \in \mathbb{R}$, *if* $\beta_1 \neq \beta_2$ *then the operator* $\boldsymbol{L}$ *has the system of root functions*

$$V_h := \left\{\nu_{s,m}(x) \in L_2(-1,1)\colon \nu_{1,m}(x) = (1+hx)\sin\left(m - \tfrac{1}{2}\right)\pi x, \nu_{0,m} = \cos\left(m - \tfrac{1}{2}\right)\pi x, m = 1, \dots\right\}, \tag{11}$$

which is the Riesz basis of the space $L_2(-1,1)$, $h = \frac{\beta_1+\beta_2}{\beta_1-\beta_2}$.

In this case, there is a biorthogonal system

$$W_h := \left\{w_{s,m}(x) \in L_2(-1,1)\colon w_{1,m}(x) = \sin\left(m - \tfrac{1}{2}\right)\pi x, w_{0,m} = (1-hx)\cos\left(m - \tfrac{1}{2}\right)\pi x, m = 1, \dots\right\}. \tag{12}$$

A.1. *Let* $\gamma = \alpha_2 = \frac{\beta_1+\beta_2}{\beta_1-\beta_2}$. *Then the operator* $\boldsymbol{L}$ *has the set of eigenvalues* $\sigma \cup \sigma_1$, *where* $\sigma := \left\{\lambda_k \in \mathbb{R}, \lambda_k = \pi^2(k - \frac{1}{2})^2, k = 1, \dots\right\}$, $\sigma_1 := \{\lambda_{1,k} \in \mathbb{R}, \lambda_{1,k} = (1-2\alpha_1)\lambda_k, \lambda_k \in \sigma, k = 1, \dots\}$ *and the system of eigenfunctions* V_h.

A.2. *Let* $\alpha_1 = 0$, $\gamma \neq \frac{\beta_1+\beta_2}{\beta_1-\beta_2}$. *Then the operator* $\boldsymbol{L}$ *has the set of double eigenvalues* σ *and the system of eigenfunctions* V_h.

A.3. *Let* $\alpha_1 = 0$, $\gamma = \frac{\beta_1+\beta_2}{\beta_1-\beta_2}$. *Then the operator* $\boldsymbol{L}$ *has the set of double eigenvalues* σ*and the system of eigenfunctions* V_h.

Let

$$f(t,x)=\sum_{k=1}^{\infty}\big(f_{0,k}(t)\,\nu_{0,k}(x)+f_{1,k}(t)\,\nu_{1,k}(x)\big),$$

$$\eta(x)=\sum_{k=1}^{\infty}\big(\eta_{0,k}\,\nu_{0,k}(x)+\eta_{1,k}\,\nu_{1,k}(x)\big).$$

Theorem 2. ***A.1.*** *Let $\gamma=\alpha_2=\frac{\beta_1+\beta_2}{\beta_1-\beta_2}$ and the following assumptions hold:*

B1. $\eta\in C^4[-1,1]$, $\beta_1\eta(-1)+\beta_2\eta(1)=0$, $\eta'(-1)+\eta'(1)=0$, $\int_{-1}^{1}x\,\eta(x)\,dx=E(0)$;

B2. $E(t)\in C^1[-1,1]$;

B3. $f(x,t)\in C(\overline{D_T})\cap C^4(D_T)$, $\beta_1 f(-1,t)+\beta_2 f(1,t)=0$, $\frac{\partial f(-1,t)}{\partial x}+\frac{\partial f(1,t)}{\partial x}=0$, $\int_{-1}^{1}x\,f(x,t)\,dx\neq 0$;

B4. $\mu_k=(2k-1)\pi h$, $k=1,\ldots$.

Then, there is an unique solution of the problem (5)–(7):

$$u(x,t)=\sum_{k=1}^{\infty}\Bigg(\bigg(\eta_{1,k}\,e^{-\lambda_{1,k}t}+\int_0^t r(\tau)\,f_{1,k}(\tau)\,e^{-\lambda_{1,k}(t-\tau)}\,d\tau\bigg)\,\nu_{1,k}(x)$$
$$+\bigg(\eta_{0,k}\,e^{-\lambda_k t}+\int_0^t r(\tau)\,f_{0,k}(\tau)\,e^{-\lambda_k(t-\tau)}\,d\tau\bigg)\,\nu_{0,k}(x)\Bigg),\quad (13)$$

and the pair of functions $\{r(t),u(x,t)\}$ is the unique solution of the inverse problem (5)–(8);

A.2. *Let $\alpha_1=0$, $\gamma\neq\frac{\beta_1-\beta_2}{\beta_2+\beta_1}$ and the Assumptions B1–B4 hold.*

Then, there is an unique solution of the problem (5)–(7)*:*

$$u(x,t)=\sum_{k=1}^{\infty}\Bigg(\bigg(\eta_{0,k}\,e^{-\lambda_k t}+\int_0^t r(\tau)\,f_{0,k}(\tau)\,e^{-\lambda_k(t-\tau)}\,d\tau\bigg)\,\nu_{0,k}(x)$$
$$-\mu_k\int_0^t\bigg(\int_0^{\tau} r(\rho)\,f_{1,k}(\rho)\,e^{-\lambda_k(\rho-\tau)}\,d\rho\bigg)\,e^{-\lambda_k(\tau-t)}\,d\tau\,\nu_{0,k}(x)$$
$$+\bigg(\eta_{1,k}\,e^{-\lambda_k t}+\int_0^t r(\tau)\,f_{1,k}(\tau)\,e^{-\lambda_k(\tau-t)}\,d\tau\bigg)\,\nu_{1,k}(x)\Bigg),\quad (14)$$

and the pair of functions $\{r(t),u(x,t)\}$ is the unique solution of the inverse problem (5)–(8);

A.3. *Let $\alpha_1=0$, $\gamma=\frac{\beta_1-\beta_2}{\beta_2+\beta_1}$, and the Assumptions B1–B3 hold. Then, there is an unique solution of the problem (5)–(7):*

$$u(x,t)=\sum_{k=1}^{\infty}\Bigg(\bigg(\eta_{0,k}\,e^{-\lambda_k t}+\int_0^t r(\tau)\,f_{0,k}(\tau)\,e^{-\lambda_k(t-\tau)}\,d\tau\bigg)\,\nu_{0,k}(x)$$
$$+\bigg(\eta_{1,k}\,e^{-\lambda_k t}+\int_0^t r(\tau)\,f_{1,k}(\tau)\,e^{-\lambda_k(t-\tau)}\,d\tau\bigg)\,\nu_{1,k}(x)\Bigg),\quad (15)$$

and the pair of functions $\{r(t),u(x,t)\}$ is the unique solution of the inverse problem (5)–(8).

3. Proof of the Theorem 1

3.1. Let us consider the eigenvalue problem for equation

$$-\nu''(x)=\lambda\,\nu(x),\quad \lambda\in\mathbb{C},\quad -1\leqslant x\leqslant 1,\quad (16)$$

with boundary conditions (10).

Determine the fundamental system of solutions for the equation (16)

$$\begin{cases}\nu_0(x,\varrho)=e^{\varrho x}+e^{-\varrho x},\\ \nu_1(x,\varrho)=e^{\varrho x}-e^{-\varrho x},\end{cases}\quad \operatorname{Re}\varrho\leqslant 0,\quad \lambda=\varrho^2,$$

and substitute the general solution

$$\nu(x,\varrho)=C_0\,\nu_0(x,\varrho)+C_1\,\nu_1(x,\varrho),\quad C_0,C_1\in\mathbb{R},$$

of the equation (16) in boundary conditions (10).

There is obtained the system of linear algebraic equations with a matrix of coefficients

$$\Omega(\varrho) = \begin{pmatrix} \omega_1(\varrho) & \omega_2(\varrho) \\ 0 & \omega_3(\varrho) \end{pmatrix}$$

to determine the parameters C_0, C_1, where $\omega_2(\varrho) = -2(\beta_1-\beta_2)(e^{\varrho}-e^{-\varrho})$, $\omega_1(\varrho) = 2(\beta_1+\beta_2)(e^{-\varrho}+e^{\varrho})$, $\omega_3(\varrho) = 2\varrho(e^{-\varrho}+e^{\varrho})$.

To determine the eigenvalues of problem (16), (10) the characteristic equation is used $\det\Omega(\varrho) = 4\varrho(\beta_1+\beta_2)(e^{-\varrho}+e^{\varrho})^2$, with the roots 0, $\pi(k-\frac{1}{2})$, $k = \pm1, \pm2, \ldots$.

Therefore, problem (15), (10) has eigenvalues $\lambda_k = \pi^2\big(k-\frac{1}{2}\big)^2$, $k = 1, \ldots$, and corresponding eigenfunctions $\nu_{0,k}(x) := \cos\pi\big(k-\frac{1}{2}\big)x$, $k = 1, \ldots$.

The associated functions of the problem are defined by relations

$$\nu_{1,k} := (1+hx)\sin\pi\left(k-\tfrac{1}{2}\right)x, \quad k = 1, \ldots.$$

Substituting these expressions into the boundary conditions (10), one can obtain $h = \frac{\beta_1-\beta_2}{\beta_2+\beta_1}$.

Therefore, the operator of problem (16), (10) has a spectrum σ and the system of functions V_h, being the root functions in the sense of ratios [20]:

$$\begin{cases} -\nu''_{0,k}(x) = \lambda_k\,\nu_{0,k}(x), \\ -\nu''_{1,k}(x) = \lambda_k\,\nu_{1,k}(x) + \mu_k\,\nu_{0,k}(x), \end{cases} \qquad k = 1, \ldots,$$

where $\mu_k = (2k-1)\pi h$, $k = 1, \ldots$.

Remark 1. Note, that in the case $\beta_2 = -\beta_1$ the boundary conditions (10) coincide with the antiperiodic conditions, $\mu_k = 0$, $k = 1, \ldots$, and the system of functions (11) is an orthonormal basis in $L_2(-1,1)$: $V_0 = \big\{\tau_{s,k}(x) \in L_2(-1,1)\colon \tau_{0,k}(x) = \cos\pi\big(k-\frac{1}{2}\big)x, \tau_{1,k}(x) = \sin\pi\big(k-\frac{1}{2}\big)x, k = 1, \ldots\big\}$.

If $\beta_2 = -\beta_1$, then the boundary conditions (10) are singular and $\det\Omega(\varrho) \equiv 0$.

The operator of the associated problem to (16), (10)

$$-w''(x) = \overline{\lambda}w(x), \quad \lambda \in \mathbb{C}, \quad -1 \leqslant x \leqslant 1,$$

$$\begin{cases} w(-1) + w(1) = 0, \\ \beta_2 w'(-1) + \beta_1 w'(1) = 0, \end{cases}$$

has the system (12) of root functions that is bi orthogonal in the sense of equalities

$$\big(\nu_{r,k}, w_{s,m}\big)_{L_2(-1,1)} = \delta_{r,s}\,\delta_{r,m}, \quad r, s = 0, 1, \quad k, m = 1, \ldots.$$

Lemma 1. *For arbitrary numbers $\beta_1, \beta_2 \in \mathbb{R}$, $\beta_1 \neq -\beta_2$ the system of functions V_h is the Riesz basis in the space $L_2(-1,1)$.*

Proof. The boundary conditions (10) are regular by Birkhoff. Therefore, systems of functions V_h, W_h are complete and minimal in space $L_2(-1,1)$.

From the definition of these systems for an arbitrary function $\varphi \in L_2(-1,1)$ we obtain Bessel inequalities [21]:

$$\begin{cases} \displaystyle\sum_{k=1}^{\infty}\sum_{r=0}^{1} \big(\varphi, \nu_{r,k}\big)^2_{L_2(-1,1)} \leqslant M_0\|\varphi\|^2_{L_2(-1,1)}, \\ \displaystyle\sum_{m=1}^{\infty}\sum_{s=0}^{1} \big(\varphi, w_{s,m}\big)^2_{L_2(-1,1)} \leqslant M_0\|\varphi\|^2_{L_2(-1,1)}, \end{cases} \qquad M_0 = 2(1+h^2).$$

Therefore, applying theorem N. K. Bari (see [21]), we obtain the statement of Lemma 1.

Thus, the statement **A.1** of Theorem 1 is proved.

3.2. Let $O(V_h, \sigma)$ be the set of operators $\boldsymbol{L}\colon L_2(-1,1) \to L_2(-1,1)$, which have the point spectrum σ and the system of root functions V_h in the sense of ratios

$$\begin{cases} \boldsymbol{L}\nu_{0,k}(x) = \lambda_k\nu_{0,k}(x), \quad k = 1\ldots, \\ \boldsymbol{L}\nu_{1,k}(x) = \lambda_k\nu_{1,k}(x) + \mu_k\nu_{0,k}(x), \quad k = 1, \ldots, \end{cases} \tag{17}$$

for some real numbers μ_k, $k = 1, \ldots$.

Let us consider the operator $\boldsymbol{L}\colon L_2(-1,1) \to L_2(-1,1)$, generated by equation

$$\boldsymbol{L}\nu := -\nu''(x) + \alpha_2\big(\nu'(x) + \nu'(-x)\big) = \lambda\nu(x) = 0, \quad \lambda \in \mathbb{C}, \quad \alpha_2 \in \mathbb{R}, \quad -1 < x < 1, \tag{18}$$

and boundary conditions (10).

By substituting functions (11) into equation (18), we obtain the relations (17), where

$$\mu_k = (\alpha_2 - h)2k\pi, \quad k = 1, \dots.$$

Therefore, $\boldsymbol{L} \in O(V_h, \sigma)$. Thus, the statement **A.2** of Theorem 1 is proved.

If equality $\alpha_2 = h$ holds, then $\mu_k = 0$. In this case the elements of system V_h are eigenfunctions of operator $\boldsymbol{L}$. Therefore, **A.3** of Theorem 1 is proved.

Let $\sigma_1 := \{\lambda_{1,k} \in \mathbb{R}, k = 1, \dots\}$ and $O(V_h, \sigma, \sigma_1)$ be the set of operators $\boldsymbol{L}\colon L_2(-1,1) \to L_2(-1,1)$, with the point spectrum $\sigma \cup \sigma_1$ and system of eigenfunctions V_h:

$$\begin{cases} \boldsymbol{L}\nu_{0,k}(x) = \lambda_k \nu_{0,k}(x), \quad \lambda_k \in \sigma, \quad k = 1, \dots, \\ \boldsymbol{L}\nu_{1,k}(x) = \lambda_{1,k}\nu_{1,k}(x), \quad \lambda_{1,k} \in \sigma_1. \end{cases}$$

Let us consider the operator $\boldsymbol{L}$ of problem (9)–(10). By substituting functions (11) into equation (9), we obtain

$$\begin{cases} \boldsymbol{L}\nu_{0,k}(x) = \lambda_k\, \nu_{0,k}(x), \\ \boldsymbol{L}\nu_{1,k}(x) = \lambda_k\, \nu_{1,k}(x) - 2\alpha_1\, \lambda_k\, (1 + \gamma x)\, \tau_{1,k}(x) + \mu_k\, \nu_{0,k}(x), \end{cases} \quad k = 1, \dots,$$

$$\mu_k = (h - \alpha_2)(2k\pi - 1), \ k = 1, \dots,$$

$$\boldsymbol{L}\nu_{1,k}(x) = \lambda_{1,k}\, \nu_{1,k}(x) + \mu_k\, \nu_{0,k}(x), \quad \lambda_{1,k} := (1 - 2\alpha_1)\lambda_k, \quad k = 1, \dots.$$

Therefore, $\boldsymbol{L} \notin O(V_h, \sigma)$.

If $\gamma = \alpha_2 = h = \frac{\beta_1 - \beta_2}{\beta_2 + \beta_1}$, then $\mu_k = 0$, $k = 1, \dots$.

Therefore,

$$\begin{cases} \boldsymbol{L}\nu_{0,k}(x) = \lambda_k\, \nu_{0,k}(x), \\ \boldsymbol{L}\nu_{1,k}(x) = \lambda_{1,k}\, \nu_{1,k}(x), \end{cases} \quad k = 1, \dots. \tag{19}$$

Thus, V_h is the system of eigenfunctions of operator $\boldsymbol{L}$ for which the equalities (19) hold, where $\sigma_1 := \{\lambda_{1,k} \in \mathbb{R}, \lambda_{1,k} = (1 - 2\alpha_1)\lambda_k, k = 1, \dots\}$.

Then, $\boldsymbol{L} \in O(V_h, \sigma, \sigma_1)$. Therefore, taking into account Lemma 1, we obtain the following statement.

Lemma 2. *For any numbers $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$, where $2\alpha_1 \neq 1$, $\beta_1 \neq -\beta_2$, the system of eigenfunctions V_h of operator $\boldsymbol{L}$ is Riesz basis of the space $L_2(-1,1)$.*

Consequently, the statement **A.1** of Theorem 1 holds.

Note, that for the case $\alpha_2 = \gamma = 0$ the spectral properties of operator $\boldsymbol{L}$ are investigated in [16,20].

4. Existence and uniqueness of solution to the problem (5)–(8)

4.1. Let the conditions $\gamma = \alpha_2 = \frac{\beta_1 - \beta_2}{\beta_2 + \beta_1}$ and Assumptions B1–B3 are true. Partial solutions of problem (5)–(7) are determined by relations

$$\begin{cases} u_{0,k}(x,t) = T_{0,k}(t)\, \nu_{0,k}(x), \\ u_{1,k}(x,t) = T_{1,k}(t)\, \nu_{1,k}(x), \end{cases} \quad k = 1, \dots.$$

To determine functions $T_{r,k}(t)$, we obtain problems that are solved sequentially

$$\begin{cases} T'_{0,k}(t) = \lambda_k\, T_{0,k}(t) + r(t)\, f_{0,k}, \quad T_{0,k}(0) = \eta_{0,k}, \\ T'_{1,k}(t) = \lambda_{1,k}\, T_{1,k}(t) + r(t)\, f_{1,k}(t), \quad T_{1,k}(0) = \eta_{1,k}, \end{cases} \quad k = 1, \dots.$$

Therefore,

$$\begin{cases} T_{1,k}(t) = \eta_{1,k}\, e^{-\lambda_{1,k} t} + \displaystyle\int_0^1 r(\tau)\, f_{1,k}(\tau)\, e^{-\lambda_{1,k}(t-\tau)}\, d\tau, \\ T_{0,k}(t) = \eta_{0,k}\, e^{-\lambda_k t} + \displaystyle\int_0^1 r(\tau)\, f_{0,k}(\tau)\, e^{-\lambda_k(t-\tau)}\, d\tau, \end{cases} \quad k = 1, \dots.$$

$$\begin{cases} u_{1,k}(x,t) = \left(\eta_{1,k}\, e^{-\lambda_{1,k} t} + \int_0^t r(\tau)\, f_{1,k}(\tau)\, e^{-\lambda_{1,k}(t-\tau)}\, d\tau\right)\nu_{1,k}(x), \\ u_{0,k}(x,t) = \left(\eta_{0,k}\, e^{-\lambda_k t} + \int_0^t r(\tau)\, f_{0,k}(\tau)\, e^{-\lambda_k(t-\tau)}\, d\tau\right)\nu_{0,k}(x), \end{cases} \quad k = 1, \ldots.$$

From the continuity of $\eta(x)$ and the boundedness of functions (11) we obtain

$$|\eta_{r,k}| \leqslant M_0,\ \max|r(t)| = M_1,\ \max|f_{r,k}(t)| = M_2,\ \max|v_{1,k}(x)| = 1 + h = M_3,\ r = 0,1,\ k = 1, \ldots.$$

Taking into account these correlations, we have the estimates

$$|u_{0,k}(x,t)| \leqslant (|\eta_{0,k}| + \max|r(t)| \cdot \max|f_{r,k}(t)|)e^{-\lambda_k \varepsilon} \leqslant M_4 e^{-\lambda_k \varepsilon}, \quad M_4 = M_0 + M_1 M_2; \tag{20}$$

$$|u_{1,k}(x,t)| \leqslant \max|v_{1,k}(x)|(|\eta_{1,k}| + \max|r(t)| \cdot \max|f_{1,k}(t)|)e^{-\lambda_{1,k}\varepsilon} \leqslant M_5 e^{-\lambda_{1,k}\varepsilon}, \quad M_5 = M_3 M_4. \tag{21}$$

Therefore given $M_6 = (1 + M_3)M_4$ the functional series

$$\sum_{k=1}^{\infty} \sum_{s=0}^{1} u_{s,k}(x,t) \tag{22}$$

is majorized with an absolutely convergent numerical series

$$M_6 \sum_{k=1}^{\infty} \big(e^{-\lambda_{1,k}\varepsilon} + e^{-\lambda_k \varepsilon}\big).$$

Therefore, according to the Weierstrass sign, the series (22) are uniformly convergent and continuous for $t \geqslant \varepsilon$ functions. Thus, the sum of the series (22) determines the continuous function $u(x,t)$, satisfying the initial condition (7).

We differentiate element-by-element the series (22) by variable t:

$$\sum_{k=1}^{\infty} \Bigg(\left(-\lambda_{1,k}\,\eta_{1,k}\, e^{-\lambda_{1,k} t} + r(t)\, f_{1,k}(t) + \int_0^t r(\tau)\, f_{1,k}(\tau)\, e^{-\lambda_{1,k}(t-\tau)}\, d\tau\right) \nu_{1,k}(x) + \left(-\lambda_k\, \eta_{0,k}\, e^{-\lambda_k t} + r(t)\, f_{0,k}(t) + \int_0^t r(\tau)\, f_{0,k}(\tau)\, e^{-\lambda_k(t-\tau)}\, d\tau\right) \nu_{0,k}(x)\Bigg).$$

Let us consider

$$\left|\frac{\partial u_{1,k}(x,t)}{\partial t}\right| \leqslant M_3\big(M_1\,|f_{1,k}(t)| + \big(|\lambda_{1,k}|\, M_0 M_2 + M_1 T M_2\big)e^{-\lambda_{1,k}\varepsilon}\big), \tag{23}$$

$$\left|\frac{\partial u_{0,k}(x,t)}{\partial t}\right| \leqslant M_1\,|f_{0,k}(t)| + \big(\lambda_k\, M_0 M_2 + M_1 T M_2\big)e^{-\lambda_k \varepsilon}, \tag{24}$$

$$t \geqslant \varepsilon > 0, \quad k = 1, \ldots.$$

Taking into account the inequalities (23), (24), we get

$$\sum_{k=1}^{\infty} \sum_{s=0}^{1} \left|\frac{\partial u_{s,k}(x,t)}{\partial t}\right| \leqslant (M_3 + 1)M_2 \sum_{k=1}^{\infty} M_0\big(|\lambda_{1,k}|\, e^{-\lambda_{1,k}\varepsilon} + \lambda_k\, e^{-\lambda_k \varepsilon}\big) + M_1 T\big(e^{-\lambda_{1,k}\varepsilon} + e^{-\lambda_k \varepsilon}\big) + M_1(M_3 + 1) \sum_{k=1}^{\infty} \big(|f_{0,k}(t)| + |f_{1,k}(t)|\big).$$

Using Assumption B3 and the Abel's theorem of convergence of functional series, uniform convergence and continuity of the sum of series is obtained $\sum_{k=1}^{\infty} \big(|f_{0,k}(t)| + |f_{1,k}(t)|\big)$ in the domain D_T. Therefore is $M_7 > 0$ an inequality that is correct

$$\sum_{k=1}^{\infty} \big(|f_{0,k}(t)| + |f_{1,k}(t)|\big) \leqslant M_7.$$

Therefore, the sum of the series

$$\sum_{k=1}^{\infty} \sum_{s=0}^{1} \frac{\partial u_{s,k}(x,t)}{\partial t}$$

is continuous function in D_T and coincides with $\frac{\partial u(x,t)}{\partial t}$.

We differentiate the series (22) element by element twice by the variable x:

$$\sum_{k=1}^{\infty}\sum_{s=0}^{1}\frac{\partial^2 u_{s,k}(x,t)}{\partial x^2}=\sum_{k=1}^{\infty}\Bigg(-\lambda_{1,k}\Bigg(\eta_{1,k}\,e^{-\lambda_{1,k}t}+\int_0^t r(\tau)\,f_{1,k}(\tau)\,e^{-\lambda_{1,k}(t-\tau)}\,d\tau\Bigg)\,\nu_{1,k}(x)$$
$$-\lambda_k\Bigg(\eta_{0,k}\,e^{-\lambda_k t}+\int_0^t r(\tau)\,f_{0,k}(\tau)\,e^{-\lambda_k(t-\tau)}\,d\tau\Bigg)\,\nu_{0,k}(x)\Bigg).$$

Taking into account the formulas (20), (21) we get

$$\left|\frac{\partial^2 u_{1,k}(x,t)}{\partial x^2}\right|\leqslant|\lambda_{1,k}|\,M_5\,e^{-\lambda_{1,k}\varepsilon},\qquad\left|\frac{\partial^2 u_{0,k}(x,t)}{\partial x^2}\right|\leqslant\lambda_k\,M_4\,e^{-\lambda_k\varepsilon},\quad k=1,\dots.$$

The resulting series is majorized given some $t\geqslant\varepsilon>0$ by series

$$M_6\sum_{k=1}^{\infty}\big(|\lambda_{1,k}|\,e^{-\lambda_{1,k}\varepsilon}+\lambda_k\,e^{-\lambda_k\varepsilon}\big).$$

Therefore, sum of this series is continuous function in D_T and coincides with $\frac{\partial^2 u(x,t)}{\partial x^2}$.

Similarly, the smoothness of the function $\frac{\partial^2 u(-x,t)}{\partial x^2}$ is investigated. Further, by the embedding theorems we obtain the continuity of functions $\frac{\partial u(x,t)}{\partial x}$, $\frac{\partial u(-x,t)}{\partial x}$ in D_T.

Therefore, the sum of the series (22) is a classical solution of the problem (5)–(7).

Let us consider the equation to define the function $r(t)$:

$$\int_0^1 x\frac{\partial u(x,t)}{\partial t}dx=E'(t)=\sum_{k=1}^{\infty}\frac{(-1)^k}{\lambda_k}\Bigg(r(t)f_{1,k}(t)-\lambda_{1,k}\Bigg(\eta_{1,k}e^{-\lambda_{1,k}t}+\int_0^t r(\tau)f_{1,k}(\tau)e^{-\lambda_{1,k}(t-\tau)}d\tau\Bigg)\Bigg);$$

$$8r(t)\sum_{k=1}^{\infty}\frac{(-1)^k}{\lambda_k}f_{1,k}(t)=E'(t)+8\sum_{k=1}^{\infty}\lambda_{1,k}\frac{(-1)^k}{\lambda_k}\Bigg(\eta_{1,k}e^{-\lambda_{1,k}t}+\int_0^t r(\tau)f_{1,k}(\tau)e^{-\lambda_{1,k}(t-\tau)}d\tau\Bigg);$$

$$8r(t)\sum_{k=1}^{\infty}\frac{(-1)^k}{\lambda_k}f_{1,k}(t)=E'(t)+(1-2\alpha_1)\sum_{k=1}^{\infty}8(-1)^k\Bigg(\eta_{1,k}e^{-\lambda_{1,k}t}+\int_0^t r(\tau)f_{1,k}(\tau)e^{-\lambda_{1,k}(t-\tau)}d\tau\Bigg);$$

$$r(t)=\frac{E'(t)+8(1-2\alpha_1)\sum_{k=1}^{\infty}(-1)^k\eta_{1,k}e^{-\lambda_{1,k}t}}{8\sum_{k=1}^{\infty}\frac{(-1)^k}{\lambda_k}f_{1,k}(t)}+\frac{(1-2\alpha_1)\sum_{k=1}^{\infty}(-1)^k\int_0^t r(\tau)f_{1,k}(\tau)e^{-\lambda_{1,k}(t-\tau)}d\tau}{\sum_{k=1}^{\infty}\frac{(-1)^k}{\lambda_k}f_{1,k}(t)}.$$

So, to determine the function $r(t)$ the Volterra integral equation of the second kind is obtained:

$$r(t)=F(t)+\int_0^t K(t,\tau)\,r(\tau)\,d\tau, \tag{25}$$

where

$$F(t)=\frac{E'(t)+8(1-2\alpha_1)\sum_{k=1}^{\infty}(-1)^k\eta_{1,k}\,e^{-\lambda_{1,k}t}}{\sum_{k=1}^{\infty}8\frac{(-1)^k}{\lambda_k}f_{1,k}(t)}, \tag{26}$$

$$K(t,\tau)=\frac{(1-2\alpha_1)\sum_{k=1}^{\infty}(-1)^k f_{1,k}(\tau)\,e^{-\lambda_{1,k}(t-\tau)}}{\sum\limits_{k=1}^{\infty}\frac{(-1)^k}{\lambda_k}f_{1,k}(t)}, \tag{27}$$

The denominator of fractions (26), (27) is not equal to zero, because the Assumption B3 is obtained

$$\int_{-1}^{1}x\,f(x,t)\,dx=8\sum_{k=1}^{\infty}\frac{(-1)^k}{\lambda_k}f_{1,k}(t)\neq0.$$

According to Assumptions B1–B3, the function $F(t)$ and the kernel $K(t,\tau)$ are continuous functions on $[0,T]$ and $[0,1]\times[0,T]$ respectively.

Therefore, equation (25) has an unique solution. This solution is a continuous function $r(t)$ on $[0,T]$, which forms an unique solution $\{r(t),u(x,t)\}$ of inverse problem (5)–(8) together with the given Fourier series (13) as a solution $u(x,t)$ of the direct problem (5)–(7).

Then, the statement **A.1** of Theorem 2 is proved.

4.2. Let $\alpha_1 = 0$. Then $\lambda_k = \lambda_{1,k}$, $k = 1, \ldots$.

In the case of $\gamma \neq \frac{\beta_1-\beta_2}{\beta_2+\beta_1}$ the elements of system V_h are the root functions of the operator $\boldsymbol{L}$ for which equalities (17) hold.

Let us consider the proof of statement **A.2** of Theorem 2.

The partial solutions of problem (5)–(7) are determined by relations (19). To find the functions $T_{r,k}(t)$ we obtain the following problems

$$\begin{cases} T'_{0,k}(t) = -\lambda_k\, T_{0,k}(t) - r(t)\, f_{0,k}(t), \quad T_{0,k}(0) = \eta_{0,k}, \\ T'_{1,k}(t) = -\lambda_k\, T_{1,k}(t) - \mu_k\, \eta_{0,k} + r(t)\, f_{0,k}(t), \quad T_{1,k}(0) = \eta_{1,k}, \end{cases} \qquad k = 1, \ldots,$$

that are solved sequentially.

Therefore,

$$\begin{cases} T_{0,k}(t) = \eta_{0,k}\, e^{-\lambda_k t} + \displaystyle\int_0^t r(\tau)\, f_{0,k}(\tau)\, e^{-\lambda_k (t-\tau)}\, d\tau, \\ T_{1,k}(t) = \eta_{1,k}\, e^{-\lambda_k t} + \displaystyle\int_0^t r(\tau)\, f_{1,k}(\tau)\, e^{-\lambda_k (t-\tau)}\, d\tau, \end{cases} \qquad k = 1, \ldots,$$

$$\begin{cases} u_{0,k}(x,t) = \left(\eta_{0,k}\, e^{-\lambda_k t} + \displaystyle\int_0^t r(\tau)\, f_{0,k}(\tau)\, e^{-\lambda_k (t-\tau)}\, d\tau \right) \nu_{0,k}(x), \\ u_{1,k}(x,t) = \left(\eta_{1,k}\, e^{-\lambda_k t} + \displaystyle\int_0^t r(\tau)\, f_{1,k}(\tau)\, e^{-\lambda_k (t-\tau)}\, d\tau \right) \nu_{1,k}(x), \end{cases} \qquad k = 1, \ldots.$$

Taking into account the Assumptions B1–B3 of the theorem and the formulas (20), (21), we obtain the estimates

$$|u_{1,k}(x,t)| \leqslant \max |v_{1,k}| \big(|\eta_{1,k}| + |f_{1,k}(t)| \cdot \max |r(t)| \big)\, e^{-\lambda_k \varepsilon} \leqslant M_5\, e^{-\lambda_k \varepsilon},$$
$$|u_{0,k}(x,t)| \leqslant \big(|\eta_{0,k}| + |f_{0,k}(t)| \cdot \max |r(t)| \big)\, e^{-\lambda_k \varepsilon} \leqslant M_4\, e^{-\lambda_k \varepsilon}.$$

Therefore, the functional series (22) is majorized by an absolutely convergent numerical series

$$2M_6 \sum_{k=1}^{\infty} e^{-\lambda_k \varepsilon}$$

for $t \geqslant \varepsilon > 0$. This is why based on the Weierstrass test, series (22) is uniformly convergent and continuous functions for $t \geqslant \varepsilon$.

Thus, the sum of the series (22) defines a continuous function that satisfies the initial condition (7).

By direct substitution, we make sure that

$$\begin{cases} \dfrac{\partial u_{0,k}(x,t)}{\partial t} = \left(r(t)\, f_{0,k}(t) - \lambda_k \left(\eta_{0,k}\, e^{-\lambda_k t} + \displaystyle\int_0^t r(\tau)\, f_{0,k}(\tau)\, e^{-\lambda_k (t-\tau)}\, d\tau \right) \right) \nu_{0,k}(x), \\ \dfrac{\partial u_{1,k}(x,t)}{\partial t} = \left(r(t)\, f_{1,k}(t) - \lambda_k \left(\eta_{1,k}\, e^{-\lambda_k t} + \displaystyle\int_0^t r(\tau)\, f_{1,k}(\tau)\, e^{-\lambda_k (t-\tau)}\, d\tau \right) \right) \nu_{1,k}(x), \end{cases}$$

for $k = 1, \ldots$.

Taking into account the formulas (23), (24) we get

$$\begin{aligned} \left| \frac{\partial u_{1,k}(x,t)}{\partial t} \right| &\leqslant \max |v_{1,k}(x)| \big(M_1\, |f_{1,k}(t)| + \lambda_k (M_0 + M_1 M_2)\, e^{-\lambda_k \varepsilon} \big) \\ &\leqslant (M_0 + M_1 M_2) M_3 \lambda_k e^{-\lambda_k \varepsilon} + M_1 M_3\, |f_{1,k}(t)|, \\ \left| \frac{\partial u_{0,k}(x,t)}{\partial t} \right| &\leqslant \big(M_0 + |f_{0,k}(t)| \cdot M_1 \big) e^{-\lambda_k \varepsilon} \leqslant M_4\, e^{-\lambda_k \varepsilon}, \quad k = 1, \ldots. \end{aligned}$$

Let us differentiate element by element series (22), by two times with regard to argument x

$$\sum_{k=1}^{\infty} \sum_{s=0}^{1} \frac{\partial^2 u_{s,k}(x,t)}{\partial x^2} = \sum_{k=1}^{\infty} \big(\big(-\lambda_k\, T_{0,k}(t) + \mu_k\, T_{1,k}(t) \big)\, \nu_{0,k}(x) - \lambda_k\, T_{1,k}(t)\, \nu_{1,k}(x) \big).$$

Taking into account the formulas (20), (21) we get

$$\left|\frac{\partial^2 u_{0,k}(x,t)}{\partial x^2}\right| \leqslant M_4\,\lambda_k\,e^{-\lambda_k\varepsilon} + |\mu_k|\,M_5\,e^{-\lambda_{1,k}\varepsilon} \leqslant 2\big(\lambda_k + |\mu_k|\big)M_5\,e^{-\lambda_{1,k}},$$

$$\left|\frac{\partial^2 u_{1,k}(x,t)}{\partial x^2}\right| \leqslant \lambda_k\,M_5\,e^{-\lambda_k\varepsilon}, \quad k = 1,\ldots.$$

For some $t \geqslant \varepsilon > 0$, $M_8 > 0$ we get

$$\sum_{k=1}^{\infty}\sum_{s=0}^{1}\left|\frac{\partial^2 u_{s,k}(x,t)}{\partial x^2}\right| \leqslant M_8\sum_{k=1}^{\infty}\big(\lambda_k + \sqrt{\lambda_k}\big)e^{-\lambda_k\varepsilon}.$$

Therefore, its sums is continuous function in the domain D_T and coincides with $\frac{\partial^2 u(x,t)}{\partial x^2}$. Similarly, the smoothness of the function $\frac{\partial^2 u(-x,t)}{\partial x^2}$ is obtained. By embedding theorems, we obtain continuity of functions $\frac{\partial u(x,t)}{\partial x}$, $\frac{\partial u(-x,t)}{\partial x}$ in D_T. Thus, defined by series (14) function $u(x,t)$ is a classical solution of the problem (5)–(7),

$$\int_0^1 x\,\frac{\partial u(x,t)}{\partial t}\,dx = E'(t) = 8\sum_{k=1}^{\infty}\frac{(-1)^k}{\lambda_k}\left(r(t)\,f_{1,k}(t) - \lambda_k\left(\eta_{1,k}\,e^{-\lambda_k t} + \int_0^t r(\tau)\,f_{1,k}(\tau)\,e^{-\lambda_k(t-\tau)}\,d\tau\right)\right);$$

$$8\sum_{k=1}^{\infty}\frac{(-1)^k}{\lambda_k}\,r(t)\,f_{1,k}(t) = E'(t) + 8\sum_{k=1}^{\infty}\left(\eta_{1,k}\,e^{-\lambda_k t} + \int_0^t r(\tau)\,f_{1,k}(\tau)\,e^{-\lambda_k(t-\tau)}\,d\tau\right);$$

$$8r(t)\sum_{k=1}^{\infty}\frac{(-1)^k}{\lambda_k}f_{1,k}(t) = E'(t) + \sum_{k=1}^{\infty}8(-1)^{k-1}\left(\eta_{1,k}\,e^{-\lambda_k t} + \int_0^t r(\tau)\,f_{1,k}(\tau)\,e^{-\lambda_k(t-\tau)}\,d\tau\right);$$

$$r(t) = \frac{E'(t) + 8\sum_{k=1}^{\infty}(-1)^k\eta_{1,k}\,e^{-\lambda_k t}}{\sum_{k=1}^{\infty}\frac{8(-1)^k}{\lambda_k}f_{1,k}(t)} + \frac{\sum_{k=1}^{\infty}(-1)^k\int_0^t r(\tau)\,f_{1,k}(\tau)\,e^{-\lambda_k(t-\tau)}\,d\tau}{\sum_{k=1}^{\infty}\frac{(-1)^k}{\lambda_k}f_{1,k}(t)}.$$

So, to determine the function $r(t)$ the Volterra integral equation of the second kind is obtained:

$$r(t) = F(t) + \int_0^t K(t,\tau)\,r(\tau)\,d\tau, \tag{28}$$

where

$$F(t) = \frac{E'(t) + 8\sum_{k=1}^{\infty}(-1)^k\,\eta_{1,k}\,e^{-\lambda_k t}}{\sum_{k=1}^{\infty}8\frac{(-1)^k}{\lambda_k}f_{1,k}(t)}, \tag{29}$$

$$K(t,\tau) = \frac{\sum_{k=1}^{\infty}(-1)^k f_{1,k}(\tau)\,e^{-\lambda_k(t-\tau)}}{\sum_{k=1}^{\infty}\frac{(-1)^k}{\lambda_k}\,f_{1,k}(t)}.$$

The denominator of fractions (28), (29) is not equal to zero, because the Assumption B3 is obtained

$$\int_{-1}^1 x\,f(x,t)\,dx = 8\sum_{k=1}^{\infty}\frac{(-1)^k}{\lambda_k}f_{1,k}(t) \neq 0.$$

According to Assumptions B1–B3, the function $F(t)$ and the kernel $K(t,\tau)$ are continuous functions on $[0,T]$ and $[0,1]\times[0,T]$ respectively.

Therefore, equation (25) has an unique solution. This solution is a continuous function $r(t)$ on $[0,T]$, which forms an unique solution $\{r(t),u(x,t)\}$ of inverse problem (5)–(8) together with the given Fourier series (13) as a solution $u(x,t)$ of the direct problem (5)–(7).

According to Assumptions B1–B3, the function $F(t)$ and the kernel $K(t,\tau)$ are continuous functions on $[0,T]$ and $[0,1]\times[0,T]$ respectively.

Therefore, equation (22) has an unique solution. This solution is a continuous function $r(t)$ on $[0,T]$, which forms an unique solution $\{r(t),u(x,t)\}$ of inverse problem (5)–(8) together with solution $u(x,t)$ of the direct problem (5)–(7).

The statement **A.2** of Theorem 2 is proved.

4.3. Let $\alpha_2 = h = \frac{\beta_2-\beta_1}{\beta_2+\beta_1}$. In this case the elements of system V_h are the eigenvalues of operator $\boldsymbol{L}$, for which the following equalities hold

$$\begin{cases} \boldsymbol{L}\nu_{0,k}(x) = \lambda_k\,\nu_{0,k}(x), \\ \boldsymbol{L}\nu_{1,k}(x) = \lambda_k\,\nu_{1,k}(x), \end{cases} \quad k = 1, \dots .$$

The proof of statement **A.3** of Theorem 2 is similar to the proof of **A.1** of this theorem.

We note that inverse problem is investigated in [22] for $\beta_1 = 1$, $\beta_2 = b$, $\gamma = 0$, $\alpha_1 = -\varepsilon$, $\alpha_2 = 0$. In addition, if $1 - 2\alpha_1 \leqslant 0$, then more research is needed.

5. Conclusion

The solutions of the inverse problem for a parabolic equation with involution and anti-periodicity conditions are constructed. The method of variables separation was used for the research. The corresponding eigenvalue problem for a differential equation with involution is studied. The eigenvalue spectrum properties are analyzed. The conditions to guarantee the existence and uniqueness of the inverse problem solution are found. The unknown coefficient is calculated using the Volterra integral equation of the second kind.

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Обернені задачі визначення невідомого залежного від часу коефіцієнта для параболічного рівняння з умовами інволюції та антиперіодичності

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Обернені задачі визначення невідомого залежно від часу коефіцієнта для параболічного рівняння з умовами інволюції та антиперіодичності. Методом розділення змінних побудовано розв'язок досліджуваної задачі з невідомим коефіцієнтом у рівнянні. Досліджено властивості індукованої спектральної задачі для диференціального рівняння другого порядку з інволюцією. Досліджено залежність спектра та його кратності, а також структуру системи кореневих функцій і часткових розв'язків задачі від інволютивної частини цього рівняння. Встановлено умови існування та єдиності розв'язку оберненої задачі. Для визначення шуканого коефіцієнта знайдено і розв'язано інтегральне рівняння Вольтера другого роду.

Ключові слова: *обернена задача; рівняння теплопровідності; метод розділення змінних; нелокальні умови; інволюція; базис Ріса.*