Study of the dynamic process in a nonlinear mathematical model of the transverse oscillations of a moving beam under perturbed boundary conditions

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The study of transverse oscillations of systems moving along their axis is a very difficult, but at the same time a very important task. Mathematical models of nonlinear transverse oscillations of a beam moving along its axis are analyzed in this paper work, both for non-resonant and resonant cases. The task becomes even more complicated if we additionally take into account the method of fastening the ends of the beam or the perturbation at its ends. We have obtained dependencies that can be used in construction, transport, industry, mechanical engineering and other domains of technology, ensuring the stability and safety of the operation of such mechanical systems. Mathematical models have been obtained for structural engineers to determine the amplitude-frequency response of relevant structures. These mathematical models are key to researching the dynamics of moving media. The obtained results allow considering not only the influence of kinematic and physical-mechanical parameters on the amplitude-amplitude frequency response of the medium, but also the fastening method. In addition, the correlations obtained in the paper make it possible to study not only the influence of the moving medium parameters on the nature of changes in the frequency and amplitude of oscillations, but also to consider the movement at the points of support of the medium. Namely, even at the stage of designing a pipeline for a liquid flowing at a certain speed, it is possible to consider the influence of the oscillation of the supports or their fastening method on the dynamics of the oscillatory process. The resulting dependencies allow designers to consider the influence of the characteristics given in the paper with a high level of accuracy and predict dynamic phenomena in them. In engineering calculations of various mechanical systems, the resulting dependencies can be used to optimize parameters to avoid negative destructive phenomena during operation.

Keywords: *transverse oscillations; mathematical model; boundary conditions; nonlinear oscillations; asymptotic method; elastic beam; resonance; fastening method.*

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1. Introduction

The oscillations and motion of objects are the basis of many moving systems and machines and are very important in the study of physical laws and behavior of such objects. One of the most difficult and at the same time important tasks is the study of transverse oscillations of systems moving along an axis. Examples of such systems are found in engineering and mechanical engineering [1]: a telescopic boom in a crane, a cable car, conveyor belts, pipes for liquids or gas, etc. All these structures can be modelled mathematically as beams or strings moving along an axis. The problem of mathematical modelling of these systems is of high scientific and practical significance and encourages scientists and engineers to research and search for new solutions.

The study of beam oscillations is a complex scientific task that requires the use of various methods of mathematical analysis, differential equations, numerical methods and experimental studies. The study of beam oscillations allows establishing dependencies of physico-mechanical and kinematic quantities on the amplitude frequency characteristics of such a system. For engineering calculations, it is enough to know the influence of speed, perturbing force, material density, modulus of elasticity and the mass of the beam on the aforesaid characteristics of the oscillating process for developing new structures and optimizing the design of various mechanisms and constructions. This makes their behaviour under different conditions and at various operation modes understandable and allows designers to improve modern designs and develop new innovative solutions. The resulting dependencies can be used in construction, transport, industry, mechanical engineering, and other fields of technology, ensuring the stability and safety of such mechanical systems. Studying their dynamic characteristics, such as oscillation period, amplitude, perturbation, and resonance allows understanding the physical laws of movement and interaction of objects as well as determining the mechanical properties of materials. The task becomes more complicated if we additionally consider the effect of the method of fastening the ends of the beam or the perturbation at its ends.

In all other cases under non-resonant conditions, the form of natural oscillations will be determined by the initial conditions and physical and mechanical properties of the medium. But in this state, they will quickly fade due to dissipative and internal forces. Therefore, during resonance, single-frequency oscillations can occur in such a system if dissipative forces are also taken into account. The form of such oscillations will be dynamic equilibrium, and the frequency will be equal to the frequency of forced oscillations of the system. Even under linear boundary conditions, scientists face significant mathematical difficulties in the study of non-stationary oscillations, which are described by boundary value problems. This is because differential equations with partial derivatives describing such processes contain time-varying coefficients [2]. It is obvious that in most cases it will not be possible to find an exact solution for such a system of equations. In this case, it is possible to apply the asymptotic methods of two parametric families of partial solutions, because they correspond to single-frequency modes of non-stationary oscillations. For a linear elastic system, the usual Fourier method can be applied. The initial problem can be reduced to the solution of differential equations or or a system of ordinary differential equations, which is not difficult. In our research, the main attention is focused on determining the effect of small perturbations at the end of the beam. The oscillating system is regarded as a moving nonlinear elastic medium with perturbing boundary conditions [3].

2. Literature analysis and problem statement

Asymptotic methods of nonlinear mechanics are effectively used to study weakly nonlinear systems with distributed parameters in single-frequency oscillation modes [2, 3]. It is known [3, 4] that single-frequency oscillations can occur under certain initial conditions in elastic systems described by mixed boundary value problems.

When studying single-frequency oscillations close to oscillations in one of the forms of dynamic equilibrium, asymptotic method of nonlinear mechanics is used to build an approximate solution, which enables us to avoid significant mathematical difficulties. In particular, in [5], the authors present a methodology that allows identifying the maximum number of solutions, even those that belong to isolation of the system's amplitude frequency response curves. However, the methodology proposed in this article leads to a potential solution of the bifurcation problem depending on any component of the truncated Fourier series without considering kinematic parameters. The authors [6] derived a system of three partial differential equations. They developed a mathematical model of curved beam oscillations and showed a numerical solution for nonlinear partial differential equations. But this case applies only to multilayer glass beams and has its own mechanical features. The fundamental study was carried out in [7], where the authors investigated the stability and dynamic characteristics of parametric oscillations of the ropes of cable-stayed bridges with an extra long span under various axial perturbations. However, such systems did not move along an axis and perturbations in the supports were not considered. The authors in [8] studied nonlinear oscillations of a beam installed in a high-speed moving structure, but the dimensions of such a beam were assumed to be small. In [9], the frequencies and amplitudes of oscillations of a profile containing concentrated structural nonlinearities represented by polynomial, free and hysteresis springs, were calculated, but such problems were solved without considering perturbations in boundary conditions. The authors of [10] considered absolute stability for an axially moving Kirchhoff beam, but at a transverse speed. A closed system was obtained, which is established using the Faedo–Galyorkin approximation in combination with some a priori estimates. The study [11] considered equations of the model for an axially moving beam in the supercritical regime, but under simple support boundary conditions. In [12], transverse oscillations that flow in parallel to longitudinal oscillations were considered, and nonlinear oscillations of the beam in the subcritical mode of bending were also investigated. The authors established complex nonlinear boundary conditions for the beam, but with some geometric restrictions. The authors of the paper [13] investigated problems for nonlinear one-dimensional wave equations with initial boundary conditions using the fixed point method. In the results, the initial and boundary conditions were considered, which made it possible to determine the nature of the classes of the nonlinear one-dimensional wave equation. However, in actual existing systems, even perturbation in one or another fastening type can lead to significant changes (both kinematic and qualitative) in the process dynamics. The authors of [14] considered a two-dimensional mixed problem, but only for a thin elastic tape. In addition, they derived a mathematical model for the longitudinal movements of the beam due to its transverse compression. The authors of [15] dealt with a similar topic, but they considered torsional oscillations using Ateb-functions.

In summary, as shown by the above analysis of literary sources, such an important problem as the impact of perturbations and movements at the points of fastening of a moving beam (or a pipeline through which liquid flows at a certain speed) on the oscillations of one-dimensional nonlinear elastic systems is yet to be considered comprehensively. Similar systems, but in a simplified form, are described in works [16–21]. As a rule, the main issue in the study of such a problem was lack of accurate analytical methods for solving corresponding nonlinear differential equations. To cover the field of research into the problem more extensively, let us consider both the non-resonant case and the resonant case.

3. Study goal and objectives

The goal of the study is to establish the pattern of changes in the frequency and amplitude of transverse oscillations caused not only by kinetic and physical-mechanical parameters of the system, but also depending on the fastening method used in the oscillating system. We will study the effect of movements and perturbations at the points of support of one-dimensional nonlinear elastic systems. Having established this dependence, we will be able to predict resonance zones and avoid dangerous modes of operation of such constructions. It is possible to consider optimal parameters of mechanisms as well as characteristics of material and supports of the oscillating system even at the stage of designing it. To achieve the goal, the following objectives have been defined:

– establish mathematical models describing the influence of boundary conditions on the dynamic processes of one-dimensional nonlinear elastic systems characterized by longitudinal motion for the non-resonant case;

– determine the laws of change of oscillation amplitude and frequency in the form of mathematical ratios, as functions of the fastening parameters of the oscillating system (perturbation, displacement);

– analyze the effect of perturbations at fastening points (ends) of a nonlinear elastic perpendicularly oscillating system moving along its axis on the nature of change in amplitude and frequency for non-resonant and resonant cases.

4. Study materials and methods

To conduct the study of transverse oscillations of a beam moving along its axis, we will make the following clarifications:

1. The beam is moving at a constant speed — V
2. The material of the beam has nonlinear-elastic characteristics
3. The system is under the influence of an external periodic force

4. Oscillations occur under small perturbing boundary conditions.

The research is based on the principle of single-frequency oscillations in nonlinear systems with many degrees of freedom and distributed parameters [22–25]. We will apply the asymptotic method of building solutions for classes of differential equations with partial derivatives [3, 26].

Let us describe basic notations and conditions of oscillation:

1) transverse oscillations of the system occur in one plane xOy – this will the plane of oscillations. Then we will measure all deviations of the points on the beam from the axis Ox.

2) at the time of transverse oscillations, the points on the beam are displaced strictly vertically, that is, perpendicular to the rectilinear axis Ox, which is an undeformed straight line. We will measure the deviation of beam elements during transverse oscillations of the beam from this axis. The displacement of these points in parallel to this axis is not taken into account.

Under such conditions, transverse oscillations of the beam can be described by a function which will depend on two variables: coordinate – x and time – t. Therefore, the deviations of the beam axis points are determined by one function $u = u(x,t)$. Let us write down a few more notations: $m(x)$ is mass of a beam length unit; E is module of elasticity (for steel $E = 2.06 \cdot 10^{11}\ \mathrm{N/m^2}$), I is the moment of inertia of the cross section of the beam relative to the neutral axis of the section. The placement of such a section is strictly parallel to the plane of oscillations.

As it is known [27], under such assumptions the differential equation that describes transverse oscillations of a beam is the following:

$$\frac{d^2u}{dt^2} + \beta^2 \frac{\partial^4 u}{\partial x^4} = \mu\, P\left(u, \psi, \frac{\partial u}{\partial t}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^3 u}{\partial x^3}, \frac{\partial^4 u}{\partial x^4}\right), \tag{1}$$

where $\beta = \sqrt{\frac{EI}{m}}$; μ is small positive parameter; $P\left(u, \psi, \frac{\partial u}{\partial t}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^3 u}{\partial x^3}, \frac{\partial^4 u}{\partial x^4}\right)$ is analytical 2π-periodical by $\phi\, t = \psi$ function. Such a function can be presented in the following form:

$$P\left(u, \psi, \frac{\partial u}{\partial t}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^3 u}{\partial x^3}, \frac{\partial^4 u}{\partial x^4}\right) = \sum_{n=-N}^{N} e^{in\phi t} P_n\left(u, \frac{\partial u}{\partial t}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^3 u}{\partial x^3}, \frac{\partial^4 u}{\partial x^4}\right). \tag{2}$$

Coefficients $P_n\left(u, \frac{\partial u}{\partial t}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^3 u}{\partial x^3}, \frac{\partial^4 u}{\partial x^4}\right)$ in the right part of equation (2) will be regarded as certain polynomials in relation to $\frac{\partial u}{\partial t}$, $\frac{\partial^2 u}{\partial x^2}$, $\frac{\partial^3 u}{\partial x^3}$, $\frac{\partial^4 u}{\partial x^4}$.

Since the beam is moving along its axis with a certain longitudinal speed V, this parameter should be considered in our equation (1). The equation will take the following form [27]:

$$\frac{\partial^2 u}{\partial t^2} + \beta^2 \frac{\partial^4 u}{\partial x^4} + 2V \frac{\partial^2 u}{\partial x \partial t} + V^2 \frac{\partial^2 u}{\partial x^2} = \mu\, P\left(u, \psi, \frac{\partial u}{\partial t}, \frac{\partial^2 u}{\partial x^2}, \ldots, \frac{\partial^4 u}{\partial x^4}\right). \tag{3}$$

The right part of equation (3) takes into consideration weakly nonlinear elastic characteristics of the medium. Besides, it includes the following: resistance forces, if these forces are small compared to nonlinear elastic forces; dissipative forces; external periodic perturbations.

Therefore, equation (3) describes transverse movements of non-stationary oscillations of the beam. It takes into consideration inertial forces, external periodic forces that are distributed along the entire length of the beam, as well as small perturbing nonlinear forces and internal frictional forces.

Let us write down the boundary conditions that will correspond to practical cases: fastening the ends of the beam includes the case of elastic fastening with a nonlinear characteristic and its set displacement in time. Boundary conditions of the non-autonomous type have the following form [11, 28]:

$$N_{1j}\left(u, \frac{\partial^2 u}{\partial x^2}\right)\bigg|_{x=0,l} = \mu \chi_j\left(\tau, \psi, \frac{\partial u}{\partial x}\right)\bigg|_{x=0,l}, \quad N_{2j}\left(u, \frac{\partial^2 u}{\partial x^2}\right)\bigg|_{x=0,l} = \mu \lambda_j\left(\tau, \psi, u\right)\big|_{x=0,l}, \tag{4}$$

where N_{ij} $(i = 1, 2;\ j = 0, l)$ is a certain linear homogeneous function at $x = j$. The right parts of (4) are 2π-periodical by ψ functions. They can be arranged in a series by powers of the small parameter μ.

The non-linear one-dimensional boundary value problem of non-autonomous type which takes into consideration the influence of small non-stationary perturbations on an oscillating system consists of a non-linear differential equation (3) and four weakly non-linear boundary conditions (4).

Under such initial conditions, the set problem meets the conditions of existence of an oscillatory process that is close to a form of unperturbed system.

5. Results of the study of transverse oscillations of a beam moving along its axis

For the linear case, when $\varepsilon = 0$ and the oscillating system is moving with a certain speed, according to asymptotic methods of linear mechanics [18], the solution of equation (3) will be searched for in the following form:

$$u_{0n}(x,t) = a_o L_n(x) \cos(\omega_o t + \varphi_o) \quad (n = 1, 2, 3, \dots), \tag{5}$$

where ω_o is natural frequency, a_o is amplitude parameter, $L_n(x)$ is fundamental functions which have the property of orthonormality, φ_o is arbitrary constant. Under undisturbed boundary conditions (this case corresponds to hinged ends), these functions will be presented as follows for the case of main oscillations [29, 30]

$$L(x) = \sin \frac{\pi x}{l}. \tag{6}$$

In case of main resonance, we will look for a solution to the perturbed boundary value problem in the form of an asymptotic expansion

$$u(x,t) = a(t) L_1(x) \cos(\gamma) + \varepsilon u_1(x, a, \gamma, \psi). \tag{7}$$

5.1. Non-resonant case

For the nonlinear case, unlike the linear, parameters a and γ ($\gamma = \omega t + \varphi$) in expression (7) will be variable. They will depend on the movement of the medium and on non-linear and periodic forces. According to [26,30], the laws of change of a and γ in equation (7) will be set with differential equations.

$$\frac{da}{dt} = \mu\, A_1(a) + \dots, \tag{8}$$
$$\frac{d\gamma}{dt} = \omega + \mu\, B_1(a) + \dots.$$

Dependencies $A_1(a)$ and $B_1(a)$ must be found in such a way that they meet the conditions of equation (5). For this, let us substitute $a(t)$ and $\gamma(t)$ with their derivatives (8) in (5). Besides, functions $A_1(a)$ and $B_1(a)$ must meet the conditions of (4) with a required degree of accuracy.

Functions $A_1(a)$, $B_1(a)$ and $u_1(x, a, \gamma)$ must be found, for the first approximation of solution the equations will have the following form:

$$\frac{\partial u}{\partial t} = \mu A_1(a) L_k(x) \cos\gamma - a L_k(x) \left(\sin\gamma(\omega + \mu\, B_1(a))\right) + \frac{\partial u_1}{\partial \gamma} \left(\omega + \mu\, B_1(a)\right), \tag{9}$$
$$\frac{\partial^2 u}{\partial t^2} = -a\,\omega^2 L_k(x)\cos\gamma - 2\mu\, a\, A_1(a) L_k(x) \sin\gamma - 2a\, B_1(a)\,\omega\, L_k(x)\cos\gamma + \frac{\partial^2 u}{\partial \gamma^2}\omega^2,$$
$$\frac{\partial u}{\partial x} = a\, L_k'(x)\cos\gamma + \mu \frac{\partial u_1}{\partial x},$$
$$\frac{\partial^2 u}{\partial x^2} = a\, L_k''(x)\cos\gamma + \mu \frac{\partial^2 u_1}{\partial x^2},$$
$$\frac{\partial^3 u}{\partial x^3} = a\, L_k'''(x)\cos\gamma + \mu \frac{\partial^3 u_1}{\partial x^3},$$
$$\frac{\partial^4 u}{\partial x^4} = a\, L_k''''(x)\cos\gamma + \mu \frac{\partial^4 u_1}{\partial x^4}.$$

Correlations (9) allow obtaining a linear differential equation. It binds the functions $u_1(x, a, \gamma)$, $A_1(a)$, and $B_1(a)$ that must be found.

By substituting boundary conditions (4) and solution (7) into equation (3), and also taking into consideration correlations (9) by equating coefficients at identical powers μ, in the left and right parts of the equation we obtain the boundary value problem with conditions (4) for the linear function $u_1(x,a,\gamma)$:

$$\frac{\partial^2 u_1}{\partial\gamma^2}\omega^2+\beta^2\frac{\partial^4 u_1}{\partial x^4}=2a\omega\,A_1(a)\sin\frac{k\pi x}{l}\sin\gamma+2\omega\,a\,B_1(a)\sin\frac{k\pi x}{l}\cos\gamma$$
$$+\left(\frac{k\pi}{l}\right)^2V^2a\sin\frac{k\pi x}{l}\cos\gamma+2\left(\frac{k\pi}{l}\right)V\,a\,\omega\cos\frac{k\pi x}{l}\sin\gamma+\overline{P}(x,a,\gamma). \tag{10}$$

Let us find the solution in the form of a Fourier series using the asymptotic method:

$$u_1(x,a,\gamma)=\sum X_m(x)\,u_{1m}(a,\gamma). \tag{11}$$

For the convenience of calculations, the multiple Fourier series are presented in the complex-exponential form. This form is equivalent to the usual form of the cosine and sine expansion. In this case, the convergence conditions will be the same. Therefore, the function $u_{1m}(a,\gamma)$ can be presented as follows:

$$u_{1m}=\sum u_{1mpr}(a)\,e^{ip(\varphi+\psi)}, \tag{12}$$

where p is mutually prime numbers, $u_{1mpr}(a)$ is complex coefficients of the Fourier series. They are related to amplitudes and are determined considering the orthonormality of the selected basis.

Let us impose an extra condition on the dependency $u_1(x,a,\gamma)$. The condition is the absence of terms that are proportional to $\sin\frac{k\pi x}{l}\cos\gamma$ and $\sin\frac{k\pi}{l}\sin\gamma$ in its extension. This will allow defining the unknown functions $A_1(a)$ and $B_1(a)$ unambiguously. Then the law of change of amplitude and phase can be obtained as expressions for the function in the following form:

$$\frac{da}{dt}=\mu\frac{1}{s}\frac{1}{4\omega\pi^2}\int_0^l\int_0^{2\pi}\int_0^{2\pi}P^*(a,x,\gamma)\sin\frac{\pi x}{l}\sin\gamma\,dx\,d\gamma\,d\psi, \tag{13}$$

$$\frac{d\gamma}{dt}=\omega-\left(\frac{k\pi}{l}\right)^2V^2+\mu\frac{1}{s}\frac{1}{4\omega\,\pi^2a}\int_0^l\int_0^{2\pi}\int_0^{2\pi}P^*(a,x,\gamma)\sin\frac{\pi x}{l}\cos\gamma\,dx\,d\gamma\,d\psi,$$

where $s=\int_0^l X^2(x)\,dx=\frac{l}{2}$.

In the non-resonance case, as seen from (13), the amplitude frequency response of the oscillating system will depend on the following values: harmonic force affecting the beam; speed of movement; amplitude a; physical and mechanical parameters.

5.2. Resonant case

If we consider such a system for the main resonant case, then the frequency of self-oscillating systems will coincide with or be close to the frequency of the perturbing force ($\omega=\phi$). Solution can be found in the form (7), as for the non-resonant case. But the difference will be that for the resonant case, the amplitude frequency response of the process significantly depends on the difference in the phases of forced oscillations and natural oscillations. Thus the functions $\frac{d\varphi}{dt}$ and $\frac{da}{dt}$ will be presented as dependencies on a and on $\varphi=\gamma-\psi$:

$$\frac{da}{dt}=\mu\,A_1(a,\varphi)+\mu^2A_2(a,\varphi)+\ldots, \tag{14}$$

$$\frac{d\varphi}{dt}=\omega-\phi+\mu\,B_1(a,\varphi)+\mu^2B_2(a,\varphi)+\ldots.$$

To solve the problem, it is necessary to define the functions $A_1(a,j)$, $B_1(a,j)$ and $u_1(a,\gamma,\psi,x)$ for the first approximation. In the same way as before, we will equate the coefficients at parameter μ and thus obtain a boundary value problem for $u_1(a,\gamma,\psi,x)$:

$$\frac{\partial^2 u_1}{\partial\gamma^2}\omega^2+2\phi\,\omega\frac{\partial u_1}{\partial\gamma\partial\psi}+\phi^2\frac{\partial^2 u_1}{\partial\psi^2}+\beta^2\frac{\partial^4 u_1}{\partial x^4}=2a\omega\,A_1(a)\sin\frac{k\pi x}{l}\sin\gamma+2\omega\,a\,B_1(a)\sin\frac{k\pi x}{l}\cos\psi$$
$$+\left(\frac{k\pi}{l}\right)^2V^2a\sin\frac{k\pi x}{l}\cos\gamma+2\left(\frac{k\pi}{l}\right)V\,a\omega\cos\frac{k\pi x}{l}\sin\gamma+\overline{P}(x,a,\gamma,\psi). \tag{15}$$

The boundary conditions will have the following form:

$$N_{1j}\left(u, \frac{\partial^2 u}{\partial x^2}\right)\Big|_{x=0,l} = \mu\,\chi_{1j}^{(0)}(x,a,\gamma,\psi)\Big|_{x=0,l},$$

$$N_{2j}\left(u, \frac{\partial^2 u}{\partial x^2}\right)\Big|_{x=0,l} = \mu\,\lambda_{2j}^{(0)}(x,a,\gamma,\psi)\Big|_{x=0,l}, \tag{16}$$

where

$$\chi = \chi_j\left(\sin\frac{k\pi x}{l}, a\cos\gamma, \psi\right)\Big|_{x=0,l}, \quad \lambda = \lambda_j\left(\sin\frac{k\pi x}{l}, a\cos\gamma, \psi\right)\Big|_{x=0,l}. \tag{17}$$

The solution should be sought as a sum to satisfy non-homogeneous boundary conditions (16),

$$u_1(x,a,\theta,\psi) = \vartheta_1(x,a,\theta,\psi) + \xi_1(x,a,\theta,\psi), \tag{18}$$

where $\xi_1(x,a,\theta,\psi)$ is auxiliary function. $\frac{\partial^4\xi}{\partial x^4} = 0$ will be the solution for the equation. Boundary conditions in relation to $\vartheta_1(x,a,\psi,\theta)$ will have the following form at a set choice of $\xi_1(x,a,\theta,\psi)$

$$N_{1j}\left(\vartheta, \frac{\partial^2\vartheta}{\partial x^2}\right)\Big|_{x=j} = 0, \quad N_{2j}\left(\vartheta, \frac{\partial^2\vartheta}{\partial x^2}\right)\Big|_{x=j} = 0, \quad (j = 0, l). \tag{19}$$

If $\frac{\partial^4\xi}{\partial x^4} = 0$, then ξ will equal $\xi = \frac{1}{6}c_1x^3 + \frac{1}{2}c_2x^2 + c_3x + c_4$.

Under the boundary conditions (19), the coefficients $c_1 - c_4$ can be determined. For the case when one end vibrates and a force acts on it, and the other end is immovably fastened, which is common in practice, the boundary conditions take the following form:

$$u(x,t)|_{x=0} = u_{xx}(x,t)|_{x=0} = 0, \quad u(x,t)|_{x=l} = Fu_x(x,t) + F_1\sin\theta, \quad u_{xx}(x,t)\big|_{x=l} = F_2u_{xxx}(x,t). \tag{20}$$

The function ξ, which will require finding, will equal

$$\xi = (Fu_x + F_1\sin\psi + xF_2u_{xxx})\frac{x^3}{(3l-2l)^2} + (Fu_x + F_1\sin\psi - xF_2u_{xxx})\frac{(5l-3l)^2}{(6l^2-4l^2)}x + F_2u_{xxx}x^2. \tag{21}$$

The only thing left will be to define the new function $\vartheta(x,a,\gamma,\psi)$ from the differential equation:

$$\frac{\partial^2\vartheta_1}{\partial\gamma^2}\omega^2 + 2\phi\omega\frac{\partial\vartheta_1}{\partial\gamma\partial\psi} + \phi^2\frac{\partial^2\vartheta_1}{\partial\psi^2} + \beta^2\frac{\partial^4\vartheta_1}{\partial x^4} = 2a\,\omega\,A_1(a)\sin\frac{k\pi x}{l}\sin\gamma + 2\omega\,a\,B_1(a)\sin\frac{k\pi x}{l}\cos\gamma$$
$$+\left(\frac{k\pi}{l}\right)^2 V^2a\sin\frac{k\pi x}{l}\cos\gamma + 2\left(\frac{k\pi}{l}\right)V\,a\,\omega\cos\frac{k\pi x}{l}\sin\gamma + P'(x,a,\gamma,\psi), \tag{22}$$

where $P' = P(x,a,\gamma,\psi) - \frac{\partial^2\xi}{\partial\gamma^2}\omega^2 - 2\phi\omega\frac{\partial^2\xi}{\partial\gamma\partial\psi} - \frac{\partial^2\xi}{\partial\psi^2}\phi^2$.

The function $\vartheta(x,a,\gamma,\psi)$ is defined from the linear non-homogeneous equation (22) and under homogeneous boundary conditions as a series:

$$\vartheta(x,a,\gamma,\psi) = \sum_{n=1}^{\infty}\vartheta_n(a,\gamma,\psi)\,L_n(x). \tag{23}$$

Case $m = 1$:

$$\frac{\partial^2\vartheta_{11}}{\partial\gamma^2}\omega^2 + 2\frac{\partial\vartheta_{11}}{\partial\gamma\partial\psi}\phi\,\omega + \phi^2\frac{\partial^2\vartheta_{11}}{\partial\psi^2} + \beta^2\left(\frac{\pi}{l}\right)^4\frac{\partial^4\vartheta_{11}}{\partial t^4} = a\,V^2\left(\frac{\pi}{l}\right)^2\sin\frac{k\pi}{l}\cos\gamma + \frac{1}{s}\int_0^l P'(a,x,\gamma,\psi)\,L_1(x)dx$$
$$+\left(\cos\gamma\left(-\frac{\partial A(a,\varphi)}{\partial\varphi}(\omega-\phi) + 2a\omega\,B\right) + \sin\gamma\left(a\frac{\partial B(a,\varphi)}{\partial\varphi}(\omega-\phi) + 2A\omega\right)\right). \tag{24}$$

case $m \neq 1$:

$$\frac{\partial^2\vartheta_{1m}}{\partial\gamma^2}\omega^2 + 2\frac{\partial\vartheta_{1m}}{\partial\gamma\partial\psi}\phi\,\omega + \phi^2\frac{\partial^2\vartheta_{1m}}{\partial\psi^2} + \beta^2\left(\frac{m\pi}{l}\right)^4\vartheta_{1m} = a\,V^2\left(\frac{m\pi}{l}\right)^2\sin\frac{m\pi}{l}\cos\gamma$$
$$+\frac{1}{s}\int_0^l P'(a,x,\gamma,\psi)\,L_m(x)\,dx. \tag{25}$$

For the main resonant case, the same as for the non-resonant case, conditions should be imposed on the function $\vartheta_{1k}(a,\gamma,\psi)$. The following will be obtained:

$$(\omega-\phi)\frac{\partial A(a,\varphi)}{\partial\varphi}-2a\omega\,B(a,\varphi)=\frac{1}{s}\frac{1}{2\pi^2}\sum_s e^{is\varphi}\int_0^l\int_0^{2\pi}P'(a,x,\varphi+\psi,\psi)\sin\frac{k\pi x}{l}\,e^{-ir\varphi}\cos\gamma\,dx\,d\psi,$$

$$a\frac{\partial B(a,\varphi)}{\partial\varphi}(\omega-\phi)-2A(a,\varphi)\,\omega+V^2\frac{\pi^2}{l^2}$$
$$=\frac{1}{s}\frac{1}{2\pi^2}\sum_s e^{ir\varphi}\int_0^l\int_0^{2\pi}P'(a,x,\varphi+\psi,\psi)\sin\frac{k\pi x}{l}\,e^{-ir\varphi}\cos\lambda\,dx\,d\psi. \quad (26)$$

We will obtain a system of differential equations for the first approximation of the solution of the problem, which binds the sought functions, in the resonant case.

5.3. Nonlinear transverse oscillations of a beam under perturbed boundary conditions

Using the example of transverse oscillations of a moving beam, consider the case when harmonic perturbation acts on the system. The material of such a medium satisfies the nonlinear technical law of elasticity [32]. The differential equation of motion, for such conditions, can be presented in the following form,

$$\frac{\partial^2u}{\partial t^2}+\beta^2\frac{\partial^4u}{\partial x^4}=-\frac{\partial^2u}{\partial x^2}V^2-2\frac{\partial^2u}{\partial x\partial t}V-\mu\frac{\partial^2u}{\partial x^2}\left[\frac{\partial^2u}{\partial x^2}\frac{\partial^4u}{\partial x^4}+2\left(\frac{\partial^3u}{\partial x^3}\right)^2\right]+\mu\,R\sin\phi\,t, \quad (27)$$

$R>0$. A single-frequency oscillatory process that occurs under the conditions: 1) when the frequency of the medium is close to the frequency of external perturbations; 2) when the boundary conditions for equation (27) correspond to hinged ends; can be described as the following dependency:

$$u(x,t)=a\sin\frac{\pi x}{l}\cos(\phi\,t+\varphi). \quad (28)$$

For the above case, the parameters a and j are defined using a system of differential equations:

- non-resonant case:

$$\frac{da}{dt}=0, \quad (29)$$
$$\frac{d\varphi}{dt}=\omega-\mu\left(\frac{9}{128}\frac{\pi^2a^2}{\omega^{-1}l^2}+\frac{(\pi V)^2}{8\omega\,l^2}+\frac{3\omega}{256\pi}\frac{Fl(\pi^2+1)+F_2\pi^4l-\pi^2l(F_2+F)}{\pi l^2}\right);$$

- resonant case:

$$\frac{da}{dt}=-\frac{4\mu R}{\pi(\omega+\phi)}\cos\varphi, \quad (30)$$
$$\frac{d\varphi}{dt}=\omega-\phi-\mu\left(\frac{9}{128}\frac{\pi^2a^2}{\omega^{-1}l^2}+\frac{(\pi V)^2}{8\omega l^2}+\frac{3\omega}{256\pi}\frac{Fl(\pi^2+1)+F_2\pi^4l-\pi^2l(F_2+F)}{\pi l^2}\right)$$
$$+4\varepsilon\frac{R}{\pi(\omega+\phi)a}\sin\varphi.$$

5.4. Analysis of the impact of perturbations at the points of fastening of the ends of a nonlinear elastic perpendicularly oscillating system on the change in amplitude and frequency for non-resonant and resonant cases

Based on the study of the mathematical model of oscillations, dependences of the frequency and amplitude of oscillations on some other parameters of the beam were obtained (Figures 1–3). Graphic dependences are built using the following numerical values: $S=0.12\,0.085\,\mathrm{m}^2$, $E=2.06\cdot10^{11}\,\mathrm{N/m}^2$, $I_0=6.1\cdot10^{-6}\,\mathrm{m}^4$, $\varepsilon H=1.0\,\mathrm{N/kg}$, $l=2\,\mathrm{m}$, $a=2\,\mathrm{cm}$, $m=80.54\,\mathrm{kg/m}$, $\rho=7900\,\mathrm{kg/m}^3$ perturbing force frequency $\phi\approx\omega$, $\omega=\left(\frac{\pi}{l}\right)^2\sqrt{\frac{EI}{\rho S}}$. With such data, the natural frequency of the beam will equal $w=308.1\,(\mathrm{Hz})$.

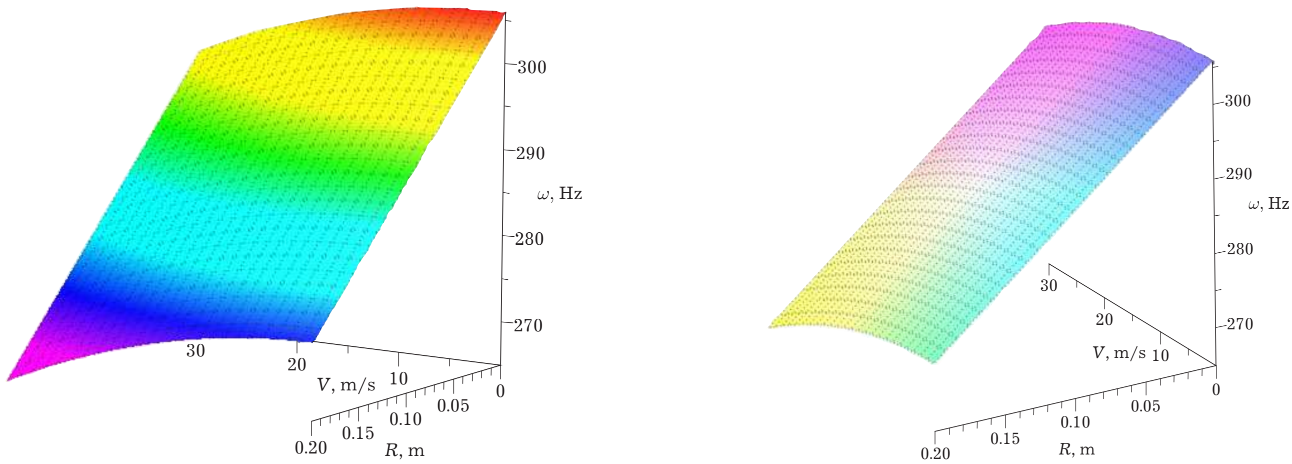

Fig. 1. Dependence of system oscillation frequency on longitudinal speed of the beam (V) and change in end oscillation amplitude (R) (for the non-resonant case).

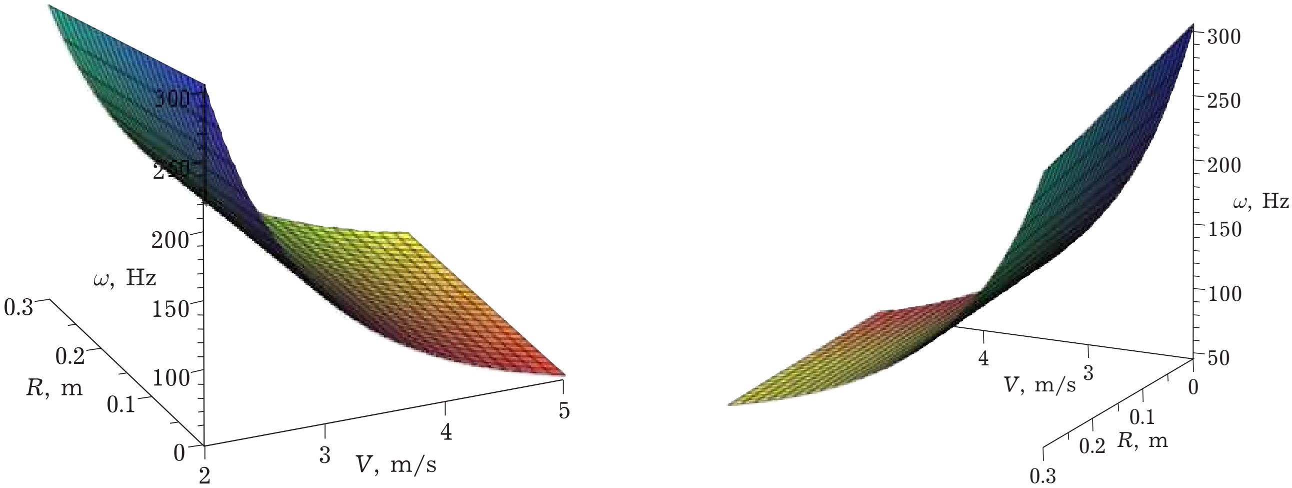

Fig. 2. Dependence of system oscillation frequency on beam length (l) and change in end oscillation amplitude (R) (for the non-resonant case) at longitudinal speed $5\,\mathrm{m/s}$.

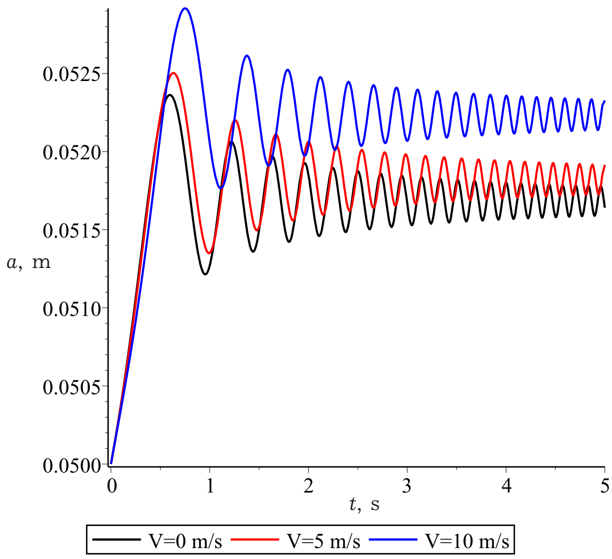

Fig. 3. Dependences of system oscillation frequency on time for different speeds for the resonant case.

6. Results of the study of kinematic and physical-mechanical parameter influence on the patterns of change in oscillation amplitude and frequency under perturbed boundary conditions

In the previous chapter of this work, a practical example of various dependences of the amplitude frequency response of oscillations on different values under perturbed boundary conditions were presented in the visual form.

The obtained dependences (29)–(30) enable us to comprehensively investigate not only the influence of kinematic and physical-mechanical parameters [27] of medium transverse oscillations on the frequency and amplitude. They also show how frequency and amplitude changes are caused by the method of fastening in resonant and non-resonant cases. For such convenient dependencies, it is easy to establish various graphic interdependencies for engineering calculations.

1. For structural engineers, mathematical models (29)–(30) were obtained. These mathematical models allow determining the amplitude frequency response of relevant structures and are crucial for researching the dynamics of moving media. The obtained results allow taking into consideration not only the influence of kinematic and physical-mechanical parameters, but also that of the fastening method, on the amplitude frequency response of the medium.

2. After substituting certain arguments, the proposed algorithms can be applied in practice to analyze transverse oscillations of a pipeline with a liquid flowing inside at a certain speed. It is possible to take into account not only the mechanical features of such a system, but also its kinematic features and vibrations at the points of support.

3. The method used in this study allows establishing the resonance zones of oscillation and, if necessary, avoid them or use them for other cases. Taking into consideration the vibrations at the points of support of the beam or pipeline, one can establish optimum values of kinematic parameters for a given structure. As a result, a rational oscillatory system and its elements will be developed as early as at the design stage.

4. After conducting numerous simulations of dependencies (29)–(30) in MAPLE 15 and obtaining graphical representations (Figures 1–3), we managed to establish the influence of vibrations at the point of support on the amplitude frequency response of the oscillating system.

5. The correlations obtained in this work not only allow studying the influence of parameters of a moving medium on the frequency and amplitude of oscillations. They also allow considering the movement at the points of support of the medium. In particular, even at the stage of designing a pipeline for liquids that will be flowing at a certain speed, it is possible to consider the fluctuations of supports or the fastening method. The resulting dependencies allow designers to take into account the influence of the characteristics given in the study with considerable accuracy and predict dynamic phenomena in them. In engineering calculations of various mechanical systems, the resulting dependencies can be used to optimize parameters and avoid negative destructive phenomena during system operation.

6. A detailed study of the influence of kinematic and some physico-mechanical parameters on the amplitude and frequency of transverse oscillations is presented in [31], but only under the condition that the ends of the beam are fastened and do not allow vibrations, that is, they are rigidly fastened.

In further research, applying a similar solution algorithm in a corresponding mathematical model would allow considering also compressive or stretching forces that affect an oscillating system.

7. Conclusions

1. If we analyze system (29) for the non-resonant case, it can be noted that the longitudinal speed of the beam affects only the frequency of its transverse oscillations. This result is explained by the fact that the system is conservative. Movements at the beam end point of support affect the frequency of transverse oscillations significantly. In particular, Figure 1 shows that the displacement of support by 0.2 m makes the frequency of oscillation decrease from 308 Hz to 275 Hz, which is almost 11%. But the effect of longitudinal speed is not significant. Thus, at a speed of 20 m/s, the oscillation frequency drops to 295 Hz, which is only 5%.

2. Given that the argument V in the ratio (29) is squared, the frequency of longitudinal oscillations of the beam decreases according to the parabolic law as the speed increases.

3. If we consider Figures 2, we can see that when the length of the beam increases, the effect of displacement at the point of support decreases. So, with a beam length of 2 m (natural frequency equal to 308 Hz) and a displacement of support by 0.2 m, the oscillation frequency will be 275 Hz, that is, the frequency decreases by 11%. But when the length of the beam is 5 m (the natural frequency is 49 Hz) and with the same displacement at the point of support, the frequency will already be about 45 Hz. Thus, the frequency of oscillations is reduced by 8%. Therefore, it is possible to compensate support vibration with increased length of the beam.

4. For the resonant case, when one end allows vibrations up to 0.2 m, the effect of the longitudinal speed is not significant. In particular, when the speed increases to 10 m/s, the amplitude of oscillation increases by almost 4%. In addition, the first amplitude of transverse oscillations at longitudinal speed occurs later, that is, the oscillation curve shifts to the right (Figure 3). Compared to a stationary beam ($V = 0$), the first amplitude value of oscillation occurs at 0.6 s, but at a speed of 10 m/s, this amplitude occurs at 0.8 s. The nature of φ change remains the same.

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Дослідження динамічного процесу в нелінійній математичній моделі поперечних коливань рухомої балки за збурених граничних умов

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Вивчення поперечних коливань систем, що рухаються вздовж своєї осі, є дуже складною, але в той же час дуже важливою задачею. У роботі проаналізовано математичні моделі нелінійних поперечних коливань балки, що рухається вздовж її осі, як для нерезонансного так й для резонансного випадку. Задача ще більше ускладнюється, якщо додатково врахувати спосіб кріплення кінців балки або збурення на її кінцях. Отримано залежності, які можна використовувати в будівництві, транспорті, промисловості, машинобудуванні та інших галузях техніки, що забезпечують стабільність і безпеку роботи таких механічних систем. Для інженерів-конструкторів отримано математичні моделі для визначення амплітудно-частотної характеристики відповідних конструкцій. Ці математичні моделі є ключовими для дослідження динаміки рухомих носіїв. Отримані результати дозволяють розглянути не тільки вплив кінематичних і фізико-механічних параметрів на амплітудно-частотну характеристику середовища, а й спосіб кріплення. Крім того, отримані в роботі кореляційні зв'язки дають змогу досліджувати не лише вплив параметрів рухомого середовища на характер зміни частоти й амплітуди коливань, а й розглядати рух у точках опори середовища. А саме, ще на етапі проектування трубопроводу для рідини, що тече з певною швидкістю, можна розглянути вплив коливання опору або способу їх кріплення на динаміку коливального процесу. Отримані залежності допоможуть проектувальникам з високою точністю враховувати вплив наведених у роботі характеристик і прогнозувати динамічні явища в них. При інженерних розрахунках різних механічних систем отримані результати можуть бути використані для оптимізації параметрів, щоб уникнути негативних руйнівних явищ під час експлуатації.

Ключові слова: *поперечні коливання; математична модель; граничні умови; нелінійні коливання; асимптотичний метод; пружна балка; резонанс; спосіб кріплення.*