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# Estimates of solutions for ambiguously solvable linear matrix equations

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The article examines the problem of estimating solutions of operator equations under conditions of uncertainty. We obtain the expressions for guaranteed errors of solutions of indefinite linear equations in the spaces of rectangular matrices in the presence of additional data with deterministic errors belonging to special sets. In a particular case, explicit formulas are obtained for guaranteed linear vector estimates and guaranteed vector estimation errors and for guaranteed posterior estimates and measurement errors. These estimation results are illustrated by a test example in the case of operators that act in the space of $2 \times 2$ matrices with a non-zero kernel.

**Keywords:** *operator equation; deterministic data errors; matrices estimates under uncertainty; linear estimate of vector; guaranteed estimate of vector; guaranteed error estimation of vector.*

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## 1. Introduction

Many publications of foreign scientists, in particular [1–8], are devoted to the problems of estimating matrices and vectors based on observational data. We investigate some problems of estimating matrices and vectors under conditions of uncertainty in works [9–13]. For observations with random errors that have unknown second moments, we obtain explicit expressions for guaranteed linear estimates and guaranteed errors of matrix parameter estimates. As a rule, in these works we assume that matrix equations, the solutions of which are observed with errors, have a unique solution when the right-hand sides of such equations are known. This article examines the problems of estimating the solutions of operator equations, which do not have a unique solution, and the measurement errors are unknown deterministic values. Linear guaranteed estimates of the solutions of such equations and their errors are found, as well as posterior guaranteed estimates and their errors under certain restrictions on the deterministic errors of the data.

## 2. Problem statement

Let $H_{p,q}$ denote the space of rectangular matrices of dimension $p \times q$ with a scalar product: $\langle X_1, X_2 \rangle \triangleq \operatorname{Sp} X_1 X_2^T$, $X_1$, $X_2 \in H_{p,q}$. Let $A$ be a linear operator that acts from the space $H_{m,n}$ of matrices to the space $H_{m_1,n_1}$.

Consider the equation:

$$A\,X = B, \tag{1}$$

where $X \in H_{m,n}$, $B \in H_{m_1,n_1}$. It is known that this equation has a solution if the matrix $B \perp \ker A^*$, where $\ker A^* \triangleq \{B_1 \colon A^* B_1 = 0\}$ (here $A^*$ is the operator conjugate to the operator $A$, and the matrix $B_1 \in H_{m_1,n_1}$).

We assume that the kernel of the operator $A$ is not empty (the condition $\{X \colon AX = 0\} \neq \varnothing$ is fulfilled) and the dimension $r, r \geqslant 1$. Therefore, an arbitrary solution of equation (1) can be written in the form:

$$X = X_0 + D, \tag{2}$$

where $X_0$ is some solution of equation (1) and $D$ is an arbitrary matrix from the kernel of the operator $A$.

 

Let $\Psi_i$, $i \in \overline{1,r}$ denote the basis matrices of the set $\ker A$. Then we can write the matrix $D$ by formula (2) in the form:

$$D = \sum_{i=1}^{r} f_i \Psi_i \triangleq \rho f, \tag{3}$$

where $f_i$, $i \in \overline{1,r}$ are some real numbers, $f = (f_1, f_2, \dots, f_r)^T$. It is clear that if the matrix $X_0$, basis $\Psi_i$, $i \in \overline{1,r}$ and vector $f$ are known, then formulas (2), (3) determine the solution of equation (1). Note also that we can choose the matrix $X_0$ as the minimum norm matrix that satisfies the equation $AX_0 = B$. We find such a matrix from the system of equations:

$$\begin{cases} AX = B, \\ \langle X, \Psi_i \rangle = 0, \quad i \in \overline{1,r}. \end{cases}$$

Further, we assume that the solution $X$ is unknown, which means that the real numbers $f_i$, $i \in \overline{1,r}$ are unknown. To determine the estimates of the numbers $f_i$, $i \in \overline{1,r}$ we set the matrices $Y_k$, $k \in \overline{1,N}$:

$$Y_k = C_k X + V_k, \quad k \in \overline{1,N}, \tag{4}$$

where $Y_k \in H_{p(k),q(k)}$, $k \in \overline{1,N}$; $C_k$, $k \in \overline{1,N}$ are known linear operators that act from the space $H_{m,n}$ to the spaces $H_{p(k),q(k)}$, $k \in \overline{1,N}$; $V_k$, $k \in \overline{1,N}$ are unknown matrices from the spaces $H_{p(k),q(k)}$, $k \in \overline{1,N}$.

We present the known restrictions on the matrix $V_1, V_2, \dots, V_N$ in the form $(V_1, V_2, \dots, V_N) \in G$, where $G \subset \prod_{k=1}^{N} H_{p(k),q(k)}$. Therefore, *the purpose of this work is to find estimates of the vector $Lf$*, where $L$ is a linear operator that acts from the space $R^r$ to the space $R^s$, $s \leqslant r$.

## 3. Guaranteed estimates of the vector and their errors

First, we give the definition of guaranteed linear and guaranteed linear posterior estimates of the vector $Lf$ and their errors.

**Definition 1.** *Vector $\widehat{Lf}$ is called a linear estimate of vector $Lf$, if it has the form:*

$$\widehat{Lf} = \sum_{k=1}^{N} U_k Y_k + c,$$

*where $U_k$, $k \in \overline{1,N}$ are operators that act from the spaces $H_{p(k),q(k)}$, $k \in \overline{1,N}$ to the space $R^s$, the vector $c$ belongs to the space $R^s$.*

**Definition 2.** *Vector $\widehat{\widehat{Lf}}$ is called a guaranteed linear estimate of vector $\widehat{Lf}$, if it has the form:*

$$\widehat{\widehat{Lf}} = \sum_{k=1}^{N} \widehat{U}_k \widehat{Y}_k + \widehat{c}, \tag{5}$$

*where $\big(\hat{U}_k, \widehat{c}\big)$, $k \in \overline{1,N}$ are found from the condition:*

$$\big(\widehat{U}_k, \widehat{c}\big), \quad k \in \overline{1,N} \in \operatorname{Arg}\min_{(U,c)} \sigma_g(U,c),$$

*where $U = (U_1, U_2, \dots, U_N)$ and the error of estimation of the vector $Lf$ is given by the formula:*

$$\sigma_g(U,c) \triangleq \sup_f \sup_G \left\| Lf - \sum_{k=1}^{N} U_k Y_k - c \right\|. \tag{6}$$

**Definition 3.** *The value $\sigma_g\big(\widehat{U}, \widehat{c}\big)$ is called the guaranteed error of the linear guaranteed estimate.*

**Definition 4.** *The vector $\widehat{\varphi} \in R^s$ is called a guaranteed posterior estimate if it is from the condition:*

$$\widehat{\varphi} \in \operatorname{Arg}\inf_{\varphi} \sigma_a(\varphi), \quad \sigma_a(\varphi) \triangleq \sup_{\varphi_1 \in G_y} \big\|\varphi - \varphi_1\big\|, \tag{7}$$

*where $G_y$ is the posterior set of vectors $Lf$ with known data (4) and has the form:*

$$\begin{aligned} G_y &= \big\{Lf \colon f \in G_y^{(1)}\big\}, \\ G_y^{(1)} &= \big\{f \colon \big(\tilde{Y}_1 - C_1 \rho f, \dots, \tilde{Y}_N - C_N \rho f\big) \in G\big\}, \end{aligned} \tag{8}$$

*where $\tilde{Y}_k = Y_k - C_k X_0$, $k \in \overline{1,N}$.*

**Definition 5.** *The value $\sigma_a(\widehat{\varphi})$ is called the guaranteed posterior error of the estimate $\hat{\varphi}$.*

Further on we assume that the set $G$ has the form:

$$G = \left\{(V_1, \ldots, V_N) : \|V_k\| \leqslant \gamma_k, \quad k \in \overline{1, N}\right\}, \tag{9}$$

where $\|V_k\| = \left(\operatorname{Sp} V_k V_k^T\right)^{1/2}$, and $\gamma_k$, $k \in \overline{1, N}$ are known positive numbers.

**Theorem 1.** *There is an equality:*

$$\sigma_g(U, c) = \begin{cases} \sigma_g^{(1)}(U, c), & \text{if} \quad W_N \neq \varnothing, \\ \infty, & \text{if} \quad W_N = \varnothing, \end{cases}$$

*where function $\sigma_g^{(1)}(U, c)$ is determined by the formula:*

$$\sigma_g^{(1)}(U, c) = \max_{\|l\|=1} \left\{ \sum_{k=1}^{N} \left\|U_k^* l\right\| \gamma_k + \left|(c, l) + \left(\psi(X_0), l\right)\right| \right\}, \tag{10}$$

*here*

$$\psi(X_0) = \sum_{k=1}^{N} U_k\, C_k\, X_0, \quad W_N = \left\{ (U_1, \ldots, U_N) \colon \sum_{k=1}^{N} \rho^* C_k^* U_k^* = L^* \right\}. \tag{11}$$

**Proof.** It is clear that:

$$\begin{aligned}
\sigma_g(U, c) &= \max_{G} \max_{\|l\|=1} \left| (L^* l, f) - \left(l, \sum_{k=1}^{N} U_k Y_k\right) - (l, c) \right| \\
&= \max_{G} \max_{\|l\|=1} \left| (L^* l, f) - \left(l, \sum_{k=1}^{N} U_k C_k \rho f\right) - \left(l, \sum_{k=1}^{N} U_k C_k X_0\right) - \sum_{k=1}^{N} \left\langle U_k^* l, V_k \right\rangle - (l, c) \right| \\
&= \max_{G} \max_{\|l\|=1} \left| \left( \left(L^* - \sum_{k=1}^{N} \rho^* C_k^* U_k^*\right) l, f \right) - \sum_{k=1}^{N} \left\langle U_k^* l, V_k \right\rangle - \left(l, \left(c + \sum_{k=1}^{N} U_k C_k X_0\right)\right) \right| \\
&= \max_{\|l\|=1} \max_{G} \left| \left( \left(L^* - \sum_{k=1}^{N} \rho^* C_k^* U_k^*\right) l, f \right) - \sum_{k=1}^{N} \left\langle U_k^* l, V_k \right\rangle - \left(l, \left(c + \sum_{k=1}^{N} U_k C_k X_0\right)\right) \right|
\end{aligned}$$

are valid.

Now, taking into account the condition $(U_1, \ldots, U_N) \in W_N$ and the equality

$$\max_{G} \left| \sum_{k=1}^{N} \left\langle U_k^* l, V_k \right\rangle + \left(c + \sum_{k=1}^{N} U_k\, C_k\, X_0, l\right) \right| = \sum_{k=1}^{N} \|U_k^* l\|\, \gamma_k + \left| \left(c + \sum_{k=1}^{N} U_k\, C_k\, X_0, l\right) \right|,$$

we conclude that Theorem 1 is valid. ■

**Corollary 1.** *If operators $\hat{U}_1, \ldots, \hat{U}_N$ are found from the condition:*

$$F(\hat{U}_1, \ldots, \hat{U}_N) = \min_{U_1, \ldots, U_N} F(U_1, \ldots, U_N),$$

*where the function $F(U_1, \ldots, U_N)$ is given by the formula $F(U_1, \ldots, U_N) \triangleq \max_{\|l\|=1} \sum_{k=1}^{N} \left\|U_k^* l\right\| \gamma_k^{-2}$, then the guaranteed estimate of the vector is calculated by the formula:*

$$\widehat{\widehat{Lf}} = \sum_{k=1}^{N} \hat{U}_k\, Y_k + \hat{c},$$

*where*

$$\hat{c} = -\sum_{k=1}^{N} \hat{U}_k\, C_k\, X_0.$$

**Corollary 2.** *If the set $W_N$ is not empty, then the following equality holds:*

$$\min_{U\in W_N,c}\sigma_g(U,c)=\min_{(\Theta_1,\dots,\Theta_N)\in\Theta}\Phi\big(\Theta_1,\dots,\Theta_N\big),$$

*where*

$$\Phi\big(\Theta_1,\dots,\Theta_N\big)\triangleq\max_{\|l\|=1}\sum_{k=1}^{N}\Big\|\big(\hat U_k^*+\Theta_k^*\big)l\Big\|,\tag{12}$$

$$\hat U_k=L\left(\rho^*\sum_{k=1}^{N}C_k^*\,C_k\,\rho\right)^{+}\rho^*\,C_k^*,\quad k\in\overline{1,N},\tag{13}$$

$$\Theta\triangleq\left\{\big(\Theta_1,\dots,\Theta_N\big)\colon\ \sum_{k=1}^{N}\Theta_k\,C_k\,\rho=0\right\}.\tag{14}$$

**Proof.** Let us write down the set of solutions of the equation $\sum_{k=1}^{N}\rho^*\,C_k^*\,U_k^*=L^*$ or its equivalent equation $\sum_{k=1}^{N}U_k\,C_k\,\rho=L$ in the next form: $U_k=S\,\rho^*\,C_k^*$, $k\in\overline{1,N}$. Then for $S$ we obtain the equation:

$$S\sum_{k=1}^{N}\rho_k^*\,C_k^*\,C_k\,\rho=L.\tag{15}$$

The minimal solution of the equation (15) according to the norm is:

$$S_0=L\left(\rho^*\sum_{k=1}^{N}C_k^*\,C_k\,\rho\right)^{+}$$

(here $(\rho)^+$ means the pseudo inverse operator to $\rho$).

Therefore, an arbitrary solution $U_k$, $k\in\overline{1,N}$ can be represented in the form: $U_k=U_k^{(0)}+\Theta_k$, where $U_k^{(0)}=\rho^*\,C_k^*\,S_0$. Now, substituting the obtained expressions $U_k$, $k\in\overline{1,N}$ into the guaranteed error, we obtain the necessary equality. ■

Note also that the following inequalities holds:

$$\min_{U,c}\sigma_g(U,c)\leqslant\max_{\|l\|=1}\sum_{k=1}^{N}\Big\|\big(U_k^{(0)}\big)^*l\Big\|\,\gamma_k\leqslant\sum_{k=1}^{N}\lambda_{\max}^{1/2}\big(U_k^{(0)}\,U_k^{(0)*}\big)\gamma_k$$

($\lambda_{\max}(Z_1)$ is the maximum eigenvalue of operator $Z_1$).

**Lemma 1.** *Let the parameter $N=1$ and the equality $Lf=(l,f)$ hold, where $l$ is an arbitrary vector from the space $R^r$. Then, if the guaranteed linear estimate of the scalar product $(l,f)$ has the form $\widehat{(l,f)}=\big(l,\hat f\big)$, where $\hat f$ is the linear estimate of the vector $f$ according to data (4), then the equality $\widehat{Lf}=L\hat f$ holds. At the same time, the guaranteed error $\sigma_g$ of such an estimate is as follows:*

$$\sigma_g=\max_{\|l\|=1}\max_{f,G}\Big|(l,f)-\big(l,\hat f\big)\Big|.$$

**Proof.** From the assumptions of Lemma 1 we get

$$\min_{U_1,c}\max_{f,G}\big\|Lf-U_1Y_1-c\big\|\geqslant\max_{\|l\|=1}\min_{U_1,c_1}\max_{f,G}\big|\big(L^*l,f\big)-\big\langle U_1,Y_1\big\rangle-c_1\big|$$
$$=\max_{\|l\|=1}\max_{f,G}\Big|\big(L^*l,f\big)-\big(L^*l,\hat f\big)\Big|=\max_{f,G}\big\|Lf-L\hat f\big\|$$

it follows that the lower limit is reached at $\hat U_1\,Y_1+\hat c=L\hat f$, which completes the proof of the lemma. ■

**Theorem 2.** *At the value of the parameter $N=1$, the condition is fulfilled*

$$\min_{U_1,c}\sigma\big(U_1,c\big)=\lambda_{\max}^{1/2}(Q)\gamma_1,$$

*for the estimation error, where $Q=L(\rho^*\,C_1^*\,C_1\,\rho)^+L^*$.*

**Proof.** Note that the following ratios:

$$\max_{f,G} \left|(l,f) - \langle U_1, Y_1\rangle - c\right| = \|U_1\|\gamma_1 + \left|c + \langle U_1, C_1 X_0\rangle\right| \geqslant \|U_1\|\gamma_1$$

are valid.

Since $\min\limits_{U_1\in W_1} \|U_1\| = \left\{\min\limits_{U_1\in W_1} \langle U_1, U_1\rangle\right\}^{1/2}$, where $W_1 \triangleq \{U_1 \colon \rho^* C_1^* U_1 = l\}$, then $\min\limits_{U_1\in W_1} \|U_1\| = \|\hat{U}_1\|$, $\hat{U}_1 = C_1 \rho \left(\rho^* C_1^* C_1 \rho\right)^+ l$. Hence we get the equality $\|\hat{U}_1\| = \left(\left(\rho^* C_1^* C_1 \rho\right)^+ l, l\right)^{1/2}$ and for the error the representation holds:

$$\sigma_g = \min_{U_1,c} \max_{f,G} \|Lf - \widehat{Lf}\| = \max_{f,G} \|Lf - L\hat{f}\|,$$

where $\hat{f} = \left(\rho^* C_1^* C_1 \rho\right)^+ \rho^* C_1^* Y_1 - \hat{c}$, $\hat{c} = \left(\rho^* C_1^* C_1 \rho\right)^+ \rho^* C_1^* X_0$.

As a result, the error is written in this form:

$$\sigma_g = \lambda_{\max}^{1/2}\left(L\left(\rho^* C_1^* C_1 \rho\right)^+ L^*\right)\gamma_1.$$ ∎

**Theorem 3.** *When $N \geqslant 1$, the inequality*

$$\inf_{U,c} \sigma_g(U,c) \geqslant \lambda_{\max}^{1/2}(Q_1)$$

*holds, where the operator* $Q_1 = L\Big(\sum\limits_{k=1}^{N} \gamma_k^{-2} \rho^* C_k^* C_k \rho\Big)^+ L^*$.

**Proof.** Note that the set $G$ contains the set $G_1$, which is defined as follows:

$$G_1 \triangleq \left\{\left(V_1, \ldots, V_N\right) \colon \sum_{k=1}^{N} \gamma_k^{-2} \langle V_k, V_k\rangle \leqslant 1\right\}.$$

Taking into account the equality $\widehat{Lf} = \sum\limits_{k=1}^{N} U_k Y_k + c$, we get the following ratios:

$$\max_{f,G} \|Lf - \widehat{Lf}\| \geqslant \max_{f,G_1} \|Lf - \widehat{Lf}\| \geqslant \min_{U,c} \max_{f,G_1} \|Lf - \widehat{Lf}\| = \min_{U\in W_N} \max_{\|l\|=1} \max_{G_1} \sum_{k=1}^{N} \langle U_k^* l, V_k\rangle.$$

Since the equalities:

$$\max_{G_1} \sum_{k=1}^{N} \langle U_k^* l, V_k\rangle = \left(\sum_{k=1}^{N} \langle U_k^* l, U_k^* l\rangle \gamma_k^2\right)^{1/2} = \left(\sum_{k=1}^{N} \left(\gamma_k^2 U_k U_k^* l, l\right)\right)^{1/2}$$

are fulfilled, we can write:

$$\max_{\|l\|=1} \max_{G_1} \sum_{k=1}^{N} \langle U_k^* l, V_k\rangle = \lambda_{\max}^{1/2} P\left(U_1, \ldots, U_N\right),$$

where the operator $P\left(U_1, \ldots, U_N\right)$ is given by the formula: $P\left(U_1, \ldots, U_N\right) = \sum\limits_{k=1}^{N} \gamma_k^2 U_k U_k^*$.

Thus, the following ratios take place:

$$\min_{U,c} \max_{f,G_1} \|Lf - \widehat{Lf}\| \geqslant \min_{U\in U_N} \lambda_{\max}^{1/2}\left(P(U_1, \ldots, U_N)\right) = \lambda_{\max}^{1/2}\left(P\left(\hat{U}_1, \ldots, \hat{U}_N\right)\right).$$

Similarly, with the proof of Theorem 2, it is possible to obtain the optimal values of $\hat{U}_k$, $k \in \overline{1,N}$:

$$\hat{U}_k = \gamma_k^{-2} C_k \rho \hat{S}_1, \quad \hat{S}_1 = \left(\sum_{k=1}^{N} \rho^* C_k^* C_k \gamma_k^{-2} \rho\right)^+ l,$$

and write the operator $P\left(\hat{U}_1, \ldots, \hat{U}_N\right)$ in the form: $P\left(\hat{U}_1, \ldots, \hat{U}_N\right) = L\Big(\sum\limits_{k=1}^{N} \rho^* C_k^* C_k \gamma_k^{-1} \rho\Big)^+ L^*$. Theorem 3 is proved. ∎

**Corollary 3.** *If the set $G$ is given in the form:*

$$G=\left\{\left(V_1,\ldots,V_N\right):\ \sum_{k=1}^{N}\left\langle V_k,V_k\right\rangle\gamma_k^{-2}\leqslant 1\right\}, \tag{16}$$

*then the guaranteed linear estimate of the vector $Lf$ has the form: $\widehat{\widehat{Lf}}=L\hat{f}$, where*

$$\hat{f}=\sum_{k=1}^{N}M_k\,Y_k+\hat{c},\quad \hat{c}=-\sum_{k=1}^{N}M_k\,C_k\,X_0,\quad M_k=\gamma_k^{-2}\left(\sum_{k=1}^{N}\rho^*\,C_k^*\,\gamma_k^{-2}\,\rho\right)^{+}\rho^*\,C_k^*$$

*and the equality*

$$\min_{U,c}\max_{f,G}\left\|Lf-\widehat{Lf}\right\|=\lambda_{\max}^{1/2}(Q_1)$$

*holds for the error.*

**Corollary 4.** *Let $L$ be a unit operator and the matrix $\left(\sum_{k=1}^{N}\rho^*\,C_k^*\,C_k\,\gamma_k^{-2}\rho\right)$ is not degenerate. Then, if the set $G$ has the form:*

$$G=\left\{(V_1,\ldots,V_N):\ \sum_{k=1}^{N}\langle V_k,V_k\rangle\gamma_k^2\leqslant 1\right\},$$

*then the guaranteed error is as follows:*

$$\sigma_g\left(\hat{U},\hat{c}\right)=\lambda_{\min}^{-1/2}\left(\sum_{k=1}^{N}\rho^*\,C_k^*\,C_k\,\gamma_k^{-2}\,\rho\right).$$

## 4. Guaranteed posterior vector estimates and their errors

**Theorem 4.** *Let the set $G$ given by formula (9). Then the set*
$G_y^{(1)}=\left\{f:\left(\tilde{Y}_1-C_1\rho\,f,\ldots,\tilde{Y}_N-C_N\,\rho f\right)\in G\right\}=\left\{f:\left(\left\|\tilde{Y}_1-C_1\,\rho\,f\right\|\leqslant\gamma_1,\ldots,\left\|\tilde{Y}_N-C_N\,\rho\,f\right\|\leqslant\gamma_N\right)\right\}$
*can be presented in the form:*

$$G_y^{(1)}=\bigcap_{k=1}^{N}\bar{G}_y^{(k)},$$

*where*

$$\bar{G}_y^{(k)}=\left\{f:\left(R_k(f-\hat{f}_k),f-\hat{f}_k\right)\leqslant\gamma_k^2-J_k(\hat{f}_k)\right\},\quad k\in\overline{1,N},$$
$$R_k=\rho^*\,C_k^*\,C_k\,\rho,\quad \hat{f}_k=\left(\rho^*\,C_k^*\,C_k\,\rho\right)^{+}\rho^*\,C_k^*\,\tilde{Y}_k,\quad J_k(f)=\left\|\tilde{Y}_k-C_k\,\rho\,f\right\|^2.$$

**Proof.** For each $k\in\overline{1,N}$, we find the optimal value of $\hat{f}_k\in\operatorname{Arg}\min\limits_{f\in G_y^{(1)}}J_k(f)$. For this, we equate the gradient of the function $J_k(f)$ to zero:

$$\frac{1}{2}\operatorname{grad}J_k(f)=-\rho^*\,C_k^*\,\tilde{Y}_k+\rho^*\,C_k^*\,C_k\,\rho\,f=0.$$

From the last equality, we find the optimal value of $\hat{f}_k$: $\hat{f}_k=\left(\rho^*\,C_k^*\,C_k\,\rho\right)^{+}\rho^*\,C_k^*\,\tilde{Y}_k$.

From Taylor's formula it follows that

$$J_k(f)=J_k(\hat{f})+\frac{1}{2}\left(J_k''(\hat{f}_k)\left(f-\hat{f}_k\right),f-\hat{f}_k\right).$$

Now, taking into account the equality $2R_k=J_k''(\hat{f}_k)$, we obtain the expression for $\bar{G}_y^{(k)}$:

$$\bar{G}_y^{(k)}=\left\{f:\left\|\tilde{Y}_k-C_k\,\rho\,f\right\|\leqslant\gamma_k\right\}.$$ ∎

**Theorem 5.** *The set $G_y^{(1)}$ is a convex closed set and if the matrix $\rho^*\sum_{k=1}^{N}\gamma_k^{-2}\,C_k^*\,C_k\,\rho\triangleq S_N$ is not degenerate, then the set $G_y^{(1)}$ is also bounded.*

**Proof.** The convexity and closure of the set $G_y^{(1)}$ is obvious. Therefore, we try to prove only its boundedness. It is clear that the set $G_y^{(1)}$ is contained in the set $G_y^{+}$:

$$G_y^{+}=\left\{f:\Phi_N(f)\leqslant N\right\},\quad \Phi_N(f)\triangleq\sum_{k=1}^{N}\gamma_k^{-2}\left\|\tilde{Y}_k-C_k\,\rho\,f\right\|^2.$$

Let $\hat{f} \in \operatorname{Arg}\min_{f} \Phi_N(f)$. Then the vector $\hat{f}$ is found from the equation:

$$\left(\sum_{k=1}^{N} \gamma_k^{-2} \rho^* \, C_k^* \, C_k \, \rho\right) f = \sum_{k=1}^{N} \rho^* \, C_k^* \, \tilde{Y}_k.$$

Since, by assumption $\det S_N \neq 0$, there is a unique solution to this equation, and the set $G_y^+$ can be written in the following form:

$$G_y^+ = \big\{ f \colon \big(S_N(f - \hat{f}), f - \hat{f}\big) \leqslant N - \Phi_N(\hat{f}) \big\}.$$

Considering that the matrix $S_N$ is also positive definite, we conclude that $G_y^+$ is a bounded set. ■

Let us introduce the concepts of upper and lower guaranteed the posterior sets and estimates.

**Definition 6.** *Let inclusions $G^- \subseteq G \subseteq G^+$ be fulfilled. Then the posterior sets:*

$$G_y^{\mp} = \Big\{ f \colon \big(\tilde{Y}_1 - C_1 \, \rho \, f, \ldots, \tilde{Y}_N - C_N \, \rho \, f\big) \in G^{\mp} \Big\}$$

*are called the lower and upper posterior sets and the corresponding guaranteed posteriori estimates and guaranteed posteriori errors are lower and upper estimates and errors, respectively.*

For the case when $G$ is given by formula (9), let us put:

$$G^- = \Big\{ (V_1, \ldots, V_N) \colon \sum_{k=1}^{N} \|V_k\|^2 \gamma_k^{-2} \leqslant 1 \Big\},$$

$$G^+ = \Big\{ (V_1, \ldots, V_N) \colon \sum_{k=1}^{N} \|V_k\|^2 \gamma_k^{-2} \leqslant N \Big\}.$$

Note that if the parameter $N = 1$, then $G^- = G^+$.

**Theorem 6.** *Let condition $\det S_N \neq 0$ be fulfilled. Then the lower and the upper guaranteed posterior estimates coincide with each other and inequalities:*

$$\sigma_y^- \leqslant \min_{f} \sigma_y(f) \leqslant \sigma_y^+,$$

*holds, where*

$$\sigma_y(f) \triangleq \max_{f_1 \in G} \|f - f_1\|, \quad \hat{f} = S_N^{-1} \sum_{k=1}^{N} \rho^* \, C_k^* \, \tilde{Y}_k, \quad 1 - \Phi_N\big(\hat{f}\big) \geqslant 0,$$

$$\sigma_y^- = \lambda_{\max}^{1/2}\big(S_N^{-1}\big)\big(1 - \Phi_N\big(\hat{f}\big)\big)^{1/2} = \lambda_{\min}^{-1/2}(S_N)\big(1 - \Phi_N\big(\hat{f}\big)\big)^{1/2},$$

$$\sigma_y^+ = \lambda_{\max}^{1/2}\big(S_N^{-1}\big)\big(N - \Phi_N\big(\hat{f}\big)\big)^{1/2}.$$

**Proof.** Since the matrix $S_N$ is nondegenerate, the sets $G_y^-$ and $G_y^+$ are bounded, and they are also convex and closed. Hence the guaranteed posterior estimates exist and, as we will see below, that the lower and upper estimates are unique.

Let us demonstrate the validity of the lower estimate for the error. Since the inequality:

$$\min_{f} \max_{f_1 \in G_y} \big\|f - f_1\big\| \geqslant \min_{f} \max_{G_y^-} \big\|f - f_1\big\|$$

holds, the formula $G_y^- = \big\{ f \colon \big(S_N(f - \hat{f}), f - \hat{f}\big) \leqslant 1 - \Phi\big(\hat{f}\big) \big\}$ implies the equality:

$$\max_{G_y^-} \big\|f - f_1\big\| = \max_{f_1 \in \bar{G}_y} \big\|f - \big(f_1 + \hat{f}\big)\big\|,$$

where $\bar{G}_y \triangleq \big\{ f_1 \colon \big(S_N f_1, f_1\big) \leqslant 1 - \Phi\big(\hat{f}\big) \big\}$.

From the ratio $\max_{f_1 \in \bar{G}_y} \big\|f - f_1 - \hat{f}\big\| = \max_{\|l\|=1} \big[\sigma_a(l) + \big(l, f - \hat{f}\big)\big]$, where $\sigma_a(l) \triangleq \max_{f_1 \in \bar{G}_y} \big(l, f_1\big)$, we obtain the inequality $\max_{G_y^-} \big\|f - f_1\big\| \geqslant \max_{\|l\|=1} \sigma_a(l)$. Now we can allow that the lower limit is reached under the condition $f_1 = \hat{f}$.

Note also that the following equalities:

$$\max_{\|l\|=1} \max_{f_1 \in G_y} (l, f_1) = \max_{\|l\|=1} \big(S_N^{-1} l, l\big)^{1/2} \big(1 - \Phi(\hat{f})\big)^{1/2} = \lambda_{\min}^{-1/2}\big(S_N\big)\big(1 - \Phi(\hat{f})\big)$$

are fulfilled.

Similarly, the validity of the upper bound estimate is proven. ■

**Corollary 5.** *At the value of the parameter $N = 1$ the equality*

$$\min_f \sigma_y(f) = \lambda_{\min}^{-1/2}\big(S_N\big)\big(1 - \Phi(\hat{f})\big)$$

*holds.*

## 5. Estimating the scalar product of vectors

Further, we find estimates of the scalar product of vectors $(l, f)$ with data (4). Let the matrices $V_k$, $k \in \overline{1, N}$ belong to the set $G$ of the form (9).

**Theorem 7.** *The guaranteed linear estimate of the scalar product of vectors $(l, f)$ has the form:*

$$\widehat{(l, f)} = \sum_{k=1}^{N} \big\langle \hat{U}_k, Y_k \big\rangle + \hat{c},$$

*where*

$$\hat{c} = -\sum_{k=1}^{N} \big\langle \hat{U}_k, C_k X_0 \big\rangle, \quad \big(\hat{U}_k, k \in \overline{1, N}\big) \in \operatorname{Arg} \min_{(U_1, \ldots, U_N) \in W_N} \Phi_1(U_1, \ldots, U_N),$$

$$\Phi_1(U_1, \ldots, U_N) \triangleq \sum_{k=1}^{N} \big\|U_k\big\| \gamma_k, \quad W_N = \Big\{(U_1, \ldots, U_N)\colon \sum_{k=1}^{N} \rho^* C_k^* U_k = l\Big\}.$$

*At the same time, the error satisfies the condition*

$$\min_{U,c} \max_{f,G} \left| (l, f) - \sum_{k=1}^{N} \big\langle U_k, Y_k \big\rangle - c \right| = \Phi_1\big(\hat{U}_1, \ldots, \hat{U}_N\big).$$

**Proof.** The proof is similar to the proof of Theorem 1 and Corollary 1. ■

**Corollary 6.** *The inequality:*

$$\Phi_1\big(\hat{U}_1, \ldots, \hat{U}_N\big) \leqslant \sum_{k=1}^{N} \big\|U_k^{(0)}\big\| \gamma_k$$

*is fulfilled, where $U_k^{(0)} = \beta_k C_k \big(\rho^* \big(\sum_{k=1}^{N} C_k^* C_k \beta_k\big) \rho\big)^+ l$, and $\beta_k$, $k \in \overline{1, N}$ are arbitrary positive numbers.*

The validity of this result follows from the fact that the equality $\sum_{k=1}^{N} \rho^* C_k^* U_k^{(0)} = l$ holds.

Next, we give the expressions for the posterior estimates and errors of the scalar product of the vectors $(l, f)$.

**Theorem 8.** *The guaranteed posteriori estimate of the scalar product of vectors $(l, f)$ has the form:*

$$\widehat{(l, f)} = \frac{1}{2}\big((l, f)_+ + (l, f)_-\big), \tag{17}$$

*where $(l, f)_+ = \min_{1 \leqslant k \leqslant N} \big((l, \hat{f}_k) + \sigma_k(l)\big)$, $(l, f)_- = \max_{1 \leqslant k \leqslant N} \big((l, \hat{f}_k) - \sigma_k(l)\big)$, and the guaranteed a posteriori error of the estimation of $\widehat{(l, f)}$ is as follows:*

$$\sigma_a = \frac{1}{2}\big((l, f)_+ - (l, f)_-\big), \tag{18}$$

*(here, vectors $\hat{f}_k$, $k \in \overline{1, N}$ and functions $\sigma_k(l)$, $k \in \overline{1, N}$ are the same as in Theorems 4, 6).*

**Proof.** From the ratios

$$G_y = \bigcap_{k=1}^{N} \bar{G}_y^{(k)}, \quad \bar{G}_y^{(k)} = \big\{V_k \colon \|V_k\| \leqslant \gamma_k\big\},$$

$$\max_{f \in G_y}(l,f) = \min_{1\leqslant k\leqslant N}\ \max_{f\in \bar{G}_y^{(k)}}(l,f),$$

$$\min_{f \in G_y}(l,f) = \max_{1\leqslant k\leqslant N}\ \min_{f\in \bar{G}_y^{(k)}}(l,f),$$

taking into account the expressions

$$\max_{f\in \bar{G}_y^{(k)}}(l,f) = \big(l,\hat{f}_k\big) + \sigma_k(l), \quad k \in \overline{1,N},$$

$$\min_{f\in \bar{G}_y^{(k)}}(l,f) = \big(l,\hat{f}_k\big) - \sigma_k(l), \quad k \in \overline{1,N},$$

the validity of equalities (17), (18) follows. ■

**Example 1.** Let the given sequence of real numbers $y_1,\ldots,y_N$, where $y_k = \operatorname{Sp} C_k^T X + v_k$, $k\in\overline{1,N}$; the matrix $X \in H_{2,2}$ is the solution of the system of equations:

$$\big\langle A_i, X\big\rangle = b_i, \quad i\in\overline{1,3}, \tag{19}$$

where $A_i$, $i\in\overline{1,3}$ are known linearly independent matrices of dimension $(2\times 2)$, $b_i$, $i\in\overline{1,3}$ are known real numbers.

Let us also assume that $C_k$, $k\in\overline{1,N}$ are known $(2\times 2)$ matrices; the scalars $v_k$, $k\in\overline{1,N}$ are unknown data errors such that $|v_k| \leqslant \gamma_k$, $\gamma_k > 0$, $k\in\overline{1,N}$.

It is necessary to find a guaranteed estimate of the matrix $X$.

**Corollary 7.** *The guaranteed posteriori estimate of the matrix $X$ has the form:*

$$\hat{X} = X_0 + \Psi \hat{f},$$

*where $X_0$ is the minimal solution of the system of equations (19) according to the norm; the matrix $\Psi$ from the space $H_{2,2}$ is such that: $\big\langle \Psi, A_i\big\rangle = 0$, $i\in\overline{1,3}$, $\|\Psi\| = 1$; $\hat{f} = \frac{1}{2}(f_+ + f_-)$, $\sigma_a = \frac{1}{2}(f_+ - f_-)$. At the same time:*

$$\max_{f\in G_y}\big\|X - \hat{X}\big\| = \|\Psi\|\sigma_a,$$

*where $G_y = [f_-; f_+]$, $f_- = \max\limits_{1\leqslant k\leqslant N}\frac{\tilde{y}_k - \gamma_k}{\langle C_k,\Psi\rangle}$, $f_+ = \min\limits_{1\leqslant k\leqslant N}\frac{\tilde{y}_k + \gamma_k}{\langle C_k,\Psi\rangle}$, if $\langle C_k,\Psi\rangle > 0$, $\tilde{y}_k = y_k - \langle C_k, X_0\rangle$.*

**Proof.** Since the posterior set for the value $f$ has the form:

$$G_y = \big\{f \colon |y_k - \langle C_k, X_0\rangle| \leqslant \gamma_k,\ k\in\overline{1,N}\big\} = \bigcap_{k=1}^{N}\big[f_k^-; f_k^+\big] = \Big[\max_{1\leqslant k\leqslant N} f_k^-;\ \min_{1\leqslant k\leqslant N} f_k^+\Big],$$

then under the condition $\langle C_k,\Psi\rangle > 0$, $k\in\overline{1,N}$ the representations

$$f_k^- = \big\langle C_k,\Psi\big\rangle^{-1}\big(\tilde{y}_k - \gamma_k\big),$$

$$f_k^+ = \big\langle C_k,\Psi\big\rangle^{-1}\big(\tilde{y}_k + \gamma_k\big), \quad k\in\overline{1,N}$$

are valid. ■

Note that if $\langle C_k,\Psi\rangle < 0$, $k\in\overline{1,N}$, then we have:

$$f_k^- = \big\langle C_k,\Psi\big\rangle^{-1}\big(\tilde{y}_k + \gamma_k\big),$$

$$f_k^+ = \big\langle C_k,\Psi\big\rangle^{-1}\big(\tilde{y}_k - \gamma_k\big), \quad k\in\overline{1,N}.$$

Further, we find the error of the guaranteed linear estimate of the scalar $f$. It follows from Theorem 7 that the error of the guaranteed linear estimate has the form:

$$\sigma\big(\hat{u}\big) = \sum_{k=1}^{N}\big|\hat{u}_k\big|\gamma_k,$$

where $\hat{u}_1,\ldots,\hat{u}_N$ are found as a solution of the optimization problem:

$$\min_{(u_1,\dots,u_N)\in W_N} \sum_{k=1}^{N} |u_k|\gamma_k = \sum_{k=1}^{N} \left|\hat{u}_k\right|\gamma_k, \quad W_N = \left\{(u_1,\dots,u_N)\colon \sum_{k=1}^{N} u_k\langle C_k,\Psi\rangle = 1\right\}.$$

Let us put $\hat{X} = X_0 + \hat{f}\Psi$, where $\hat{f} = \sum_{k=1}^{N} \hat{u}_k\tilde{y}_k$. Then we can write the equality

$$\max_{f,G} \left\|X - \hat{X}\right\| = \|\Psi\|\sigma\left(\hat{u}\right).$$

Now we turn to the specific calculations of estimates and their errors. To do this, we specify a triple of mutually perpendicular matrices $A_i$, $i \in \overline{1,3}$:

$$A_1 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} -3 & 1 \\ 1 & 1 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & -2 \\ 1 & 1 \end{pmatrix},$$

test matrix $X_* = \left(\begin{smallmatrix} 2 & 1 \\ 0 & -1 \end{smallmatrix}\right)$ and we calculate the numbers $b_i$, $i \in \overline{1,3}$ according to the formula $\langle A_i, X_*\rangle = b_i$, $i \in \overline{1,3}$: $b_1 = 2$, $b_2 = -6$, $b_3 = -3$, and the matrix $\Psi = \left(\begin{smallmatrix} 0 & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{smallmatrix}\right)$ with unit norm, which is perpendicular to the matrices $A_i$, $i \in \overline{1,3}$. We find the matrices $C_k$, $k \in \overline{1,4}$, that satisfy the inequalities $\langle C_k, \Psi\rangle > 0$, $k \in \overline{1,4}$:

$$C_1 = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}, \quad C_2 = \begin{pmatrix} 0 & -1 \\ 1 & 3 \end{pmatrix}, \quad C_3 = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \quad C_4 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

We calculate the norm-minimum solution $X_0$ of the system of equations (19): $X_0 = \left(\begin{smallmatrix} 2 & 1 \\ -0.5 & -0.5 \end{smallmatrix}\right)$.

In accordance with Corollary 7, the results of calculations of the guaranteed a posteriori estimates of the scalar $f$, the matrix $X$ and of the a posteriori guaranteed errors $\sigma_\alpha$ of these estimates for four options of specifying the deterministic data errors $v_k$, $k \in \overline{1,4}$ are shown in Table 1.

**Table 1.** Guaranteed posterior estimates and their errors.

| | Option 1 | | | | Option 2 | | | |
|---|---|---|---|---|---|---|---|---|
| | $k=1$ | $k=2$ | $k=3$ | $k=4$ | $k=1$ | $k=2$ | $k=3$ | $k=4$ |
| $v_k$ | -0.2 | 0.28 | 0.3 | -0.24 | 0.1 | 0.1 | -0.2 | -0.2 |
| $y_k$ | -1.2 | -3.72 | 1.3 | 1.76 | -0.9 | -3.9 | 0.8 | 1.8 |
| $\gamma_k$ | 0.3 | 0.3 | 0.3 | 0.3 | 0.2 | 0.2 | 0.2 | 0.2 |
| $\tilde{y}_k$ | -0.7 | -0.72 | -0.7 | -0.74 | -0.4 | -0.9 | -1.2 | -0.7 |
| $f_-$ | -0.707107 | | | | -0.777817 | | | |
| $f_+$ | -0.622254 | | | | -0.707107 | | | |
| $\hat{f}$ | -0.664681 | | | | -0.742462 | | | |
| $\sigma_a$ | 0.04243 | | | | 0.03536 | | | |
| $\hat{X}$ | $\left(\begin{smallmatrix} 2 & 1 \\ 0.03 & -0.97 \end{smallmatrix}\right)$ | | | | $\left(\begin{smallmatrix} 2 & 1 \\ 0.025 & -1.025 \end{smallmatrix}\right)$ | | | |

| | Option 3 | | | | Option 4 | | | |
|---|---|---|---|---|---|---|---|---|
| | $k=1$ | $k=2$ | $k=3$ | $k=4$ | $k=1$ | $k=2$ | $k=3$ | $k=4$ |
| $v_k$ | 0.05 | 0.05 | -0.1 | -0.1 | 0.02 | 0.02 | -0.03 | -0.03 |
| $y_k$ | -0.95 | -3.95 | 0.9 | 1.9 | -0.98 | -3.98 | 0.97 | 1.97 |
| $\gamma_k$ | 0.1 | 0.1 | 0.1 | 0.1 | 0.03 | 0.03 | 0.03 | 0.03 |
| $\tilde{y}_k$ | -0.45 | -0.95 | -1.1 | -0.6 | -0.48 | -0.98 | -1.03 | -0.53 |
| $f_-$ | -0.742462 | | | | -0.714178 | | | |
| $f_+$ | -0.707107 | | | | -0.707107 | | | |
| $\hat{f}$ | -0.724784 | | | | -0.710642 | | | |
| $\sigma_a$ | 0.017678 | | | | 0.003535 | | | |
| $\hat{X}$ | $\left(\begin{smallmatrix} 2 & 1 \\ 0.0125 & -1.0125 \end{smallmatrix}\right)$ | | | | $\left(\begin{smallmatrix} 2 & 1 \\ -0.0025 & -1.0025 \end{smallmatrix}\right)$ | | | |

To find a guaranteed linear estimate of the scalar $f$ and its error, we solve the optimization problem: find

$$\min_{u_k, k\in\overline{1,4}} \sum_{k=1}^{4} |u_k|\gamma_k = \sum_{k=1}^{4} \left|\hat{u}_k\right|\gamma_k$$

under the condition $\sum_{k=1}^{4} u_k\left\langle C_k, \Psi\right\rangle = 1$.

The solution to this problem was obtained using the generalized gradient descent method:

$$\hat{u} = (0.4; 0.25355339; 0.25355339; 0)^T .$$

The results of calculations of the guaranteed linear estimates of the scalar $f$ based on the data of $\tilde{y}_k$, $k \in \overline{1,4}$ from Table 1, its errors $\sigma\big(\hat{u}\big)$ and the guaranteed linear estimates of the matrix $X$ are shown in Table 2.

**Table 2.** Guaranteed linear estimates of the scalar $f$, its errors, and the $X$ matrix.

| | Option 1 | Option 2 | Option 3 | Option 4 |
|---|---|---|---|---|
| $\hat{f}$ | -0.64005 | -0.69246 | -0.69978 | -0.70164 |
| $\sigma(\hat{u})$ | 0.2721 | 0.1814 | 0.0907 | 0.0272 |
| $\hat{X}$ | $\begin{pmatrix} 2 & 1 \\ -0.0474 & -0.9526 \end{pmatrix}$ | $\begin{pmatrix} 2 & 1 \\ -0.010 & -0.990 \end{pmatrix}$ | $\begin{pmatrix} 2 & 1 \\ -0.0052 & -0.9948 \end{pmatrix}$ | $\begin{pmatrix} 2 & 1 \\ -0.0039 & -0.9961 \end{pmatrix}$ |

Analysing the results presented in Tables 1 and 2, we conclude that when the deterministic data errors $v_k$, $k \in \overline{1,4}$ are reduced (by absolute value), the estimates of the matrix $\hat{X}$ becomes closer to the test matrix $X_*$ and the error of the posterior guaranteed estimate $\sigma_a$ does not exceed the error of the guaranteed linear estimate $\sigma\big(\hat{u}\big)$ of the scalar $f$.

## 6. Conclusions

In the article the new methods are developed for estimating the solutions of linear operator equations in the presence of additional data with unknown deterministic errors belonging to specific sets of a special form. It is assumed that linear operator equations have non-unique solutions and the measurement errors are unknown deterministic values. Explicit expressions are derived for both the general case and certain specific cases of relevant operators to determine linear guaranteed estimates and their errors, as well as a posteriori guaranteed estimates and corresponding errors under specified conditions on deterministic measurement errors. The theoretical results are illustrated with a test example involving linear operators acting on the space of second-order square matrices with a non-zero kernel.


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## Оцінки розв'язків лінійних неоднозначних розв'язних матричних рівнянь

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У статті розглядається проблема оцінювання розв'язків операторних рівнянь за умов невизначеності. Отримано вирази для гарантованих похибок розв'язків невизначених лінійних рівнянь у просторах прямокутних матриць за наявності додаткових даних із детермінованими похибками, що належать до спеціальних множин. У частковому випадку отримано явні формули для гарантованих оцінок лінійних векторів та гарантованих похибок оцінок векторів, а також для гарантованих апостеріорних оцінок та гарантованих похибок апостеріорних вимірювань. Наведені результати оцінювання ілюструються тестовим прикладом у випадку операторів, які діють у просторі матриць розміру $2 \times 2$ з ненульовим ядром.

**Ключові слова:** *операторне рівняння; детерміновані похибки даних; оцінки матриць в умовах невизначеності; лінійна оцінка вектора; гарантована оцінка вектора; гарантована похибка оцінки вектора.*