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Numerical modeling of surface subsidence due to compaction of soil with fine inclusions

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A mathematical model of filtration consolidation of an inhomogeneous soil mass was formed taking into account the change in the size of the area during the compaction process. The inhomogeneity is considered as the presence of fine inclusions (geobarriers) the physical and mechanical characteristics of which differ from those of the main soil. From a mathematical viewpoint, the model is described by a one-phase Stefan problem that has a kinematic boundary condition on the upper moving boundary as its component. The purpose of the research is to find out the effect of fine inclusion on the dynamics of subsidence of the soil surface in the process of compaction. The change in the dimensions of the solution area is physically determined by the change in the volume of the pores of the porous medium in the process of dissipating excess pressure. If the permeability of the geobarrier is low, it affects the dynamics of consolidation processes and, accordingly, the magnitude of subsidence. Finite element solutions of the initial-boundary value problem for the nonlinear parabolic equation in the heterogeneous region with the conjugation condition of non-ideal contact were found. Numerical time discretization methods, a method for determining the change in the position of the upper boundary at discrete moments of time, and an algorithm for determining the physical and mechanical characteristics of a porous medium depending on the degree of consolidation are given. A number of test examples were considered, and the effect of a thin inclusion on the dynamics of the change in the position of the upper boundary of the problem solution area was investigated.

Keywords: *consolidation; subsidence; conjugation condition; finite element method.*

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1. Introduction

Even though the mathematical model of the consolidation problem formulated in terms of the equations of mathematical physics is already about a hundred years old, its research remains relevant to this day [1]. Moreover, new approaches to the mathematical description of consolidation processes are being developed, which take into account the physical features of porous media and processes in them, as well as methods of researching relevant boundary value problems. Such developments include taking into account the effect of chemical substances and the temperature state of the porous medium [2,3]; taking into account the rheology of the porous medium and the corresponding models [4,5]; development of the theory of inverse consolidation problems, including using machine learning methods [6]; development of the theory of consolidation using the fractional derivatives apparatus [5, 7]; taking into account the multi-layered porous medium in the consolidation problem [5, 8], to name a few.

Research on the problem of consolidation of inhomogeneous environments is also developing from the viewpoint of the presence of natural or artificial fine inclusions (so-called geobarriers) in the environments. The importance of considering such inclusions is especially relevant in the context of industrial and household waste storage facilities [3]. The possibility of a significant effect of thin geobarriers on the course of processes in porous media, especially if we account for the influence of physico-chemical and biological factors on the hydrological parameters of a thin inclusion, is shown in [9–13].

In addition to determining the level of excess pressures in the pore fluid, which is an important factor in disrupting the stability of soil masses [14], the consolidation problem has another goal. This is to determine the level of subsidence of the soil surface in the process of consolidation [15]. Taking subsidence into account significantly complicates the mathematical model of the problem. The model becomes a Stefan problem and requires setting the so-called kinematic boundary condition on the soil surface [16]. The derivation of such a condition is also a separate task. A method of deriving such a condition was proposed in [17] taking into account interrelated physico-chemical processes in porous media in mathematical models [18]. However, the complex task of soil consolidation in the presence of fine inclusions taking into account surface subsidence was not considered yet. It is natural that the consolidation problems in the presence of fine inclusions did not investigate the influence of geobarriers on the magnitude and dynamics of subsidence of the soil surface. To perform this task is the main goal of this work.

2. Formulation of the problem, integral kinematic boundary condition, and conjugation condition

We consider the process of filtration consolidation of a soil layer of total thickness $l = l(t)$, $t \geqslant 0$, with a thin inclusion ω of thickness d, located at depth $x = \xi$ (Figure 1). The general initial-boundary value problem as a mathematical model of the studied consolidation processes accounts for the presence of a thin inclusion through the so-called conjugation conditions for the unknown function, and the processes in the inclusion itself are not investigated. The material of a thin inclusion differs in its physical and mechanical characteristics from those of the main soil which can lead to the discontinuity of the solutions of the initial-boundary value problem at the inclusion.

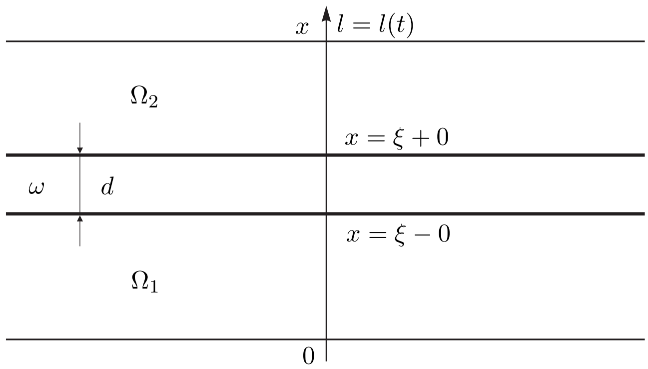

Fig. 1. A soil layer of thickness l with a thin inclusion ω of thickness d ($d \ll l$).

The main unknown function in the problem of filtration consolidation of a fully saturated porous medium is the function of excess pressures h in the pore fluid. They result from applying an external load to the soil surface. Over time, the pressures dissipate and the loads are transformed into stresses in the skeleton of the porous medium. A change in stress in the skeleton, according to the compression dependences for soils, leads to a change in the porosity of the soil. Consequently, the filtration coefficient also changes [3, 19, 20]. Since the change in porosity from the very beginning in the filtration consolidation problem is due to the change in pressure h in the pore fluid, an indirect nonlinear dependence of the filtration coefficient on h can be defined. The presence of such nonlinear dependences for the material of a thin inclusion requires their consideration in the conjugation conditions. The method of modification of conjugation conditions for such cases is proposed in [9–11, 13]. Particularly, for the problem considered in the article,

$$u^{\pm}\big|_{x=\xi} = \left(-k(h)\frac{\partial h}{\partial x}\right)^{\pm}\Bigg|_{x=\xi} = -\frac{[h]}{\int_0^d \frac{dx}{k_\omega(h)}}.$$

Here h is pressure; k, k_ω are the filtration coefficients of the main soil and inclusion soil, respectively; u is the filtration rate which is determined from the filtration law; $u^\pm$ are the values of filtration rates at $x=\xi-0$ and $x=\xi+0$, respectively; $[h]=h^+-h^-$ is a pressure jump at the thin inclusion.

Assume that in the classical conjugation conditions [21, 22]

$$u^\pm\big|_{x=\xi}=-\frac{k_\omega}{d}\,[h]$$

the filtration coefficient k_ω is constant. Taking into account the nonlinearities in the conjugation conditions of this type is problematic, because it requires justification which pressure at the thin inclusion, h^+ or h^-, is applied in the dependence $k_\omega=k_\omega(h)$. If $h^+\neq h^-$, then also $k_\omega\left(h^+\right)\neq k_\omega\left(h^-\right)$.

The change in porosity has another effect, i.e. a change in the size of the area over which the process is being investigated. Here, the increase in the pore fluid pressure causes swelling (increase in size), whereas the decrease in pressure leads to subsidence (decrease in size). Hereafter, we will use the term "subsidence", and refer to swelling as subsidence with a minus sign. Taking into account subsidence requires setting a kinematic boundary condition on the soil surface which describes the change in the boundary position $l=l(t)$, $t\in(0,T]$. Here $T>0$ is the specified time interval. Also $l(t)|_{t=0}=l_0>0$, where l_0 is the set initial thickness of the soil mass. A method of deriving such conditions for the effects of physico-chemical and mechanical factors is proposed in [17]. As we consider only the change in excess pressures as the influencing factors, and taking into account that the problem is one-dimensional, from the condition [17, formula (6)], we get

$$\frac{dl(t)}{dt}=-\int_0^{l(t)}\frac{\partial u}{\partial x}dz.$$

3. Mathematical model of the filtration consolidation of a porous medium with a thin inclusion for the case of a moving upper boundary

The mathematical model of the above problem is described by the following boundary value problem:

$$\frac{\partial h}{\partial t}=\frac{1+e}{\gamma a}\frac{\partial}{\partial x}\left(k(h)\frac{\partial h}{\partial x}\right),\quad x\in\Omega_1\cup\Omega_2,\quad t\in(0,T],\tag{1}$$

$$h(x,t)|_{x=l(t)}=0,\quad t\in[0,T],\tag{2}$$

$$u(x,t)|_{x=0}=\left(-k(h)\frac{\partial h}{\partial x}\right)\bigg|_{x=0}=0,\quad t\in[0,T],\tag{3}$$

$$h(x,0)=h_0(x),\quad x\in\overline{\Omega}_1\cup\overline{\Omega}_2,\tag{4}$$

$$u^\pm\big|_{x=\xi}=\left(-k(h)\frac{\partial h}{\partial x}\right)^\pm\bigg|_{x=\xi}=-\frac{[h]}{\int_0^d\frac{dx}{k_\omega(h)}},\tag{5}$$

$$\frac{dl(t)}{dt}=-\int_0^{l(t)}\frac{\partial u}{\partial x}dz,\tag{6}$$

$$l(t)|_{t=0}=l_0>0.\tag{7}$$

Here, $\Omega_1=(0;\xi)$, $\Omega_2=(\xi;l(t))$, $0<\xi<l(t)$; $\Omega=\Omega_1\cup\Omega_2$; h is the pressure; $T>0$ is the set length of the time interval; $h_0(x)$ is the known function; l_0 is the set initial thickness of the soil mass; a is the soil compressibility coefficient; $e=\frac{n}{1-n}$ is the soil void ratio; n, n_ω is the porosity of soil and inclusion material, respectively; γ is the specific gravity of the liquid; k, k_ω are filtration coefficients of the main soil and inclusion soil, respectively, that may nonlinearly depend on the pressure; u is the filtration rate that is determined according to Darcy's law $u=-k(h)\frac{\partial h}{\partial x}$; $u^\pm$ are the values of filtration rates at $x=\xi-0$ and $x=\xi+0$, respectively; $[h]=h^+-h^-$ is the pressure jump on a thin inclusion.

Similarly to [13, 22], we introduce the following notations: $Q_T=\Omega\times(0;T]$, $Q_T^1=\Omega_1\times(0;T]$, $Q_T^2=\Omega_2\times(0;T]$.

Definition 1. *The classical solution of the initial-boundary value problem (1)–(7) which admits a discontinuity of the first kind at the point $x = \xi$ is a function $h(x,t) \in \Psi$ that $\forall (x,t) \in \overline{Q}_T$ satisfies equation (1) and the initial condition (4).*

Here Ψ is the set of functions $\psi(x,t)$ which together with $\frac{\partial \psi}{\partial x}$ are continuous on each of the closures $\overline{Q}^1_T$, $\overline{Q}^2_T$, have bounded continuous partial derivatives $\frac{\partial \psi}{\partial t}$, $\frac{\partial^2 \psi}{\partial x^2}$ on Q^1_T, Q^2_T and satisfy conditions (2), (3), (5).

4. Generalized solution of the problem (1)–(7)

Similarly to [13,22], let H_0 be the space of functions $s(x)$ that in each of the regions Ω_i belong to the Sobolev space $W^1_2(\Omega_i)$, $i = 1, 2$, and they acquire zero values at the ends of the segment $[0; l(t)]$ where boundary conditions of the first kind are set for the function $h(x,t)$.

Let $h(x,t) \in \Psi$ be the classical solution of the initial-boundary value problem (1)–(7). Take $s(x) \in H_0$. We multiply equation (1) and initial condition (4) by $s(x)$. Integrating them over the segment $[0; l(t)]$ and taking into account the conjugation conditions (5), we get

$$\int_0^{l(t)} \frac{\gamma a}{1+e} \frac{\partial h}{\partial t} s(x)\, dx + \int_0^{l(t)} k(h) \frac{\partial h}{\partial x} \frac{ds}{dx}\, dx + \frac{[h][s]}{\int_0^d \frac{dx}{k_\omega(h)}} = 0, \tag{8}$$

$$\int_0^{l(t)} h(x,0) s(x)\, dx = \int_0^{l(t)} h_0(x) s(x)\, dx. \tag{9}$$

So, if $h(x,t) \in \Psi$ is a classical solution of the initial-boundary value problem (1)–(7), then $h(x,t)$ is a solution of problem (8), (9) in the weak formulation.

Let H be the space of functions $v(x,t)$ that are square integrable together with their first derivatives $\frac{\partial v}{\partial t}$, $\frac{\partial v}{\partial x}$ on each of the intervals $(0;\xi)$, $(\xi; l(t))$, $\forall t \in (0;T]$, $T > 0$, and they satisfy the same boundary conditions of the first kind as the function $h(x,t)$.

Definition 2. *A function $h(x,t) \in H$ that for any $s(x) \in H_0$ satisfies the integral relations (8), (9) is called a generalized solution of the initial boundary value problem (1)–(7).*

We will seek an approximate generalized solution of the initial boundary value problem (1)–(7) in the form

$$\widehat{h}(x,t) = \sum_{i=1}^{N} h_i(t) \varphi_i(x), \tag{10}$$

where $\{\varphi_i(x)\}_{i=1}^N$ is the basis of the finite-dimensional subspace $M_0 \subset H_0$; $h_i(t)$, $i = \overline{1,N}$ are unknown coefficients that depend only on time.

A set of functions that can be represented in the form (10) generate a finite-dimensional subspace $M_1 \subset H_1$.

Definition 3. *An approximate generalized solution of the initial-boundary value problem (1)–(7) is a function $\widehat{h}(x,t) \in M_1$ which for an arbitrary function $S(x) \in M_0$ satisfies the integral relations*

$$\int_0^{l(t)} \frac{\gamma a}{1+e} \frac{\partial \widehat{h}}{\partial t} S(x)\, dx + \int_0^{l(t)} k(\widehat{h}) \frac{\partial \widehat{h}}{\partial x} \frac{dS}{dx}\, dx + \frac{[\widehat{h}][S]}{\int_0^d \frac{dx}{k_\omega(\widehat{h})}} = 0, \tag{11}$$

$$\int_0^{l(t)} \widehat{h}(x,0) S(x)\, dx = \int_0^{l(t)} h_0(x) S(x)\, dx. \tag{12}$$

Next, from the weak formulation (11), (12), setting the function $S(x)$ equal to each basis function $\varphi_i(x)$, $i = \overline{1,N}$, we obtain the Cauchy problem for the system of nonlinear differential equations

$$\mathbf{M}(\mathbf{H}) \frac{d\mathbf{H}}{dt} + \mathbf{L}(\mathbf{H})\, \mathbf{H}(t) = \mathbf{0}, \tag{13}$$

$$\widetilde{\mathbf{M}}\,\mathbf{H}^{(0)} = \widetilde{\mathbf{F}}, \tag{14}$$

where

$$\widetilde{\mathbf{F}} = \big(\tilde{f}_i\big)_{i=1}^{N}, \quad \widetilde{\mathbf{M}} = (\tilde{m}_{ij})_{i,j=1}^{N}, \quad \tilde{m}_{ij} = \int_0^{l_0} \varphi_i\,\varphi_j\,dx, \quad \mathbf{M} = (m_{ij})_{i,j=1}^{N},$$

$$\mathbf{L} = (l_{ij})_{i,j=1}^{N}, \quad \mathbf{H} = (h_i(t))_{i=1}^{N}, \quad \mathbf{H}^{(0)} = (h_i(0))_{i=1}^{N},$$

$$m_{ij} = \int_0^{l(t)} \frac{\gamma a}{1+e}\,\varphi_i\varphi_j\,dx,$$

$$l_{ij} = \int_0^{l(t)} k(\widehat{h})\frac{d\varphi_i}{dx}\frac{d\varphi_j}{dx}\,dx + \frac{[\varphi_i]\,[\varphi_j]}{\int_0^d \frac{dx}{k_\omega(\widehat{h})}}.$$

Problem (13), (14) is a Cauchy problem for a system of nonlinear differential equations of the first order. Finding its solution also requires the use of appropriate discretization methods. We can use of the Crank–Nicolson method [13, 22], the predictor–corrector method [13, 22], or a fully implicit linearized difference scheme [2].

5. Approximation of the kinematic boundary condition

Condition (6) is not directly convenient for the numerical calculation of the subsidence of the soil surface. Therefore, we substitute in (6) the filtration rate determined according to the filtration law

$$u = -k(h)\frac{\partial h}{\partial x}.$$

We have

$$\frac{dl(t)}{dt} = \int_0^{l(t)} \frac{\partial}{\partial x}\left(k(h)\frac{\partial h}{\partial x}\right) dz.$$

Next, using equation (1), from the above equality we have

$$\frac{dl(t)}{dt} = \int_0^{l(t)} \frac{\gamma a}{1+e}\frac{\partial h}{\partial t}\,dz. \tag{15}$$

Applying time discretization to (15) according to the implicit difference scheme, we get

$$\frac{l^{(j+1)} - l^{(j)}}{\tau} = \int_0^{l^{(j)}} \frac{\gamma a}{1+e^{(j+1)}}\frac{h^{(j+1)} - h^{(j)}}{\tau}\,dz, \quad j = 0, 1, 2, \dots.$$

Then

$$l^{(j+1)} = l^{(j)} + \int_0^{l^{(j)}} \frac{\gamma a}{1+e^{(j+1)}}\big(h^{(j+1)} - h^{(j)}\big)dz, \quad j = 0, 1, 2, \dots. \tag{16}$$

Here the time segment $[0,T]$ is split into m_τ equal parts with step $\tau = \frac{T}{m_\tau}$; $t_j = j\tau$; $l^{(j)} = l(t_j)$, $j = 0, 1, 2, \dots$. The position $l^{(j)}$ is known in (16), as are the values of $h^{(j+1)}$, $e^{(j+1)}$, and $h^{(j)}$. Note that, according to the algorithm, the value of pressure on the time layer $(j+1)$ is sought using the position of the upper limit $l^{(j)}$. The integral in (16) can be obtained using numerical integration formulas.

According to the linear compression dependence for soils

$$e = -a\,\sigma + \text{const}.$$

Here σ are vertical stresses in the soil skeleton (in one-dimensional case). Further,

$$\frac{\partial e}{\partial t} = -a\frac{\partial \sigma}{\partial t}.$$

Also, according to the principle of effective stress,

$$\frac{\partial \sigma}{\partial t} = -\gamma\frac{\partial h}{\partial t}.$$

That is,

$$\frac{\partial e}{\partial t} = a\gamma \frac{\partial h}{\partial t}.$$

From the latter, we get

$$\frac{e^{(j+1)} - e^{(j)}}{\tau} = a\gamma \frac{h^{(j+1)} - h^{(j)}}{\tau}, \quad j = 0, 1, 2, \ldots, m_\tau - 1,$$

or

$$e^{(j+1)} = a\gamma\big(h^{(j+1)} - h^{(j)}\big) + e^{(j)}, \quad j = 0, 1, 2, \ldots, m_\tau - 1.$$

6. Results of numerical experiments and their analysis

Soil parameters for numerical experiments are taken from the Hydrus-1D freeware. Specifically, Sandy Clay was considered as the main soil, with $k_0 = 0.0288\,\frac{\text{m}}{\text{day}}$, $n_0 = 0.38$. Silty Clay was used as the inclusion soil, with $k_{0\omega} = 0.0048\,\frac{\text{m}}{\text{day}}$, $n_{0\omega} = 0.46$. The index "0" indicates initial values.

The filtration coefficient in the calculation process was assumed to be dependent on the porosity coefficient according to the Kozeny–Carman formula [19]

$$k = k_0 \frac{1 + e_0}{1 + e} \left(\frac{e}{e_0}\right)^3,$$

where k_0, e_0 are the initial values of the filtration coefficient and the void ratio; k, e are their current variable values.

The soil compressibility coefficient in (1) is $a = 2 \cdot 10^{-7}\,\frac{\text{m}^2}{\text{H}}$, $a_\omega = 9 \cdot 10^{-7}\,\frac{\text{m}^2}{\text{H}}$, specific gravity of the pore fluid $\gamma = 10^4\,\frac{\text{H}}{\text{m}^2}$. The initial distribution of pressure $h_0(x) = 20\,\text{m}$ is corresponding to the application of the respective load to the soil surface. Unobstructed outflow of pore fluid is ensured at the upper boundary, with no drainage at the lower boundary.

For the model problem, a soil layer with initial thickness $l = 40\,\text{m}$ was considered. The initial depth of the inclusion varied in the numerical experiments at $\xi = 20\,\text{m}$, $\xi = 30\,\text{m}$, $\xi = 33\,\text{m}$, and $\xi = 35\,\text{m}$, and its thickness was $d = 0.2\,\text{m}$. Variable x step was $0.04\,\text{m}$, time step $\tau = 10\,\text{day}$. Piecewise-quadratic functions were used as the basis functions of finite element method. The results of numerical experiments are presented in Tables 1–4 (Modified conjugation condition, I; Classical conjugation condition, II; Relative difference in subsidence, III). The relative difference in subsidence was calculated as a ratio of the absolute difference to the subsidence under the classical conjugation condition.

Table 1. Subsidence of the upper boundary of the soil for $\xi = 20\,\text{m}$.

Time moment	I	II	III
100 days	0.3856 m	0.3862 m	0.16%
200 days	0.5227 m	0.5277 m	0.95%
300 days	0.6162 m	0.6278 m	1.85%
400 days	0.6874 m	0.7054 m	2.55%
500 days	0.7438 m	0.7668 m	3.00%
600 days	0.7891 m	0.8156 m	3.25%
720 days	0.8324 m	0.8613 m	3.36%

Both the subsidence values themselves and the relative differences depend on the depth of the thin weakly permeable inclusion. The closer the inclusion is to the lower limit, the smaller its influence on subsidence. The closer the inclusion is to the upper drained boundary, the smaller the subsidences become, while the relative difference between the subsidences in the case of classical and modified coupling conditions increases. Subsidences in the case of the modified conjugation condition are smaller than for the classical one. This is because the filtering coefficient in the integral conjugation condition is variable and depends on the consolidation degree. Its value decreases in the process of consolidation

Table 2. Subsidence of the upper boundary of the soil for $\xi = 30\,\text{m}$.

Time moment	I	II	III
100 days	0.3290 m	0.3401 m	3.26 %
200 days	0.4364 m	0.4665 m	6.45 %
300 days	0.5211 m	0.5656 m	7.87 %
400 days	0.5910 m	0.6459 m	8.50 %
500 days	0.6495 m	0.7113 m	8.69 %
600 days	0.6988 m	0.7647 m	8.62 %
720 days	0.7481 m	0.8162 m	8.34 %

Table 3. Subsidence of the upper boundary of the soil for $\xi = 33\,\text{m}$.

Time moment	I	II	III
100 days	0.2910 m	0.3123 m	6.82 %
200 days	0.3962 m	0.4403 m	10.02 %
300 days	0.4816 m	0.5419 m	11.13 %
400 days	0.5531 m	0.6245 m	11.43 %
500 days	0.6135 m	0.6921 m	11.36 %
600 days	0.6649 m	0.7477 m	11.07 %
720 days	0.7169 m	0.8015 m	10.56 %

Table 4. Subsidence of the upper boundary of the soil for $\xi = 35\,\text{m}$.

Time moment	I	II	III
100 days	0.2602 m	0.2904 m	10.40 %
200 days	0.3658 m	0.4211 m	13.13 %
300 days	0.4523 m	0.5249 m	13.83 %
400 days	0.5252 m	0.6095 m	13.83 %
500 days	0.5872 m	0.6788 m	13.49 %
600 days	0.6402 m	0.7359 m	13.00 %
720 days	0.6943 m	0.7914 m	12.27 %

and as a result, over time, the inclusion increasingly plays the role of a watertight boundary. Therefore, pressures in the soil mass below the inclusion are dissipated more slowly and in general the soil subsides less than in the case of classical condition. Such results indicate the qualitative correspondence of the model with the modified conjugation condition to the physics of the process and, hence, to the improvement of the qualitative adequacy of the mathematical model. At the same time, the maximum relative difference in subsidence can reach 14%. Clearly, with time the pressures continue to dissipate even through a weakly permeable inclusion, and the tendency of decreasing relative differences in subsidence can already be traced through the last two lines in Tables 2–4. However, the presence of the inclusion and taking into account the change in its characteristics in the process of consolidation changes the overall dynamic picture of both the dissipation of excess pressure and subsidence of the soil surface.

[1] Xiao T., Ni P., Chen Z., Feng J., Chen D., Mei G. A semi-analytical solution for consolidation of ground with local permeable pipe pile. Computers and Geotechnics. **143**, 104590 (2022).

[2] Vlasyuk A. P., Martynyuk P. M., Fursovych O. R. Numerical solution of a one-dimensional problem of filtration consolidation of saline soils in a nonisothermal regime. Journal of Mathematical Sciences. **160** (4), 525–535 (2009).

[3] Masum S. A., Zhang Z., Tian G., Sultana M. Three-dimensional fully coupled hydro-mechanical-chemical model for solute transport under mechanical and osmotic loading conditions. Environmental Science and Pollution Research. **30** (3), 5983–6000 (2023).

[4] Jozefiak K., Zbiciak A., Brzezinski K., Maslakowski M. A Novel Approach to the Analysis of the Soil Consolidation Problem by Using Non-Classical Rheological Schemes. Applied Sciences. **11** (5), 1980 (2021).

[5] Ding P., Xu R., Zhu Y., Wen M. Fractional derivative modelling for rheological consolidation of multilayered soil under time-dependent loadings and continuous permeable boundary conditions. Acta Geotechnica. **17** (6), 2287–2304 (2022).

[6] Bekele Y. W. Physics-informed deep learning for one-dimensional consolidation. Journal of Rock Mechanics and Geotechnical Engineering. **13** (2), 420–430 (2021).

[7] Bulavatsky V. M., Bohaienko V. O. Some Consolidation Dynamics Problems within the Framework of the Biparabolic Mathematical Model and its Fractional-Differential Analog. Cybernetics and Systems Analysis. **56**, 770–783 (2020).

[8] Li L., Qin A., Jiang L. Semi-analytical solution for one-dimensional consolidation of a two-layered soil system with unsaturated and saturated conditions. International Journal for Numerical and Analytical Methods in Geomechanics. **45** (15), 2284–2300 (2021).

[9] Chui Y., Martyniuk P., Kuzlo M., Ulianchuk-Martyniuk O. The conditions of conjugation in the tasks of moisture transfer on a thin clay inclusion taing into account salt solutions and themperature. Journal of Theoretical and Applied Mechanics, Sofia. **49** (1), 28–38 (2019).

[10] Ulianchuk-Martyniuk O. V. Numerical simulation of the effect of semi–permeable properties of clay on the value of concentration jumps of contaminants in a thin geochemical barrier. Eurasian Journal of Mathematical and Computer Applications. **8** (1), 91–104 (2020).

[11] Ulianchuk-Martyniuk O. V., Michuta O. R. Conjugation conditions in the problem of filtering chemical solutions in the case of structural changes to the material and chemical suffusion in the geobarrier. JP Journal of Heat and Mass Transfer. **19** (1), 141–154 (2020).

[12] Ulianchuk-Martyniuk O. V., Michuta O. R., Ivanchuk N. V. Finite element analysis of the diffusion model of the bioclogging of the geobarrier. Eurasian Journal Of Mathematical and Computer Applications. **9** (4), 100–114 (2021).

[13] Michuta O. R., Martyniuk P. M. Nonlinear evolutionary problem of filtration consolidation with the nonclassical conjugation condition. Journal of Optimization, Differential Equations and their Applications. **30** (1), 71–87 (2022).

[14] Kutya T. V., Martynyuk P. N. Mathematical Simulation of Humidification of Earth on a Slope and Calculation of Its Safety Factor. Journal of Engineering Physics and Thermophysics. **91**, 1189–1198 (2018).

[15] Zhang Q., Zhou Q., Ye F., Yan J., Liu N., Lou Y. Excess Pore Water Pressure and Ground Consolidation Settlement Caused by Grouting of Shield Tunnelling. Mathematical Problems in Engineering. **2022**, 1854234 (2022).

[16] Ivanchuk N., Martynyuk P., Tsvetkova T., Michuta O. Mathematical modeling and computer simulation of the filtration processes in earth dams. Eastern-European Journal of Enterprise Technologies. **2** (6), 63–69 (2017).

[17] Gerus V., Martynyuk P., Michuta O. General kinematic boundary conditionin the theory of soil filtration consolidation. Physico-mathematical modeling and informational technologies. **22**, 23–31 (2015).

[18] Herus V. A., Ivanchuk N. V., Martyniuk P. M. A System Approach to Mathematical and Computer Modeling of Geomigration Processes Using Freefem++ and Parallelization of Computations. Cybernetics and Systems Analysis. **54**, 284–294 (2018).

[19] Francisca F. M., Glatstein D. A. Long term hydraulic conductivity of compacted soils permeated with landfill leachate. Applied Clay Science. **49** (3), 187–193 (2010).

[20] Bomba A., Klymiuk Yu., Prysiazhniuk I., Prysiazhniuk O., Safonyk A. Mathematical modeling of wastewater treatment from multicomponent pollution by using microporous particles. AIP Conference Proceedings. **1773** (1), 040003 (2016).

[21] Sergienko I. V., Deineka V. S. Optimal Control of Distributed Systems with Conjugation Conditions. Springer-Verlag (2005).

[22] Sergienko I. V., Skopetskiy V. V., Deineka V. S. Mathematical modeling and research of processes in inhomogeneous environments. Naukova Dumka (1991).

Числове моделювання просідання поверхні як наслідок ущільнення грунту з тонким включенням

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Сформовано математичну модель фільтраційної консолідації неоднорідного масиву ґрунту з урахуванням зміни розмірів області в процесі ущільнення. Неоднорідність в статті розглянуто з точки зору наявності тонких включень (геобар'єрів), фізико-механічні характеристики яких відрізняються від аналогічних характеристик основного ґрунту. З математичної точки зору модель описується однофазною задачею Стефана і своєю складовою містить кінематичну граничну умову на верхній рухомій межі. Метою дослідження задачі є з'ясування впливу тонкого включення на зміну динаміки просідань поверхні ґрунту в процесі ущільнення. Зміна розмірів області розв'язання задачі фізично обумовлюється зміною об'єму пор пористого середовища в процесі розсіювання надлишкових напорів. Якщо проникність геобар'єра є низькою, то це впливає на динаміку консолідаційних процесів і, відповідно, — на значення просідань. Знайдено скінченноелементні розв'язки початково-крайової задачі для нелінійного параболічного рівняння в неоднорідній області з умовою спряження неідеального контакту. Наведено числові схеми дискретизації в часі, схема визначення зміни положення верхньої межі в дискретні моменти часу та алгоритм визначення фізико-механічних характеристик пористого середовища залежно від ступеня консолідації. Розглянуто ряд тестових прикладів та досліджено вплив тонкого включення на динаміку зміни положення верхньої межі області розв'язання задачі.

Ключові слова: *консолідація; просідання; умова спряження; метод скінченних елементів.*