MATHEMATICAL MODELING AND COMPUTING, Vol. 12, No. 1, pp. 49–56 (2025)

Classical aspect of spin angular momentum in geometric quantum mechanics

Ahmad H. A. S.[1], Umair H.[2,3], Nurisya M. S.[1,3], Chan K. T.[1,3]

[1] *Institute for Mathematical Research (INSPEM), Universiti Putra Malaysia (UPM), 43400 UPM Serdang, Selangor Darul Ehsan, Malaysia*

[2] *Centre for Foundation Studies in Science of University Putra Malaysia, University Putra Malaysia, 43400, Selangor, Malaysia*

[3] *Faculty of Science, University Putra Malaysia, 43400, Selangor, Malaysia*

(Received 15 October 2024; Revised 17 December 2024; Accepted 21 December 2024)

Geometric Quantum Mechanics is a formulation demonstrating how quantum theory may be cast in the language of Hamiltonian phase-space dynamics. Within this framework, the classical properties of spin $\frac{1}{2}$, spin 1 and spin $\frac{3}{2}$ particles have been studied. The correspondence between the Poisson bracket and commutator algebras for these systems was shown by explicitly computing the value of the commutator of spin operators and comparing it with the Poisson bracket of the corresponding classical observables. This study was extended by comparing the Casimir operator and its classical counterpart. The results showed that there exists a correspondence between classical and quantum Casimir operators at least for the case of spin $\frac{1}{2}$. This research clearly shows the limit of classical notions to describe the purely quantum concept.

Keywords: *differential geometry; geometric quantum mechanics; classical observables.*

2010 MSC: 53D22, 53Z05, 81S07, 81Q65 **DOI:** 10.23939/mmc2025.01.049

1. Introduction

The connection between classical and quantum representation is one of the greatest problems in understanding microscopic systems. Although quantum mechanics and classical mechanics have several points in common, they are quite different in several aspects. The most striking one is that classical mechanics is based on geometry and most of the systems are non-linear, whereas quantum mechanics is intrinsically formulated as algebraic and linear. The linearity seems to be a necessary condition since none of the standard quantum mechanics postulates can be stated without referring to it. This distinction is quite strange since in general, linear structure in physics arises as approximations to more accurate non-linear ones, but in this case, the situation happens in opposite way. Thus, it is difficult to make a smooth connection between classical mechanics and quantum mechanics.

This problem has motivated some physicists [1–15] to develop a formulation that does not involve the quantization process as such but acknowledges quantum theory as provided. This research line which is referred as Geometric Quantum Mechanics, demonstrates that quantum theory may be formulated in the language of Hamiltonian phase-space dynamics. The deeper investigation shows that the Hilbert space $\mathcal{H}$ is not the true space of states, since any two-state vectors $\Psi, \Phi \in \mathcal{H}$ such that $\Psi = \alpha\Phi$ $(\alpha \in \mathbb{C})$ are physically equivalent $(\Psi \backsim \Phi)$. Thus, the proper quantum space of pure states is the set of rays through the origin in $\mathcal{H}$, i.e. $P(\mathcal{H}) := \mathcal{H}/\backsim$ which is known as the complex projective Hilbert space or the quantum phase space for both finite and infinite dimensional $\mathcal{H}$. Furthermore, the existence of Hermitian inner product in $\mathcal{H}$ endows $P(\mathcal{H})$ with the structure of Kähler manifold (ω, g, j) where ω is non-degenerate, closed symplectic two-form, g is Riemannian metric and j is the compatible complex structure satisfying $j^2 = -1$ [3]. Thus, similar to classical mechanics, the correct quantum state space is also can be regarded as a symplectic manifold. In terms of self-adjoint operators on $\mathcal{H}$, via

This work was supported by Putra Fast Track Grant of University Putra Malaysia, Vot No. 9752900.

its expectation value, one can obtain a real-valued function on $\mathcal{H}$, which has well-defined projection h to $P(\mathcal{H})$ [4]. Note that, every phase space function induced a flow along its Hamiltonian vector field X_h [5]. Hence, on the Hilbert space, the flow is certainly defined by the Schrödinger equation of the quantum theory. In other words, Schrödinger evolution is exactly similar to the Hamiltonian flow on quantum phase space $P(\mathcal{H})$. Here, one can directly see that classical mechanics and quantum mechanics have many similarities. However, the fact that the Riemannian metric in quantum phase space is closely related to the notion of probability provides us with several main features, that are missing in classical mechanics such as uncertainty principle and state vector reduction in quantum measurement processes.

In this study, the examination of the correspondence between quantum and classical aspects of geometric quantum mechanics has been focused, and the classical properties of the observables have been studied as the literature was identified did not discussed at a deeper level. This study is important to identify the limitation of the classical notion of geometric quantum mechanics to describe purely quantum concepts such as the commutator of two spin operators and the Casimir operator.

The structure of the paper is the following: in Section 2 we make a quick review of the classical observable formulation in geometric quantum mechanics. Then we study the correspondence between commutators and Poisson brackets and compare the classical analog of the Casimir operator and the standard one in Section 3. Finally, we discuss (Section 4) our results and we end with conclusions.

2. Classical observable of geometric quantum mechanics

Let us start with the geometric construction of quantum mechanics by considering $\mathcal{H}$ as n-complex dimensional Hilbert space equipped with a Hermitian inner product

$$\langle * | * \rangle \colon \mathcal{H} \times \mathcal{H} \to \mathbb{C},$$

where it can be explicitly decomposed into real and imaginary parts

$$\langle \Psi | \Phi \rangle = G(\Psi, \Phi) + i\,\Omega(\Psi, \Phi)$$

for all $|\Psi\rangle, |\Phi\rangle \in \mathcal{H}$. The real part $G(\Psi, \Phi)$ is Riemannian metric, where

$$G(\Psi, \Phi) = G(\Phi, \Psi)$$

and Ω is symplectic form that satisfies the following relation

$$\Omega(\Psi, \Phi) = -\Omega(\Phi, \Psi).$$

Equivalently, if we consider $\mathcal{H}$ as m-dimensional real Hilbert space, then there exists a complex structure J on $\mathcal{H}$ that allows one to define multiplication by complex scalar. Technically, for $\xi = \alpha + i\beta \in \mathbb{C}$ where $\alpha, \beta \in \mathbb{R}$ and $q \in \mathcal{H}$, one can define

$$\xi q := \alpha q + \beta J q.$$

Besides, it is obvious that this vector space necessarily has an even dimension; the fact that $J^2 := -\mathbf{1}_m$ and $\det(J^2) = (\det J)^2 = (-1)^m$ imply $m = 2n$. Thus, a real, $2n$-dimensional Hilbert space together with a complex structure J corresponds to a complex n-dimensional Hilbert space. In this context, we can relate G, Ω and J as

$$G(\Psi, \Phi) = \Omega(\Psi, J(\Phi)) = -\Omega(J(\Psi), \Phi),$$

which define the Kähler structure on $\mathcal{H}$. Furthermore, since $\mathcal{H}$ is equipped with symplectic structure, it is clear, that we can regard $\mathcal{H}$ as a symplectic manifold.

Now, let A be a smooth function on $\mathcal{H}$ corresponding to any self-adjoint operator $\hat{A}$:

$$\hat{A} \to A \in C^\infty(\mathcal{H})$$

written as

$$A(\psi) := \langle \Psi | \hat{A}\Psi \rangle = G(\Psi, \hat{A}\Psi), \tag{1}$$

which is called an evaluation function. We can define the expectation value function $\langle \hat{A} \rangle$ associated with $\hat{A}$ as follows

$$\langle \hat{A} \rangle_\Psi := \frac{\langle \Psi | \hat{A}\Psi \rangle}{\langle \Psi | \Psi \rangle},$$

and for the case of Ψ is normalized, the equation becomes

$$A(\Psi) = \langle \hat{A} \rangle_\Psi.$$

Here, we regard $A(\Psi)$ as classical observable corresponds to self-adjoint operator $\hat{A}$.

Moreover, we can also define the classical observables a and b on projective Hilbert space $P(\mathcal{H})$. Let a: $\hat{A} \to \mathbb{R}$ and b: $\hat{B} \to \mathbb{R}$ be two functions on $\mathrm{P}(\mathcal{H})$ of the operators $\hat{A}$ and $\hat{B}$ respectively, then one can define

$$a \circ \Pi = \langle \hat{A} \rangle = A, \quad b \circ \Pi = \langle \hat{B} \rangle = B,$$

where $\Pi\colon \mathcal{H} \to P(\mathcal{H})$.

3. Poisson bracket and classical Casimir operator

Here the classical aspect of spin $\frac{1}{2}$, spin 1 and spin $\frac{3}{2}$ particles in geometric quantum mechanics has been studied. The research starts with examining the correspondence between the Poisson bracket and commutator followed by a comparison between the Casimir operator and its classical counterpart for these systems.

3.1. Spin $\frac{1}{2}$ particle

Let the Hilbert space $\mathcal{H} \cong \mathbb{C}^2$ and (e_1, e_2) represents the orthonormal basis in $\mathbb{C}^2$ satisfying

$$\langle e_i | e_j \rangle = \delta_{ij},$$

for $i, j = 1, 2$. Then the state of spin $\frac{1}{2}$ particle in $\mathcal{H}$ is expressed as

$$|\Psi\rangle = Z_1 |e_1\rangle + Z_2 |e_2\rangle,$$

where $Z_1, Z_2 \in \mathbb{C}$. Consider S_x, S_y and S_z be classical observables define by (1) as follows

$$\begin{aligned}
S_x(\Psi) &= \langle \Psi | \hat{S}_x | \Psi \rangle = \frac{\hbar}{2}(Z_1 \bar{Z}_2 + \bar{Z}_1 Z_2),\\
S_y(\Psi) &= \langle \Psi | \hat{S}_y | \Psi \rangle = \frac{i\hbar}{2}(Z_1 \bar{Z}_2 - \bar{Z}_1 Z_2),\\
S_z(\Psi) &= \langle \Psi | \hat{S}_z | \Psi \rangle = \frac{\hbar}{2}(|Z_1|^2 - |Z_2|^2),
\end{aligned}$$

where the self-adjoint operators $\hat{S}_x$, $\hat{S}_y$ and $\hat{S}_z$ are Pauli matrices

$$\hat{S}_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{S}_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \hat{S}_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The Poisson bracket for classical observables S_x and S_y is defined as

$$\{S_x, S_y\} = \frac{2}{i}\left(\frac{\partial S_x}{\partial Z_1}\frac{\partial S_y}{\partial \bar{Z}_1} - \frac{\partial S_x}{\partial \bar{Z}_1}\frac{\partial S_y}{\partial Z_1} + \frac{\partial S_x}{\partial Z_2}\frac{\partial S_y}{\partial \bar{Z}_2} - \frac{\partial S_x}{\partial \bar{Z}_2}\frac{\partial S_y}{\partial Z_2}\right).$$

Thus, we get

$$\begin{aligned}
\{S_x, S_y\} &= \frac{2}{i}\left(-\frac{i\hbar^2|Z_2|^2}{4} - \frac{i\hbar^2|Z_2|^2}{4} + \frac{i\hbar^2|Z_1|^2}{4} + \frac{i\hbar^2|Z_1|^2}{4}\right)\\
&= \frac{2}{i}\left(\frac{i\hbar^2}{2}\left[|Z_1|^2 - |Z_2|^2\right]\right) = 2\hbar S_z.
\end{aligned}$$

In this way, one also obtains

$$\begin{aligned}
\{S_x, S_z\} &= 2\hbar S_y,\\
\{S_y, S_z\} &= 2\hbar S_x.
\end{aligned}$$

The Poisson bracket related to commutator has been showed as follows

$$\begin{aligned}
\{S_y, S_z\} &\leftrightarrow \frac{2}{i}[\hat{S}_y, \hat{S}_z],\\
\{S_x, S_z\} &\leftrightarrow \frac{2}{i}[\hat{S}_x, \hat{S}_z],\\
\{S_x, S_y\} &\leftrightarrow \frac{2}{i}[\hat{S}_x, \hat{S}_y].
\end{aligned}$$

Now, the classical analog of the Casimir operator has been computed and compared with its quantum counterpart. Classically the Casimir operator is defined as

$$S^2 = S_x^2 + S_y^2 + S_z^2,$$

where

$$\begin{aligned}
S_x^2 &= \frac{\hbar^2}{4}(Z_1\bar{Z}_2 + \bar{Z}_1 Z_2)(Z_1\bar{Z}_2 + \bar{Z}_1 Z_2) = \frac{\hbar^2}{4}\left(Z_1^2\bar{Z}_2^2 + 2|Z_1|^2|Z_2|^2 + \bar{Z}^2 Z_2^2\right), \\
S_y^2 &= -\frac{\hbar^2}{4}(Z_1\bar{Z}_2 - \bar{Z}_1 Z_2)(Z_1\bar{Z}_2 - \bar{Z}_1 Z_2) = \frac{\hbar^2}{4}\left(Z_1^2\bar{Z}_2^2 - 2|Z_1|^2|Z_2|^2 + \bar{Z}^2 Z_2^2\right), \\
S_z^2 &= -\frac{\hbar^2}{4}(Z_1\bar{Z}_2 - \bar{Z}_1 Z_2)(Z_1\bar{Z}_2 - \bar{Z}_1 Z_2) = \frac{\hbar^2}{4}\left(|Z_1|^4 - 2|Z_1|^2|Z_2|^2 + |Z_2|^4\right).
\end{aligned}$$

and one obtains

$$\begin{aligned}
S_x^2 + S_y^2 + S_z^2 &= \frac{\hbar^2}{4}\big(Z_1^2\bar{Z}_2^2 + 2|Z_1|^2|Z_2|^2 + \bar{Z}^2 Z_2^2 - Z_1^2\bar{Z}_2^2 + 2|Z_1|^2|Z_2|^2 \\
&\quad - \bar{Z}^2 Z_2^2 |Z_1|^4 - 2|Z_1|^2|Z_2|^2 + |Z_2|^4\big) \\
&= \frac{\hbar^2}{4}\left(|Z_1|^4 + 2|Z_1|^2|Z_2|^2 + |Z_2|^4\right) \\
&= \frac{\hbar^2}{2}\left(|Z_1|^2 + |Z_2|^2\right)^2.
\end{aligned}$$

Since $|Z_1|^2 + |Z_2|^2 = 1$ due to the normalization condition, then the above equation becomes

$$S^2 = S_x^2 + S_y^2 + S_z^2 = \frac{\hbar^2}{4}.$$

In comparison with the standard Casimir operator [17], its relationship can be expressed as

$$\hat{S}^2 = 3S^2 = \frac{3\hbar^2}{4}.$$

3.2. Spin 1 particle

The corresponding self-adjoint operators for this case are defined as

$$\hat{S}_x = \frac{\hbar}{\sqrt{2}}\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}; \quad \hat{S}_y = \frac{\hbar}{\sqrt{2}}\begin{pmatrix}0 & -i & 0\\ i & 0 & -i\\ 0 & i & 0\end{pmatrix}; \quad \hat{S}_z = \frac{\hbar}{\sqrt{2}}\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -1\end{pmatrix},$$

and the orthonormal basis in Hilbert space $\mathcal{H} \cong \mathbb{C}^3$ is represented by (e_1, e_2, e_3) satisfies

$$\langle e_i | e_j \rangle = \delta_{ij},$$

for $i, j = 1, 2, 3$. Then, the state of spin 1 is expressed as

$$|\Psi\rangle = Z_1|e_1\rangle + Z_2|e_2\rangle + Z_3|e_3\rangle,$$

where $Z_1, Z_2, Z_3 \in \mathbb{C}$. Let us consider S_x, S_y and S_z be classical observables defined by

$$\begin{aligned}
S_x(\Psi) &= \langle\Psi|\hat{S}_x|\Psi\rangle = \frac{\hbar}{\sqrt{2}}\left(Z_1\bar{Z}_2 + \bar{Z}_1 Z_2 + Z_2\bar{Z}_3 + \bar{Z}_2 Z_3\right), \\
S_y(\Psi) &= \langle\Psi|\hat{S}_y|\Psi\rangle = \frac{i\hbar}{\sqrt{2}}\left(Z_1\bar{Z}_2 - \bar{Z}_1 Z_2 + Z_2\bar{Z}_3 - \bar{Z}_2 Z_3\right), \\
S_z(\Psi) &= \langle\Psi|\hat{S}_z|\Psi\rangle = \hbar\left(|Z_1|^2 - |Z_3|^2\right).
\end{aligned}$$

The Poisson bracket for classical observables S_x and S_y is stated as follows

$$\{S_x, S_y\} = \frac{2}{i}\left(\frac{\partial S_x}{\partial Z_1}\frac{\partial S_y}{\partial \bar{Z}_1} - \frac{\partial S_x}{\partial \bar{Z}_1}\frac{\partial S_y}{\partial Z_1} + \frac{\partial S_x}{\partial Z_2}\frac{\partial S_y}{\partial \bar{Z}_2} - \frac{\partial S_x}{\partial \bar{Z}_2}\frac{\partial S_y}{\partial Z_2} + \frac{\partial S_x}{\partial Z_3}\frac{\partial S_y}{\partial \bar{Z}_3} - \frac{\partial S_x}{\partial \bar{Z}_3}\frac{\partial S_y}{\partial Z_3}\right).$$

Therefore, one obtains

$$\begin{aligned}
\{S_x, S_y\} = -\hbar^2|Z_2|^2 - \hbar^2|Z_2|^2 + \hbar^2\left(|Z_1|^2 - \bar{Z}_1 Z_3 + Z_1\bar{Z}_3 - |Z_3|^2\right) \\
- \hbar^2\left(-|Z_1|^2 - \bar{Z}_1 Z_3 + Z_1\bar{Z}_3 + |Z_3|^2\right) + \hbar^2|Z_2|^2 2 + \hbar^2|Z_2|^2 = 2\hbar S_z.
\end{aligned}$$

Using the same approach, one also gets

$$\{S_x, S_z\} = 2\hbar S_y,$$
$$\{S_y, S_z\} = 2\hbar S_x.$$

Therefore, the Poisson bracket related to commutator has been showed by the following expressions

$$\{S_y, S_z\} \leftrightarrow \frac{2}{i}[\hat{S}_y, \hat{S}_z],$$
$$\{S_x, S_z\} \leftrightarrow \frac{2}{i}[\hat{S}_x, \hat{S}_z],$$
$$\{S_x, S_y\} \leftrightarrow \frac{2}{i}[\hat{S}_x, \hat{S}_y].$$

Besides, the classical analog of the Casimir operator S^2 for this case has been calculated and compared it with the standard one. Firstly, S_x^2, S_y^2 and S_z^2 are computed

$$\begin{aligned}
S_x^2 &= \frac{\hbar^2}{2}\left(Z_1^2\bar{Z}_2^2 + |Z_1|^2|Z_2|^2 + Z_1\bar{Z}_3|Z_2|^2 + Z_1Z_3\bar{Z}_2^2 + |Z_1|^2|Z_2|^2 + \bar{Z}_1^2Z_2^2\right)\\
&+ \frac{\hbar^2}{2}\left(\bar{Z}_1\bar{Z}_3Z_2^2 + \bar{Z}_1Z_3|Z_2|^2 + Z_1\bar{Z}_3|Z_2|^2 + \bar{Z}_1\bar{Z}_3Z_2^2 + Z_2^2\bar{Z}_3^2\right)\\
&+ \frac{\hbar^2}{2}\left(|Z_2|^2|Z_3|^2 + Z_1Z_3\bar{Z}_2^2 + \bar{Z}_1Z_3|Z_2|^2 + |Z_2|^2|Z_3|^2 + \bar{Z}_2^2Z_3^2\right),\\
S_y^2 &= -\frac{\hbar^2}{2}\left(Z_1^2\bar{Z}_2^2 - |Z_1|^2|Z_2|^2 + Z_1\bar{Z}_3|Z_2|^2 - Z_1Z_3\bar{Z}_2^2 - |Z_1|^2|Z_2|^2 + \bar{Z}_1^2Z_2^2\right)\\
&- \frac{\hbar^2}{2}\left(-\bar{Z}_1\bar{Z}_3Z_2^2 + \bar{Z}_1Z_3|Z_2|^2 + Z_1\bar{Z}_3|Z_2|^2 - \bar{Z}_1\bar{Z}_3Z_2^2 + Z_2^2\bar{Z}_3^2\right)\\
&- \frac{\hbar^2}{2}\left(-|Z_2|^2|Z_3|^2 - Z_1Z_3\bar{Z}_2^2 + \bar{Z}_1Z_3|Z_2|^2 - |Z_2|^2|Z_3|^2 + \bar{Z}_2^2Z_3^2\right),\\
S_z^2 &= \hbar^2\left(|Z_1|^4 - 2|Z_1|^2|Z_3|^2 + |Z_3|^4\right),
\end{aligned}$$

and then we demonstrate that

$$\begin{aligned}
S^2 &= S_x^2 + S_y^2 + S_z^2\\
&= \hbar^2\left(|Z_1|^4 + 2|Z_1|^2|Z_2|^2 + 2|Z_2|^2|Z_3|^2 - 2|Z_1|^2|Z_3|^2 + 2Z_1Z_3\bar{Z}_2^2 + 2\bar{Z}_1\bar{Z}_3Z_2^2 + |Z_3|^4\right).
\end{aligned}$$

Unlike spin $\frac{1}{2}$ case, rather than a constant, the classical analog of the Casimir operator for spin 1 has functional dependence. In this context, there is an inability to relate it with the standard Casimir operator with a value equal to $2\hbar$. To understand this, recall that the classical Casimir operator is defined as

$$\begin{aligned}
S^2 &= S_x^2 + S_y^2 + S_z^2\\
&= \langle\Psi|\hat{S}_x|\Psi\rangle^2 + \langle\Psi|\hat{S}_y|\Psi\rangle^2 + \langle\Psi|\hat{S}_z|\Psi\rangle^2,
\end{aligned}$$

and the standard Casimir operator is

$$\begin{aligned}
\langle\hat{S}^2\rangle &= \langle\hat{S}_x^2\rangle + \langle\hat{S}_y^2\rangle + \langle\hat{S}_z^2\rangle\\
&= \langle\Psi|\hat{S}_x^2|\Psi\rangle + \langle\Psi|\hat{S}_y^2|\Psi\rangle + \langle\Psi|\hat{S}_z^2|\Psi\rangle.
\end{aligned}$$

Now in the point of view of matrix algebra, let us define functions f and g as follows

$$f\colon A \to B = \langle\Psi|A|\Psi\rangle,$$
$$g\colon C \to D = C^2,$$

where in this case A, C and D are 3×3 matrices and B is 1×1 matrix (a number).

Here, the classical and quantum Casimir operator can be stated as $\sum(g\circ f)(\hat{S}_i)$ and $\sum(f\circ g)(\hat{S}_i)$, respectively where $i = x, y, z$. Thus, since in general the composition of functions is non-commutative i.e. $f\circ g \neq g\circ f$, it is clear that the classical and standard Casimir operators lead to different results.

3.3. Spin $\frac{3}{2}$ particle

Let us define the self-adjoint operators for this case as

$$\hat{S}_x = \frac{\hbar}{2}\begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}, \quad \hat{S}_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i\sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & -2i & 0 \\ 0 & 2i & 0 & -i\sqrt{3} \\ 0 & 0 & i\sqrt{3} & 0 \end{pmatrix}, \quad \hat{S}_z = \frac{\hbar}{2}\begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix},$$

and the orthonormal basis in Hilbert space $\mathcal{H} \cong \mathbb{C}^4$ is represented by (e_1, e_2, e_3, e_4) satisfies

$$\langle e_i | e_j \rangle = \delta_{ij},$$

for $i, j = 1, 2, 3, 4$. Then, the state of spin $\frac{3}{2}$ is expressed as

$$|\Psi\rangle = Z_1|e_1\rangle + Z_2|e_2\rangle + Z_3|e_3\rangle + Z_4|e_4\rangle,$$

where $Z_1, Z_2, Z_3, Z_4 \in \mathbb{C}$. Consider S_x, S_y and S_z be classical observables defined as

$$S_x(\Psi) = \langle\Psi|\hat{S}_x|\Psi\rangle = \hbar(Z_2\bar{Z}_3 + \bar{Z}_2 Z_3) + \frac{\hbar\sqrt{3}}{2}(Z_1\bar{Z}_2 + \bar{Z}_1 Z_2 + Z_3\bar{Z}_4 + \bar{Z}_3 Z_4),$$
$$S_y(\Psi) = \langle\Psi|\hat{S}_y|\Psi\rangle = i\hbar(Z_2\bar{Z}_3 - \bar{Z}_2 Z_3) + \frac{i\hbar\sqrt{3}}{2}(Z_1\bar{Z}_2 - \bar{Z}_1 Z_2 + Z_3\bar{Z}_4 - \bar{Z}_3 Z_4),$$
$$S_z(\Psi) = \langle\Psi|\hat{S}_z|\Psi\rangle = \frac{3\hbar}{2}(|Z_1|^2 - |Z_4|^2) + \frac{\hbar}{2}(|Z_2|^2 - |Z_3|^2).$$

The Poisson bracket for classical observables S_x and S_y is stated as follows

$$\{S_x, S_y\} = \frac{2}{i}\left(\frac{\partial S_x}{\partial Z_1}\frac{\partial S_y}{\partial \bar{Z}_1} - \frac{\partial S_x}{\partial \bar{Z}_1}\frac{\partial S_y}{\partial Z_1} + \frac{\partial S_x}{\partial Z_2}\frac{\partial S_y}{\partial \bar{Z}_2} - \frac{\partial S_x}{\partial \bar{Z}_2}\frac{\partial S_y}{\partial Z_2}\right) + \frac{2}{i}\left(\frac{\partial S_x}{\partial Z_3}\frac{\partial S_y}{\partial \bar{Z}_3} - \frac{\partial S_x}{\partial \bar{Z}_3}\frac{\partial S_y}{\partial Z_3} + \frac{\partial S_x}{\partial Z_4}\frac{\partial S_y}{\partial \bar{Z}_4} - \frac{\partial S_x}{\partial \bar{Z}_4}\frac{\partial S_y}{\partial Z_4}\right).$$

Therefore, one obtains

$$\{S_x, S_y\} = 3\hbar^2|Z_1|^2 - 3\hbar^2|Z_2|^2 + 3\hbar^2|Z_3|^2 - 3\hbar^2|Z_4|^2 + 4\hbar^2|Z_2|^2 - 4\hbar^2|Z_3|^2 = 2\hbar S_z.$$

Using the same approach, one also gets

$$\{S_x, S_z\} = 2\hbar S_y,$$
$$\{S_y, S_z\} = 2\hbar S_x.$$

Therefore, showing that the Poisson bracket is related to the commutator by the following expressions

$$\{S_y, S_z\} \leftrightarrow \frac{2}{i}[\hat{S}_y, \hat{S}_z];$$
$$\{S_x, S_z\} \leftrightarrow \frac{2}{i}[\hat{S}_x, \hat{S}_z];$$
$$\{S_x, S_y\} \leftrightarrow \frac{2}{i}[\hat{S}_x, \hat{S}_y].$$

Besides, the classical analog of Casimir operator S^2 has been calculated for this case and compared it with the standard one. Firstly, we compute S_x^2, S_y^2 and S_z^2

$$\begin{aligned} S_x^2 = {} & \frac{3\hbar^2}{4}\bar{Z}_1^2 Z_2^2 + \frac{3\hbar^2}{2}|Z_1|^2|Z_2|^2 + \hbar^2\sqrt{3}\bar{Z}_1 Z_3|Z_2|^2 + \hbar^2\sqrt{3}\bar{Z}_1\bar{Z}_3 Z_2^2 + \frac{3\hbar^2}{2}\bar{Z}_1 Z_2 Z_3\bar{Z}_4 \\ & + \frac{3\hbar^2}{2}\bar{Z}_1 Z_2\bar{Z}_3 Z_4 + \frac{3\hbar^2}{4}Z_1^2\bar{Z}_2^2 + \hbar^2\sqrt{3}Z_1 Z_3\bar{Z}_2^2 + \hbar^2\sqrt{3}Z_1\bar{Z}_3|Z_2|^2 + \frac{3\hbar^2}{2}Z_1\bar{Z}_2 Z_3\bar{Z}_4 \\ & + \frac{3\hbar^2}{2}Z_1\bar{Z}_2\bar{Z}_3 Z_4 + \hbar^2\bar{Z}_2^2 Z_3^2 + 2\hbar^2|Z_2|^2|Z_3|^2 + \hbar^2\sqrt{3}\bar{Z}_2\bar{Z}_4 Z_3^2 + \hbar^2\sqrt{3}\bar{Z}_2 Z_4|Z_3|^2 \\ & + \hbar^2 Z_2^2\bar{Z}_3^2 + \hbar^2\sqrt{3}Z_2\bar{Z}_4|Z_3|^2 + \hbar^2\sqrt{3}Z_2 Z_4\bar{Z}_3^2 + \frac{3\hbar^2}{4}Z_3^2\bar{Z}_4^2 + \frac{3\hbar^2}{2}|Z_3|^2|Z_4|^2 + \frac{3\hbar^2}{4}\bar{Z}_3^2 Z_4^2, \end{aligned}$$

$$
\begin{aligned}
S_y^2 = &-\frac{3\hbar^2}{4}\bar{Z}_1^2 Z_2^2 + \frac{3\hbar^2}{2}|Z_1|^2|Z_2|^2 - \hbar^2\sqrt{3}\bar{Z}_1 Z_3|Z_2|^2 + \hbar^2\sqrt{3}\bar{Z}_1\bar{Z}_3 Z_2^2 - \frac{3\hbar^2}{2}\bar{Z}_1 Z_2 Z_3 \bar{Z}_4 \\
&+ \frac{3\hbar^2}{2}\bar{Z}_1 Z_2 \bar{Z}_3 Z_4 - \frac{3\hbar^2}{4} Z_1^2 \bar{Z}_2^2 + \hbar^2\sqrt{3} Z_1 Z_3 \bar{Z}_2^2 - \hbar^2\sqrt{3} Z_1 \bar{Z}_3 |Z_2|^2 + \frac{3\hbar^2}{2} Z_1 \bar{Z}_2 Z_3 \bar{Z}_4 \\
&- \frac{3\hbar^2}{2} Z_1 \bar{Z}_2 \bar{Z}_3 Z_4 - \hbar^2 \bar{Z}_2^2 Z_3^2 + 2\hbar^2 |Z_2|^2|Z_3|^2 - \hbar^2\sqrt{3}\bar{Z}_2\bar{Z}_4 Z_3^2 + \hbar^2\sqrt{3}\bar{Z}_2 Z_4 |Z_3|^2 \\
&- \hbar^2 Z_2^2 \bar{Z}_3^2 + \hbar^2\sqrt{3} Z_2 \bar{Z}_4 |Z_3|^2 - \hbar^2\sqrt{3} Z_2 Z_4 \bar{Z}_3^2 - \frac{3\hbar^2}{4} Z_3^2 \bar{Z}_4^2 + \frac{3\hbar^2}{2}|Z_3|^2|Z_4|^2 - \frac{3\hbar^2}{4}\bar{Z}_3^2 Z_4^2,
\end{aligned}
$$

$$
\begin{aligned}
S_z^2 = &\frac{9\hbar^2}{4}|Z_1|^4 - \frac{9\hbar^2}{4}|Z_1|^2|Z_4|^2 + \frac{3\hbar^2}{4}|Z_1|^2|Z_2|^2 - \frac{3\hbar^2}{4}|Z_1|^2|Z_3|^2 - \frac{9\hbar^2}{4}|Z_1|^2|Z_4|^2 \\
&+ \frac{9\hbar^2}{4}|Z_4|^4 - \frac{3\hbar^2}{4}|Z_2|^2|Z_4|^4 + \frac{3\hbar^2}{4}|Z_3|^2|Z_4|^2 + \frac{3\hbar^2}{4}|Z_1|^2|Z_2|^2 - \frac{3\hbar^2}{4}|Z_2|^2|Z_4|^2 \\
&+ \frac{\hbar^2}{4}|Z_2|^4 - \frac{\hbar^2}{4}|Z_2|^2|Z_3|^2 - \frac{3\hbar^2}{4}|Z_1|^2|Z_3|^2 + \frac{3\hbar^2}{4}|Z_3|^2|Z_4|^2 - \frac{\hbar^2}{4}|Z_2|^2|Z_3|^2 + \frac{\hbar^2}{4}|Z_3|^4,
\end{aligned}
$$

and then we demonstrate that

$$
\begin{aligned}
S^2 = S_x^2 + S_y^2 + S_z^2 = &\frac{9\hbar^2}{4}|Z_1|^4 + \frac{\hbar^2}{4}|Z_2|^4 + \frac{\hbar^2}{4}|Z_3|^4 + \frac{9\hbar^2}{4}|Z_4|^4 + \frac{9\hbar^2}{2}|Z_1|^2|Z_2|^2 \\
&+ \frac{7\hbar^2}{2}|Z_2|^2|Z_3|^2 + \frac{9\hbar^2}{2}|Z_3|^2|Z_4|^2 - \frac{9\hbar^2}{2}|Z_1|^2|Z_4|^2 - \frac{3\hbar^2}{2}|Z_1|^2|Z_3|^2 \\
&- \frac{3\hbar^2}{2}|Z_2|^2|Z_4|^2 + 2\hbar^2\sqrt{3} Z_1 Z_3 \bar{Z}_2^2 + 3\hbar^2 Z_1 \bar{Z}_2 \bar{Z}_3 Z_4 + 2\hbar^2\sqrt{3}\bar{Z}_1\bar{Z}_3 Z_2^2 \\
&+ 3\hbar^2 \bar{Z}_1 Z_2 Z_3 \bar{Z}_4 + 2\hbar^2\sqrt{3} Z_2 Z_4 \bar{Z}_3^2 + 2\hbar^2\sqrt{3}\bar{Z}_2\bar{Z}_4 Z_3^2.
\end{aligned}
$$

Acorrding to the computation of classical Casimir operator for spin $\frac{1}{2}$, spin 1 and spin $\frac{3}{2}$, the operator could be generalized for the case of $\mathbb{C}^n$ as follows.

Conjecture. *The classical Casimir operator for Hilbert space $\mathcal{H} \cong \mathbb{C}^n$ is defined as*

$$
S^2 = S_x^2 + S_y^2 + S_z^2 = \sum_{i,j,k,l}^{n} H_{ij} H_{kl} Z_i \bar{Z}_j Z_k \bar{Z}_l,
$$

where $H_{ij}H_{kl} = \sum \langle e_i|\hat{S}_\alpha|e_j\rangle\langle e_k|\hat{S}_\alpha|e_l\rangle$, $\alpha = x, y, z$ and $i, j, k, l = 1, 2, 3, \ldots, n$.

4. Discussion and conclusion

In this study, we show that, although the correspondence between the Poisson bracket and commutator is consistent for all spin systems, the correlation between the Casimir operator and its classical counterpart is not well defined. Unlike spin $\frac{1}{2}$ case, whose Casimir is a constant, the classical Casimir operator for higher spins is expressed in terms of functions. In this context, it fails to be identical to the standard Casimir operator with a value equal to constant. Thus, this result clearly demonstrates the limitation of the classical notion to describe the purely quantum concept. It is consistent with Gleason's theorem [16] statement that the classical representation is not entirely available to describe quantum mechanical systems.

[1] Heslot A. Quantum mechanics as a classical theory. Physical Review D. **31** (6), 1341–1348 (1985).

[2] Varadarajan V. S. Geometry of Quantum Theory. Vol. 1. Princeton, N.J., van Nostrand (1968).

[3] Ashtekar A., Schilling T. A. Geometry of quantum mechanics. AIP Conference Proceedings. **342** (1), 471–478 (1995).

[4] Kibble T. W. B. Geometrization of quantum mechanics. Communications in Mathematical Physics. **65**, 189–201 (1979).

[5] Cirelli R., Lanzavecchia P. Hamiltonian vector fields in quantum mechanics. II Nuovo Cimento B. **79**, 271–283 (1984).

[6] Anandan J. A geometric approach to quantum mechanics. Foundations of Physics. **21**, 1265–1284 (1991).

[7] Brody D. C., Hughston L. P. Geometric quantum mechanics. Journal of Geometry and Physics. **38** (1), 19–53 (2001).

[8] Chruściński D., Jamiolkowski A. Geometric Phases in Classical and Quantum Mechanics. Progress in Mathematical Physics. Vol. 36. Boston, Birkhäser (2004).

[9] Benvegnù A., Sansonetto N., Spera M. Remarks on geometric quantum mechanics. Journal of Geometry and Physics. **51** (2), 229–243 (2004).

[10] Chruściński D. Geometric Aspects of Quantum Mechanics and Quantum Entanglement. Journal of Physics: Conference Series. **30**, 9–16 (2006).

[11] Marmo G., Volkert G. Geometrical description of quantum mechanics — transformations and dynamics. Physica Scripta. **82**, 038117 (2010).

[12] Clemente-Gallardo J. The geometrical formulation of quantum mechanics. Revista Real Academia de Ciencias de Zaragoza. **67**, 51–103 (2012).

[13] Heydari H. Geometric formulation of quantum mechanics. Preprint ArXiv:1503.00238v2 (2016).

[14] Anandan J., Aharonov Y. Geometry of quantum evolution. Physical Review Letters. **65** (14), 1697–1700 (1990).

[15] Heydari H. A geometric framework for mixed quantum states based on a Kähler structure. Journal of Physics A: Mathematical and Theoretical. **48**, 255301 (2015).

[16] Gleason A. M. Measures on the Closed Subspaces of a Hilbert Space. Journal of Mathematics and Mechanics. **6** (6), 885–893 (1957).

[17] Griffiths D. J. Introduction to Quantum Mechanics. Prentice Hall. 147–149 (1995).

Класичний аспект кутового моменту обертання в геометричній квантовій механіці

Ахмад Х. А. С.[1], Умайр Х.[2,3], Нурісяя М.С.[1,3], Чан К. Т.[1,3]

[1]*Інститут математичних досліджень (INSPEM), Університет Путра Малайзії (UPM), 43400 UPM Серданг, Селангор Дарул Ехсан, Малайзія*

[2]*Центр фундаментальних досліджень у галузі науки Університету Путра Малайзії, Університет Путра Малайзія, 43400, Селангор, Малайзія*

[3]*Інститут математичних досліджень, Університет Путра Малайзія, 43400, Селангор, Малайзія*

Геометрична квантова механіка — це формулювання, яке демонструє, як квантову теорію можна подати мовою гамільтонової динаміки фазового простору. У межах цього дослідження були вивчені класичні властивості частинок зі спіном $\frac{1}{2}$, спіном 1 і спіном $\frac{3}{2}$. Відповідність між дужкою Пуассона та комутаторними алгебрами для цих систем було показано шляхом явного обчислення значення комутатора операторів спіну та порівняння його з дужкою Пуассона відповідних класичних спостережуваних. Це дослідження було розширено шляхом порівняння оператора Казимира та його класичного аналога. Результати показали, що існує відповідність між класичними та квантовими операторами Казимира принаймні для випадку спіну $\frac{1}{2}$. Це дослідження чітко показує межу класичних понять для опису суто квантової концепції.

Ключові слова: *диференціальна геометрія; геометрична квантова механіка; класичні спостережувані.*