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Time delay and nonlinear incidence effects on the stochastic SIRC epidemic model

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This paper presents theoretical and numerical study of a stochastic SIRC epidemic model with time delay and nonlinear incidence. The existence and uniqueness of a global positive solution is proved. The Lyapunov analysis method is used to obtain sufficient conditions for the existence of a stationary distribution and the disease extinction under certain assumptions. Numerical simulations are also elaborated for the considered stochastic model in order to corroborate the theoretical findings.

Keywords: *SIRC model; stochastic; time delay; Lyapunov function; nonlinear incidence; stationary distribution.*

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1. Introduction

Mathematical models are frequently used to study disease transmission dynamics as well as epidemiological characteristics. Based on these potential models, control and mitigation strategies for the epidemics can be developed. Actually, epidemic disease has received much attention in recent years due to its impact for humanity, which offer many deaths, problems of system health and affected economy [1, 2]. Many epidemiological models are actually expressed in system of mathematical deterministic concepts [3–8], and deterministic models have traditionally served as the foundation of mathematical epidemiology. The SIR and SEIR models have shaped much of current understanding of recurring epidemics [8–15]. The concept of SDE was successfully implemented in various field such as population systems, economics and finance, and neural networks. Recently, it was derived that mathematical models with a white noise type are mostly used to describe the universal laws.

This paper focusses on the SIRC model and in the considered SIRC model, the population is divided into four distinct classes: the susceptible $S(t)$, the infected $I(t)$, the recovered $R(t)$, and the cross-immune $C(t)$ individuals. Motivated by the above works, we conclude that combining nonlinear incidence and random environmental factors in the deterministic delayed SIRC model will be more efficient to handle physical phenomena. So, the considered stochastic SIRC epidemic model takes the following form

$$\begin{cases} \mathrm{d}S(t) = \left[\gamma(1-S(t)) - \dfrac{\beta S(t)\, I(t-\varkappa)}{\phi(I(t))} + \eta\, C(t)\right]\mathrm{d}t - \sigma_1 S(t)\,\mathrm{d}\mathcal{W}_1(t) - \sigma_5 \dfrac{S(t)\, I(t-\varkappa)}{\phi(I(t))}\,\mathrm{d}\mathcal{W}_5(t), \\ \mathrm{d}I(t) = \left[\dfrac{\beta S(t)\, I(t-\varkappa)}{\phi(I(t))} + \mu\,\beta\, C(t)\, I(t) - (\gamma+\alpha) I(t)\right]\mathrm{d}t - \sigma_2 I(t)\,\mathrm{d}\mathcal{W}_2(t) \\ \qquad\qquad + \sigma_5 \dfrac{S(t) I(t-\varkappa)}{\phi(I(t))}\mathrm{d}\mathcal{W}_5(t), \\ \mathrm{d}R(t) = [(1-\mu)\beta\, C(t)\, I(t) + \alpha\, I(t) - (\gamma+\delta) R(t)]\,\mathrm{d}t - \sigma_3 R(t)\,\mathrm{d}\mathcal{W}_3(t), \\ \mathrm{d}C(t) = [\delta R(t) - \beta\, C(t)\, I(t) - (\gamma+\eta)\, C(t)]\,\mathrm{d}t - \sigma_4 C(t)\,\mathrm{d}\mathcal{W}_4(t). \end{cases} \tag{1}$$

In the model (1), the biological significance of dynamic properties and parameters can be seen in [3, 16], which we have omitted here. The Brownian motion $\mathcal{W}_j(t)$, $j = 1, \ldots, 5$ are mutually independent standard defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$ satisfying the usual conditions, σ_j^2, $j = 1, \ldots, 5$ denote the intensities of the environmental factor. The initial conditions of the model (1) are

$$\begin{cases} S(\chi) = \Phi_1(\chi), \quad I(\chi) = \Phi_2(\chi), \\ R(\chi) = \Phi_3(\chi), \quad C(\chi) = \Phi_4(\chi), \\ \Phi_j(\chi) \geqslant 0, \quad \chi \in [-\varkappa, 0], \\ \Phi_j \in \mathcal{C}, \quad j = 1, \ldots, 4, \end{cases} \tag{2}$$

where $\mathcal{C}$ is continuous function space $\mathcal{C}\left([-\varkappa, 0]; \mathbb{R}_+^4\right)$, and $\mathbb{R}_+^4 = \{(x_1, x_2, x_3, x_4) | x_j > 0, j = 1, \ldots, 4\}$. By a biological meaning, we further assume that $\Phi_j(0) > 0$, for $i = 1, \ldots, 4$.

This model is theoretically analyzed and some theorems about global existence and uniqueness of positive solution are given in section 2. Extinction of disease in section 3, as well as the proof is provided for both the existence and uniqueness of an ergodic stationary distribution in section 4. A numerical code is elaborated for numerical solution and here after some of the obtained numerical solutions of the considered SIRC epidemic model with specific parameters.

2. Existence of the global and positive solution

In order to study dynamics of the SIRC epidemic model with delay, the first concern is whether the positive solution of the system exists globally. Within this section, proper Lyapunov function is constructed to establish the existence and uniqueness of a positive solution of the system [17].

Theorem 1. *For the given condition initial (2), there exists unique positive solution $X(t) = (S(t), I(t), R(t), C(t))$ on $[-\varkappa, -\infty)$ for the system (1) and $X(t) \in \mathbb{R}_+^4$ with probability one. In other words, for any $t \geqslant -\varkappa$, $X(t) \in \mathbb{R}_+^4$ almost surly (a.s.).*

Proof. The local Lipschitz conditions are met by the coefficients of the system (1) are locally Lipschitz, then for any initial value (2), system (1) has unique local solution $X(t)$ on $t \in [-\varkappa, \varkappa_e)$, where $\varkappa_e$ is the explosion time [17]. To show that solution is global, we only need to demonstrate it for $\varkappa_e = \infty$ a.s. Define the stopping time $\varkappa^*$ by

$$\varkappa^* = \inf\left\{t \in [\varkappa, \varkappa_e) : S(t) \leqslant 0 \text{ or } I(t) \leqslant 0 \text{ or } R(t) \leqslant 0 \text{ or } C(t) \leqslant 0\right\},$$

with the usual setting $\inf \varnothing = \infty$, where $\varnothing$ is the empty set. Obviously, $\varkappa^* \leqslant \varkappa_e$. So, if we can prove that $\varkappa^* = \infty$ a.s., thus $\varkappa_e = \infty$ and $X(t) \in \mathbb{R}_+^4$ a.s. for all $t \geqslant 0$. Suppose that $\varkappa^* < \infty$, then there exists $T > 0$ such that $P\{\varkappa^* < T\} > 0$.

Define a C^2-function $V : \mathbb{R}_+^4 \to \mathbb{R}_+$ by

$$V(X(t)) = \ln\big(S(t)\, I(t)\, R(t)\, C(t)\big) + \beta \int_t^{t+\varkappa} I(u - \varkappa)\, \mathrm{d}u. \tag{3}$$

Using Itô's lemma, we obtain,

$$\begin{aligned} \mathrm{d}V(X(t)) &= \mathcal{L}\,\mathrm{V}(X(t))\,\mathrm{d}t + \sigma_5\,(S-1)\,\frac{I(t-\varkappa)}{\phi(I)}\,\mathrm{d}\mathcal{W}_5(t) \\ &\quad - \sigma_1\,\mathrm{d}\mathcal{W}_1(t) - \sigma_2\,\mathrm{d}\mathcal{W}_2(t) - \sigma_3\,\mathrm{d}\mathcal{W}_3(t) - \sigma_4\,\mathrm{d}\mathcal{W}_4(t), \end{aligned}$$

where

$$\begin{aligned} \mathcal{L}\,\mathrm{V}(X(t)) &= \beta\, I(t) - \beta\, I(t-\varkappa) + \frac{\gamma}{S} - \gamma - \beta\,\frac{I(t-\varkappa)}{\phi(I)} + \eta\,\frac{C}{S} \\ &\quad + \beta\,\frac{S\,I(t-\varkappa)}{I\,\phi(I)} + \mu\,\beta\, C - (\gamma+\alpha) + (1-\mu)\,\beta\,\frac{C\,I}{R} + \alpha\,\frac{I}{R} - (\gamma+\delta) \\ &\quad + \delta\,\frac{R}{C} - \beta\, I - (\delta+\eta) - \frac{\sigma_5^2}{2}\,\frac{I^2(t-\varkappa)}{\phi^2(I)} - \frac{\sigma_1^2+\sigma_2^2+\sigma_3^2+\sigma_4^2}{2} - \frac{\sigma_5^2}{2}\,\frac{S^2 I^2(t-\varkappa)}{\phi^2(I)\, I^2}. \end{aligned}$$

If $X(t)$ is positive and $\phi(I) \geqslant 1$, then it implies that

$$\begin{aligned}\mathcal{L}\,\mathrm{V}(X(t)) \geqslant{}& -\beta\, I(t-\varkappa) - \beta\,\frac{I(t-\varkappa)}{\phi(I)} - 4\gamma - (\alpha+\delta+\eta)\\ &- \frac{\sigma_5^2}{2}\,\frac{I^2(t-\varkappa)}{\phi^2(I)} - \frac{\sigma_1^2+\sigma_2^2+\sigma_3^2+\sigma_4^2}{2} - \frac{\sigma_5^2}{2}\,\frac{S^2I^2(t-\varkappa)}{\phi^2(I)\,I^2}\\ :={}& \mathrm{K}(X(t)).\end{aligned}$$

So, we have

$$\begin{aligned}\mathrm{dV}(X(t)) \geqslant{}& \mathrm{K}(X(t))\,\mathrm{d}t - \sigma_1\, I(t-\varkappa)\,\mathrm{d}\mathcal{W}_1(t) + \sigma_5\,(S-1)\,\frac{I(t-\varkappa)}{\phi(I)}\,\mathrm{d}\mathcal{W}_5(t)\\ &- \sigma_1\,\mathrm{d}\mathcal{W}_1(t) - \sigma_2\,\mathrm{d}\mathcal{W}_2(t) - \sigma_3\,\mathrm{d}\mathcal{W}_3(t) - \sigma_4\,\mathrm{d}\mathcal{W}_4(t),\end{aligned}$$

then

$$\begin{aligned}\mathrm{V}(X(t)) \geqslant{}& \int_0^t \mathrm{K}(X(u))\,\mathrm{d}u + \sigma_5\int_0^t (S(u)-1)\,I(u-\varkappa)\,\mathrm{d}\mathcal{W}_5(u)\\ &+ \mathrm{V}(X(0)) - \sigma_1\mathcal{W}_1(t) - \sigma_2\mathcal{W}_2(t) - \sigma_3\mathcal{W}_3(t) - \sigma_4\mathcal{W}_4(t).\end{aligned} \tag{4}$$

Note that $X(\varkappa^*) = 0$. Therefore,

$$\lim_{t\to\varkappa^*} V(X(t)) = -\infty.$$

Letting $t \to \varkappa^*$ in (4), we have

$$\begin{aligned}-\infty \geqslant{}& \mathrm{V}(X(0)) + \int_0^{\varkappa^*} \mathrm{K}(X(u))\,\mathrm{d}u - \sigma_1\mathcal{W}_1(\varkappa^*) - \sigma_2\mathcal{W}_2(\varkappa^*) - \sigma_3\mathcal{W}_3(\varkappa^*)\\ &- \sigma_4\mathcal{W}_4(\varkappa^*) + \sigma_5\int_0^{\varkappa^*} (S(u)-1)\,I(u-\varkappa)\,\mathrm{d}\mathcal{W}_5(u) > -\infty,\end{aligned}$$

which results in a contradiction. Therefore, it can be concluded that $\varkappa^* = \infty$ a.s., thereby completing the proof of Theorem 1. ■

3. Disease extinction

Let us note by

$$\mathcal{R}_E = \frac{\beta}{\left(\gamma+\alpha+\frac{\sigma_1^2}{2}+\frac{\sigma_2^2}{2}\right)\wedge\left(\gamma+\delta+\frac{\sigma_3^2}{2}\right)\wedge\left(\gamma+\eta+\frac{\sigma_4^2}{2}\right)} = \frac{\beta}{Q}, \quad a\wedge b = \min\{a,b\}.$$

Theorem 2. *Let $(S(t), I(t), R(t), C(t))$ be the solution of the system (1) with the initial conditions (2). If $\mathcal{R}_E < 1$, then the solution of the system satisfies*

$$\lim_{t\to\infty}\sup\left(\frac{\ln(I(t)+R(t)+C(t))}{t}\right) \leqslant Q\,(\mathcal{R}_E - 1) \quad \text{a.s.}$$

Furthermore, $\lim_{t\to\infty}\sup\langle S(t)\rangle = 1$ *a.s.*

Proof. Let $Z(t) = I(t)+C(t)+R(t)$ and $M(t) = S(t)+I(t)+C(t)+R(t)$. By applying the Itô's lemma, we obtain

$$\begin{aligned}\mathrm{d}\ln Z(t) ={}& \left\{\frac{1}{I+R+C}\left[\frac{\beta SI(t-\varkappa)}{\phi(I)}\right] - \frac{1}{2(I+R+C)^2}\left[\sigma_2^2 I^2 + \sigma_5^2\left[\frac{SI(t-\varkappa)}{\phi(I)}\right]^2 + R^2\sigma_3^2 + C^2\sigma_4^2\right]\right\}\mathrm{d}t\\ &- \frac{1}{I+R+C}\left[\sigma_2\, I\,\mathrm{d}\mathcal{W}_2(t) + \frac{S\,I(t-\varkappa)}{\phi(I)}\,\mathrm{d}\mathcal{W}_5(t) + \sigma_3\, R\,\mathrm{d}\mathcal{W}_3(t) + \sigma_4\, C\,\mathrm{d}\mathcal{W}_4(t)\right]\\ \leqslant{}& \left\{\frac{\beta\, S\, I(t-\varkappa)}{\phi(I)} - \frac{1}{2(I+R+C)^2}\left[\sigma_2^2\, I^2 + \sigma_5^2\left(\frac{S\,I(t-\varkappa)}{\phi(I)}\right)^2 + R^2\sigma_3^2 + C^2\sigma_4^2\right]\right\}\mathrm{d}t\end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{I+R+C}\left[\sigma_2\, I\,\mathrm{d}\mathcal{W}_2(t)+\frac{S\,I(t-\varkappa)}{\phi(I)}\,\mathrm{d}\mathcal{W}_5(t)+\sigma_3\,R\,\mathrm{d}\mathcal{W}_3(t)+\sigma_4\,C\,\mathrm{d}\mathcal{W}_4(t)\right]\\
&\leqslant\left[\beta-\frac{\left(\sigma_2^2+\sigma_5^2\right)I^2+R^2\sigma_3^2+C^2\sigma_4^2}{2(I+R+C)^2}\right]\mathrm{d}t\\
&-\frac{1}{I+R+C}\left[\sigma_2\, I\,d\mathcal{W}_2(t)+\frac{S\,I(t-\varkappa)}{\phi(I)}\,\mathrm{d}\mathcal{W}_5(t)+\sigma_3\,R\,\mathrm{d}\mathcal{W}_3(t)+\sigma_4\,C\,\mathrm{d}\mathcal{W}_4(t)\right]\\
&\leqslant\left\{\beta-\frac{1}{(I+R+C)^2}\left[\left(\gamma+\alpha+\frac{\sigma_2^2}{2}+\frac{\sigma_5^2}{2}\right)I^2+\left(\gamma+\delta+\frac{\sigma_3^2}{2}\right)R^2+\left(\gamma+\eta+\frac{\sigma_4^2}{2}\right)C^2\right]\right\}\mathrm{d}t\\
&-\frac{1}{I+R+C}\left[\sigma_2\, I\,\mathrm{d}\mathcal{W}_2(t)+\frac{S\,I(t-\varkappa)}{\phi(I)}\,\mathrm{d}\mathcal{W}_5(t)+\sigma_3\,R\,\mathrm{d}\mathcal{W}_3(t)+\sigma_4\,C\,\mathrm{d}\mathcal{W}_4(t)\right]\\
&\leqslant\left\{\beta-\frac{1}{(I+R+C)^2}\left[\left(\gamma+\alpha+\frac{\sigma_2^2}{2}+\frac{\sigma_5^2}{2}\right)\wedge\left(\gamma+\delta+\frac{\sigma_3^2}{2}\right)\wedge\left(\gamma+\eta+\frac{\sigma_4^2}{2}\right)\right.\right.\\
&\left.\left.\times\left(I^2+R^2+C^2\right)\right]\right\}\mathrm{d}t\\
&-\frac{1}{I+R+C}\left[\sigma_2\, I\,\mathrm{d}\mathcal{W}_2(t)+\frac{S\,I(t-\varkappa)}{\phi(I)}\,\mathrm{d}\mathcal{W}_5(t)+\sigma_3\,R\,\mathrm{d}\mathcal{W}_3(t)+\sigma_4\,C\,\mathrm{d}\mathcal{W}_4(t)\right]\\
&\leqslant\left\{\beta-\left[\left(\gamma+\alpha+\frac{\sigma_2^2}{2}+\frac{\sigma_5^2}{2}\right)\wedge\left(\gamma+\delta+\frac{\sigma_3^2}{2}\right)\wedge\left(\gamma+\eta+\frac{\sigma_4^2}{2}\right)\left(I^2+R^2+C^2\right)\right]\right\}\mathrm{d}t\\
&-\frac{1}{I+R+C}\left[\sigma_2\, I\,\mathrm{d}\mathcal{W}_2(t)+\frac{S\,I(t-\varkappa)}{\phi(I)}\,\mathrm{d}\mathcal{W}_5(t)+\sigma_3\,R\,\mathrm{d}\mathcal{W}_3(t)+\sigma_4\,C\,\mathrm{d}\mathcal{W}_4(t)\right]\\
&=Q\left(\mathcal{R}_E-1\right)\mathrm{d}t\\
&-\frac{1}{I+R+C}\left[\sigma_2\, I\,\mathrm{d}\mathcal{W}_2(t)+\frac{S\,I(t-\varkappa)}{\phi(I)}\,\mathrm{d}\mathcal{W}_5(t)+\sigma_3\,R\,\mathrm{d}\mathcal{W}_3(t)+\sigma_4\,C\,\mathrm{d}\mathcal{W}_4(t)\right].
\end{aligned}
$$

By integrating the preceding inequality from 0 to t, and dividing both sides by t, we achieve the following result:

$$
\frac{\ln Z(t)}{t}\leqslant Q\left(\mathcal{R}_E-1\right)-\Psi_1(t), \tag{5}
$$

where

$$
\begin{aligned}
\Psi_1(t)=\frac{1}{t}\int_0^t&\frac{1}{I(s)+R(s)+C(s)}\\
&\times\left[\sigma_2 I(s)d\mathcal{W}_2(s)+\frac{S(s)\,I(s-\varkappa)}{\phi(I)}\,\mathrm{d}\mathcal{W}_5(s)+\sigma_3\,R(s)\,\mathrm{d}\mathcal{W}_3(s)+\sigma_4\,C(s)\,\mathrm{d}\mathcal{W}_4(s)\right].
\end{aligned}
$$

By applying the strong law of large numbers to the Brownian motion [17] we obtain that, $\lim\limits_{t\to\infty}\Psi_1(t)=0$ a.s., which indicates $\lim\limits_{t\to\infty}I(t)=R(t)=C(t)=0$ a.s., when $\mathcal{R}_E<1$.

From the model (1), we arrive at

$$
\mathrm{d}M(t)=(\eta-\eta X(t))\,\mathrm{d}t-\sigma_1\,S(t)\,\mathrm{d}\mathcal{W}_1-\sigma_2\,I(t)\,\mathrm{d}\mathcal{W}_2-\sigma_3\,R(t)\,\mathrm{d}\mathcal{W}_3-\sigma_4\,C(t)\,\mathrm{d}\mathcal{W}_4. \tag{6}
$$

Taking the integration of (6), from 0 to t, one gets

$$
\langle M(t)\rangle=1+\Psi_2(t), \tag{7}
$$

where

$$
\begin{aligned}
\Psi_2(t)=\frac{1}{\eta}\Bigg[&\frac{1}{t}M(0)-\frac{1}{t}M(t)-\frac{\sigma_1\int_0^t S(s)\,\mathrm{d}\mathcal{W}_1(s)}{t}\\
&-\frac{\sigma_2\int_0^t I(s)\,\mathrm{d}\mathcal{W}_2(s)}{t}-\frac{\sigma_3\int_0^t R(s)\,\mathrm{d}\mathcal{W}_3(s)}{t}-\frac{\sigma_4\int_0^t C(s)\,d\mathcal{W}_4(s)}{t}\Bigg].
\end{aligned}
$$

Thus, we have $\lim\limits_{t\to\infty}\Psi_2(t)=0$ a.s., then $\lim\limits_{t\to\infty}\langle S(t)\rangle=1$ a.s. This completes the proof of Theorem 2. ■

4. The existence of unique ergodic stationary distribution

Our interest is the study of the epidemic dynamical system when the disease persists and prevails in the host population. In this section, we show that the system has a stationary distribution using the Khasminskii's theory [18].

Lemma 1 (Ref. [18]). *If a bounded domain $\mathcal{D} \subset \mathbb{R}^d$ with a regular boundary Γ exists, then the Markov process $X(t)$ will has a unique ergodic stationary distribution $\pi(\cdot)$, and*
1) A positive number exists $\mathcal{K}$ such that $\sum_{i,j=1}^{d} a_{ij}(x)\,\nu_i\,\nu_j \geqslant \mathcal{K}|\nu|^2$, $x \in \mathcal{D}$, $\nu \in \mathbb{R}^d$.
2) A non-negativeC^2-function V exists such that $\mathcal{L}V$ is negative in $\mathbb{R}^d \backslash \mathcal{D}$.
Then

$$\mathcal{P}_x \left\{ \lim_{T\to\infty} \frac{1}{T} \int_0^T f(X(t))\,dt = \int_{\mathbb{R}^d} f(x)\pi(dx) \right\} = 1,$$

for all $x \in \mathbb{R}^d$, where $f(\cdot)$ is an integrable function with respect to the measure π.

The stochastic model's reproduction number is defined by:

$$\mathcal{R}_s = \frac{\gamma\beta^2\delta(1-\mu)}{\left(\gamma + \frac{\sigma_1^2}{2} + \frac{\sigma_5^2}{2}\right)\left(\gamma + \alpha + \frac{\sigma_2^2}{2} + \frac{\sigma_5^2}{2}\right)\left(\gamma + \delta + \frac{\sigma_3^2}{2}\right)\left(\gamma + \eta + \frac{\sigma_4^2}{2}\right)}.$$

Theorem 3. *If $\mathcal{R}_s > 1$, then the solution $X(t) = (S(t), I(t), R(t), C(t))$ of system (1) is ergodic and has a unique stationary distribution.*

Proof. Let $Z = S + I + R + C$. The diffusion matrix of the model (1) is given below:

$$A = \begin{pmatrix} \sigma_1^2 S^2 + \sigma_5^2 \left(\frac{SI}{\phi(I)}\right)^2 & -\sigma_5^2 \left(\frac{SI}{\phi(I)}\right)^2 & 0 & 0 \\ -\sigma_5^2 \left(\frac{SI}{\phi(I)}\right)^2 & \sigma_2^2 I^2 + \sigma_5^2 \left(\frac{SI}{\phi(I)}\right)^2 & 0 & 0 \\ 0 & 0 & \sigma_3^2 R^2 & 0 \\ 0 & 0 & 0 & \sigma_4^2 C^2 \end{pmatrix}.$$

By choosing

$$\mathcal{K} = \min_{X \in \bar{\mathcal{D}} \subset \mathbb{R}_+^4} \left\{ \sigma_1^2 S^2 + \sigma_5^2 \left(\frac{SI}{\phi(I)}\right)^2, \sigma_2^2 I^2 + \sigma_5^2 \left(\frac{SI}{\phi(I)}\right)^2, \sigma_3^2 R^2, \sigma_4^2 C^2 \right\},$$

one gets for all $X \in \bar{\mathcal{D}}$, and $\nu = (\nu_1, \nu_2, \nu_3, \nu_4) \in \mathbb{R}_+^4$ that

$$\begin{aligned} \sum_{i,j=1}^{4} a_{ij}(X)\,\nu_i\,\nu_j &= \left(\sigma_1^2 S^2 + \sigma_5^2 \left(\frac{SI}{\phi(I)}\right)^2\right)\nu_1^2 + \left(\sigma_2^2 I^2 + \sigma_5^2 \left(\frac{SI}{\phi(I)}\right)^2\right)\nu_2^2 \\ &\quad + \sigma_3^2 R^2 \nu_3^2 + \sigma_4^2 C^2 \nu_4^2 - 2\sigma_5^2 \left(\frac{SI}{\phi(I)}\right)^2 \nu_1\,\nu_2 \\ &\geqslant \mathcal{K}|\nu|^2, \end{aligned}$$

Thus, the first condition of Lemma 1 is satisfied.

Let us $\mathcal{C}^2$-function $V\colon \mathbb{R}_+^4 \to \mathbb{R}$ is in the following form

$$\begin{aligned} V(X) &= M\left(-c_1 \ln S - c_2 \ln I - c_3 \ln R - c_4 \ln C + \beta \int_t^{t+\varkappa} I(t-\varkappa)\,\mathrm{d}s\right) \\ &\quad + \frac{1}{\theta+1}(S+I+R+C)^{\theta+1} - \ln S - \ln I + \beta \int_t^{t+\varkappa} I(t-\varkappa)\,\mathrm{d}s - \ln R - \ln C \\ &= M\Upsilon_1 + \Upsilon_2 + \Upsilon_3 + \Upsilon_4 + \Upsilon_5 + \Upsilon_6, \end{aligned}$$

where c_j, $j = 1, \ldots, 4$ are positive constants that will be specified later. θ is a constant satisfying $1 < \theta < \frac{2\gamma}{\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2 \vee 2\sigma_5^2}$, $a \vee b = \max\{a, b\}$, and $M > 0$ satisfies

$$-M\lambda + G \leqslant -2, \tag{8}$$

where $\lambda = 4\gamma\big[\mathcal{R}_s^{\frac{1}{4}} - 1\big] > 0$, and

$$G = \sup_{X\in\mathbb{R}_+^4} \left\{ -\frac{1}{2}\left[\gamma - \frac{\theta}{2}\left(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2 \vee 2\sigma_5^2\right)\right] I^{\theta+1} \right.$$
$$\left. +4\gamma + 2\beta I + \alpha + \delta + \eta + B + \frac{\sigma_1^2+\sigma_2^2+\sigma_3^2+\sigma_4^2}{2} + \frac{\sigma_5^2}{2}\frac{I^2+S^2}{\phi^2(I)} \right\}. \tag{9}$$

and

$$B = \sup_{X\in\mathbb{R}_+^4} \left\{ \gamma(S+I+R+C)^\theta - \frac{1}{2}\left[\gamma - \frac{\theta}{2}\left(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2 \vee 2\sigma_5^2\right)\right] \times (S+I+R+C)^{\theta+1} \right\} < \infty. \tag{10}$$

It can be easily verified that

$$\liminf_{k\to\infty,\, X\in\mathbb{R}_+^4\setminus\mathcal{D}_k} V(X) = \infty,$$

where $\mathcal{D}_k = \left(\frac{1}{k},k\right) \times \left(\frac{1}{k},k\right) \times \left(\frac{1}{k},k\right) \times \left(\frac{1}{k},k\right)$. Furthermore, $V(X)$ is a continuous function, that must have a minimum point $V(X(0))$ in the interior of $\mathbb{R}_+^4$. Then, the $\mathcal{C}^2$-function $\mathcal{V}\colon \mathbb{R}_+^4 \to \mathbb{R}_+$ is non-negative and defined as follows

$$\begin{aligned} \mathcal{V}(X) &= V(X) - V(X(0)) \\ &= M\Upsilon_1 + \Upsilon_2 + \Upsilon_3 + \Upsilon_4 + \Upsilon_5 + \Upsilon_6 - V(X(0)), \end{aligned} \tag{11}$$

where

$$\begin{aligned} \Upsilon_1 &= -c_1 \ln S - c_2 \ln I - c_3 \ln R - c_4 \ln C + \beta \int_t^{t+\varkappa} I(s-\varkappa)\,\mathrm{d}s, \\ \Upsilon_2 &= \frac{1}{\theta+1}(S+I+R+C)^{\theta+1} = \frac{Z^{\theta+1}}{\theta+1}, \quad \Upsilon_3 = -\ln S, \\ \Upsilon_4 &= -\ln I + \beta\int_t^{t+\varkappa} I(s-\varkappa)\,\mathrm{d}s, \quad \Upsilon_5 = -\ln R, \quad \Upsilon_6 = -\ln C, \end{aligned}$$

The application of Itô's lemma to various Υ_j $(j=1,2,\dots,6)$ provides that

$$\begin{aligned} \mathcal{L}\Upsilon_1 &= -c_1\frac{\gamma}{S} + c_1\gamma + c_1\beta\frac{I(t-\varkappa)}{\phi(I)} - c_1\eta\frac{C}{S} + c_1\frac{\sigma_1^2}{2} + c_1\frac{\sigma_5^2}{2}\left(\frac{I(t-\varkappa)}{\phi(I)}\right)^2 \\ &\quad - c_2\beta\frac{SI(t-\varkappa)}{\phi(I)I} - c_2\mu\beta C - c_2(\gamma+\alpha) + c_2\frac{\sigma_2^2}{2} + c_2\frac{\sigma_5^2}{2}\left(\frac{SI(t-\varkappa)}{\phi(I)I}\right)^2 \\ &\quad - c_3(1-\mu)\beta\frac{CI}{R} - c_3\alpha\frac{I}{R} + c_3(\gamma+\delta) + c_3\frac{\sigma_3^2}{2} \\ &\quad - c_4\delta\frac{R}{C} + c_4\beta I + c_4(\gamma+\eta) + c_4\frac{\sigma_4^2}{2} \\ &\leqslant -\left(c_1\frac{\gamma}{S} + c_2\beta\frac{S}{\phi(I)} + c_3(1-\mu)\beta\frac{CI}{R} + c_4\delta\frac{R}{C}\right) + c_1\left(\gamma + \frac{\sigma_1^2}{2} + \frac{\sigma_5^2}{2}\right) \\ &\quad + c_2\left(\gamma + \frac{\sigma_2^2}{2} + \frac{\sigma_5^2}{2}\right) + c_3\left(\gamma+\delta+\frac{\sigma_3^2}{2}\right) + c_4\left(\gamma+\eta+\frac{\sigma_4^2}{2}\right) + \beta\left(c_1+c_4\right)I, \\ &\leqslant -4\big[c_1c_2c_3c_4\gamma\beta^2\delta\left(1-\mu\right)\big]^{\frac{1}{4}} + \beta\left(c_1+c_4\right)I + c_1\left(\gamma+\frac{\sigma_1^2}{2}+\frac{\sigma_5^2}{2}\right) \\ &\quad + c_2\left(\gamma + \frac{\sigma_2^2}{2} + \frac{\sigma_5^2}{2}\right) + c_3\left(\gamma+\delta+\frac{\sigma_3^2}{2}\right) + c_4\left(\gamma+\eta+\frac{\sigma_4^2}{2}\right). \end{aligned}$$

Now, taking c_j $(j=1,\dots,4)$ such that

$$c_1\left(\gamma+\frac{\sigma_1^2}{2}+\frac{\sigma_5^2}{2}\right) = c_2\left(\gamma+\frac{\sigma_2^2}{2}+\frac{\sigma_5^2}{2}\right) = c_3\left(\gamma+\delta+\frac{\sigma_3^2}{2}\right) = c_4\left(\gamma+\eta+\frac{\sigma_4^2}{2}\right) = \gamma,$$

then

$$
\begin{aligned}
\mathcal{L}\Upsilon_1 &\leqslant -4\gamma\left\{\left[\frac{\gamma\beta^2\delta\,(1-\mu)}{\left(\gamma+\frac{\sigma_1^2}{2}+\frac{\sigma_5^2}{2}\right)\left(\gamma+\frac{\sigma_2^2}{2}+\frac{\sigma_5^2}{2}\right)\left(\gamma+\delta+\frac{\sigma_3^2}{2}\right)\left(\gamma+\eta+\frac{\sigma_4^2}{2}\right)}\right]^{\frac{1}{4}}-1\right\}+\beta\,(c_1+c_4)\,I \\
&= -\lambda+\beta\,(c_1+c4)\,I; \qquad (12) \\
\mathcal{L}\Upsilon_2 &= Z^\theta[\gamma-\gamma Z]+\frac{\theta}{2}Z^{\theta-1}\left[\sigma_1^2S^2+\sigma_2^2I^2+\sigma_3^2R^2+\sigma_4^2C^2+2\sigma_5^2\left(\frac{SI}{\phi(I)}\right)^2\right] \\
&\leqslant Z^\theta[\gamma-\gamma Z]+\frac{\theta}{2}Z^{\theta+1}\left(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2\vee\sigma_4^2\vee 2\sigma_5^2\right) \\
&\leqslant \gamma Z^\theta - Z^{\theta+1}\left[\eta-\frac{\theta}{2}\left(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2\vee\sigma_4^2\vee 2\sigma_5^2\right)\right] \\
&\leqslant B-\frac{1}{2}\left[\gamma-\frac{\theta}{2}\left(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2\vee\sigma_4^2\vee 2\sigma_5^2\right)\right]Z^{\theta+1} \\
&\leqslant B-\frac{1}{2}\left[\gamma-\frac{\theta}{2}\left(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2\vee\sigma_4^2\vee 2\sigma_5^2\right)\right]\left(S^{\theta+1}+I^{\theta+1}+R^{\theta+1}+C^{\theta+1}\right), \qquad (13)
\end{aligned}
$$

where B is defined by (10).

$$\mathcal{L}\Upsilon_3 = -\frac{\gamma}{S}+\gamma+\beta\frac{I(t-\varkappa)}{\phi(I)}-\eta\frac{C}{S}+\frac{\sigma_1^2}{2}+\frac{\sigma_5^2}{2}\left(\frac{I(t-\varkappa)}{\phi(I)}\right)^2 \qquad (14)$$

$$\mathcal{L}\Upsilon_4 = -\beta\frac{SI(t-\varkappa)}{\phi(I)I}-\mu\beta C-(\gamma+\alpha)+\frac{\sigma_2^2}{2}+\frac{\sigma_5^2}{2}\left(\frac{SI(t-\varkappa)}{\phi(I)I}\right)^2 \qquad (15)$$

$$\mathcal{L}\Upsilon_5 = -\,(1-\mu)\,\beta\frac{CI}{R}-\alpha\frac{I}{R}+(\gamma+\delta)+\frac{\sigma_3^2}{2} \qquad (16)$$

$$\mathcal{L}\Upsilon_6 = -\delta\frac{R}{C}+\beta I+(\gamma+\eta)+\frac{\sigma_4^2}{2} \qquad (17)$$

From (14)–(17) one gets:

$$
\begin{aligned}
\mathcal{L}\mathcal{V} &\leqslant -M\lambda+M\beta\,(c_1+c_4)\,I-\frac{1}{2}\left[\gamma-\frac{\theta}{2}\left(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2\vee\sigma_4^2\vee 2\sigma_5^2\right)\right]\left(S^{\theta+1}+I^{\theta+1}+R^{\theta+1}+C^{\theta+1}\right) \\
&\quad -\frac{\gamma}{S}+4\gamma-\eta\frac{C}{S}-\beta\frac{SI(t-\varkappa)}{\phi(I)}-\mu BC-(1-\mu)\,\beta\frac{CI}{R}-\alpha\frac{I}{R} \\
&\quad -\gamma\frac{R}{C}+2\beta I+\alpha+\delta+\eta+\frac{\sigma_1^2+\sigma_2^2+\sigma_3^2+\sigma_4^2}{2}+\frac{\sigma_5^2}{2}\frac{I^2+S^2}{\phi^2(I)}.
\end{aligned}
$$

The bounded close set is defined by

$$\mathcal{D}=\left\{X\in\mathbb{R}_+^4 : \varepsilon_1\leqslant S\leqslant\frac{1}{\varepsilon_1},\varepsilon_2\leqslant I\leqslant\frac{1}{\varepsilon_2},\varepsilon_3\leqslant R\leqslant\frac{1}{\varepsilon_3},\varepsilon_4\leqslant C\leqslant\frac{1}{\varepsilon_4}\right\},$$

where $\varepsilon_j>0$ $(j=1,2,3,4)$ are sufficiently small constants satisfying the following conditions

$$\max\left\{-\frac{\gamma}{\varepsilon_1},-\alpha\frac{\varepsilon_2}{\varepsilon_3},-\gamma\frac{\varepsilon_3}{\varepsilon_4}\right\}+F\leqslant -1, \qquad (18)$$

$$-M\lambda+M\beta\,(c_1+c_4)\,\varepsilon_2+G\leqslant -1, \qquad (19)$$

$$-\frac{1}{4}\left[\gamma-\frac{\theta}{2}\left(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2\vee\sigma_4^2\vee 2\sigma_5^2\right)\right]\frac{1}{\varepsilon_1^{\theta+1}}+H\leqslant -1, \qquad (20)$$

$$-\frac{1}{4}\left[\gamma-\frac{\theta}{2}\left(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2\vee\sigma_4^2\vee 2\sigma_5^2\right)\right]\frac{1}{\varepsilon_2^{\theta+1}}+J\leqslant -1, \qquad (21)$$

$$-\frac{1}{4}\left[\gamma-\frac{\theta}{2}\left(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2\vee\sigma_4^2\vee 2\sigma_5^2\right)\right]\frac{1}{\varepsilon_3^{\theta+1}}+K\leqslant -1, \qquad (22)$$

$$-\frac{1}{4}\left[\gamma-\frac{\theta}{2}\left(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2\vee\sigma_4^2\vee 2\sigma_5^2\right)\right]\frac{1}{\varepsilon_4^{\theta+1}}+L\leqslant -1, \qquad (23)$$

where

$$F=\sup_{X\in\mathbb{R}^4_+}\left\{M\beta\left(c_1+c_4\right)I+B-\frac{1}{2}\left[\gamma-\frac{\theta}{2}\left(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2\vee\sigma_4^2\vee 2\sigma_5^2\right)\right]\left(S^{\theta+1}+I^{\theta+1}+R^{\theta+1}+C^{\theta+1}\right)\right.$$
$$\left.+4\gamma+2\beta I+\alpha+\delta+\eta+\frac{\sigma_1^2+\sigma_2^2+\sigma_3^2+\sigma_4^2}{2}+\frac{\sigma_5^2}{2}\frac{I^2+S^2}{\phi^2(I)}\right\},$$

$$H=\sup_{X\in\mathbb{R}^4_+}\left\{M\beta\left(c_1+c_4\right)I+B-\frac{1}{4}\left[\gamma-\frac{\theta}{2}\left(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2\vee\sigma_4^2\vee 2\sigma_5^2\right)\right]S^{\theta+1}\right.$$
$$-\frac{1}{2}\left[\gamma-\frac{\theta}{2}\left(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2\vee\sigma_4^2\vee 2\sigma_5^2\right)\right]\left(I^{\theta+1}+R^{\theta+1}+C^{\theta+1}\right)$$
$$\left.+4\gamma+2\beta I+\alpha+\delta+\eta+\frac{\sigma_1^2+\sigma_2^2+\sigma_3^2+\sigma_4^2}{2}+\frac{\sigma_5^2}{2}\frac{I^2+S^2}{\phi^2(I)}\right\},$$

$$J=\sup_{X\in\mathbb{R}^4_+}\left\{M\beta\left(c_1+c_4\right)I+B-\frac{1}{4}\left[\gamma-\frac{\theta}{2}\left(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2\vee\sigma_4^2\vee 2\sigma_5^2\right)\right]I^{\theta+1}\right.$$
$$-\frac{1}{2}\left[\gamma-\frac{\theta}{2}\left(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2\vee\sigma_4^2\vee 2\sigma_5^2\right)\right]\left(S^{\theta+1}+R^{\theta+1}+C^{\theta+1}\right)$$
$$\left.+4\gamma+2\beta I+\alpha+\delta+\eta+\frac{\sigma_1^2+\sigma_2^2+\sigma_3^2+\sigma_4^2}{2}+\frac{\sigma_5^2}{2}\frac{I^2+S^2}{\phi^2(I)}\right\},$$

$$K=\sup_{X\in\mathbb{R}^4_+}\left\{M\beta\left(c_1+c_4\right)I+B-\frac{1}{4}\left[\gamma-\frac{\theta}{2}\left(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2\vee\sigma_4^2\vee 2\sigma_5^2\right)\right]R^{\theta+1}\right.$$
$$-\frac{1}{2}\left[\gamma-\frac{\theta}{2}\left(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2\vee\sigma_4^2\vee 2\sigma_5^2\right)\right]\left(S^{\theta+1}+I^{\theta+1}+C^{\theta+1}\right)$$
$$\left.+4\gamma+2\beta I+\alpha+\delta+\eta+\frac{\sigma_1^2+\sigma_2^2+\sigma_3^2+\sigma_4^2}{2}+\frac{\sigma_5^2}{2}\frac{I^2+S^2}{\phi^2(I)}\right\},$$

$$L=\sup_{X\in\mathbb{R}^4_+}\left\{M\beta\left(c_1+c_4\right)I+B-\frac{1}{4}\left[\gamma-\frac{\theta}{2}\left(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2\vee\sigma_4^2\vee 2\sigma_5^2\right)\right]C^{\theta+1}\right.$$
$$-\frac{1}{2}\left[\gamma-\frac{\theta}{2}\left(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2\vee\sigma_4^2\vee 2\sigma_5^2\right)\right]\left(S^{\theta+1}+I^{\theta+1}+R^{\theta+1}\right)$$
$$\left.+4\gamma+2\beta I+\alpha+\delta+\eta+\frac{\sigma_1^2+\sigma_2^2+\sigma_3^2+\sigma_4^2}{2}+\frac{\sigma_5^2}{2}\frac{I^2+S^2}{\phi^2(I)}\right\},$$

To complete the proof, we must demonstrate that $\mathcal{LV}\leqslant -1$ for any $X\in\mathbb{R}^4_+\backslash\mathcal{D}=\mathbb{R}^4_+\setminus\cup_{i=1}^8\mathcal{D}_i$, where

$$\mathcal{D}_1=\left\{X\in\mathbb{R}^4_+\colon 0<S<\varepsilon_1\right\},\qquad \mathcal{D}_2=\left\{X\in\mathbb{R}^4_+\colon 0<I<\varepsilon_2\right\},$$
$$\mathcal{D}_3=\left\{X\in\mathbb{R}^4_+\colon 0<R<\varepsilon_3, I\geqslant\varepsilon_2\right\},\qquad \mathcal{D}_4=\left\{X\in\mathbb{R}^4_+\colon 0<C<\varepsilon_4, R\geqslant\varepsilon_3\right\},$$
$$\mathcal{D}_5=\left\{X\in\mathbb{R}^4_+\colon S>\frac{1}{\varepsilon_1}\right\},\qquad \mathcal{D}_6=\left\{X\in\mathbb{R}^4_+\colon I>\frac{1}{\varepsilon_2}\right\},$$
$$\mathcal{D}_7=\left\{X\in\mathbb{R}^4_+\colon R>\frac{1}{\varepsilon_3}\right\},\qquad \mathcal{D}_8=\left\{X\in\mathbb{R}^4_+\colon C>\frac{1}{\varepsilon_4}\right\}.$$

Case 1: For all $X\in\mathcal{D}_1$,

$$\mathcal{LV}\leqslant-\frac{\gamma}{S}-\frac{1}{2}\left[\gamma-\frac{\theta}{2}\left(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2\vee\sigma_4^2\vee 2\sigma_5^2\right)\right]\left(S^{\theta+1}+I^{\theta+1}+R^{\theta+1}+C^{\theta+1}\right)$$
$$+M\beta\left(c_1+c_4\right)I+B+4\gamma+2\beta I+\alpha+\delta+\eta+\frac{\sigma_1^2+\sigma_2^2+\sigma_3^2+\sigma_4^2}{2}+\frac{\sigma_5^2}{2}\frac{I^2+S^2}{\phi^2(I)}$$
$$\leqslant-\frac{\gamma}{S}+F$$
$$\leqslant-\frac{\gamma}{\varepsilon_1}+F\leqslant -1,$$

which is obtained from (18). Therefore, $\mathcal{LV}\leqslant -1$ for any $X\in\mathcal{D}_1$.

Case 2: For all $X \in \mathcal{D}_2$,

$$\begin{aligned}\mathcal{L}\mathcal{V} &\leqslant -M\lambda + M\beta\left(c_1+c_4\right)I - \frac{1}{2}\left[\gamma - \frac{\theta}{2}\left(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2 \vee 2\sigma_5^2\right)\right]I^{\theta+1} \\ &\quad + B + 4\gamma + 2\beta I + \alpha + \delta + \eta + \frac{\sigma_1^2+\sigma_2^2+\sigma_3^2+\sigma_4^2}{2} + \frac{\sigma_5^2}{2}\frac{I^2+S^2}{\phi^2(I)} \\ &\leqslant -M\lambda + M\beta\left(c_1+c_4\right)I + G \\ &\leqslant -M\lambda + M\beta\left(c_1+c_4\right)\varepsilon_2 + G \leqslant -1,\end{aligned}$$

which is obtained from (19). Then, $\mathcal{L}\mathcal{V} \leqslant -1$ for any $X \in \mathcal{D}_2$.

Case 3: For all $X \in \mathcal{D}_3$,

$$\begin{aligned}\mathcal{L}\mathcal{V} &\leqslant -\alpha\frac{I}{R} - \frac{1}{2}\left[\gamma - \frac{\theta}{2}\left(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2 \vee 2\sigma_5^2\right)\right]\left(S^{\theta+1}+I^{\theta+1}+R^{\theta+1}+C^{\theta+1}\right) \\ &\quad + M\beta\left(c_1+c_4\right)I + B + 4\gamma + 2\beta I + \alpha + \delta + \eta + \frac{\sigma_1^2+\sigma_2^2+\sigma_3^2+\sigma_4^2}{2} + \frac{\sigma_5^2}{2}\frac{I^2+S^2}{\phi^2(I)} \\ &\leqslant -\alpha\frac{I}{R} + F \\ &\leqslant -\alpha\frac{\varepsilon_2}{\varepsilon_3} + F \leqslant -1,\end{aligned}$$

which is obtained from (18). Thus, $\mathcal{L}\mathcal{V} \leqslant -1$ for any $X \in \mathcal{D}_3$.

Case 4: For all $X \in \mathcal{D}_4$,

$$\begin{aligned}\mathcal{L}\mathcal{V} &\leqslant -\gamma\frac{R}{C} - \frac{1}{2}\left[\gamma - \frac{\theta}{2}\left(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2 \vee 2\sigma_5^2\right)\right]\left(S^{\theta+1}+I^{\theta+1}+R^{\theta+1}+C^{\theta+1}\right) \\ &\quad + M\beta\left(c_1+c_4\right)I + B + 4\gamma + 2\beta I + \alpha + \delta + \eta + \frac{\sigma_1^2+\sigma_2^2+\sigma_3^2+\sigma_4^2}{2} + \frac{\sigma_5^2}{2}\frac{I^2+S^2}{\phi^2(I)} \\ &\leqslant -\gamma\frac{R}{C} + F \\ &\leqslant -\gamma\frac{\varepsilon_3}{\varepsilon_4} + F \leqslant -1,\end{aligned}$$

which is obtained from (18). So that, $\mathcal{L}\mathcal{V} \leqslant -1$ for any $X \in \mathcal{D}_4$.

Case 5: For all $X \in \mathcal{D}_5$,

$$\begin{aligned}\mathcal{L}\mathcal{V} &\leqslant -\frac{1}{4}\left[\gamma - \frac{\theta}{2}\left(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2 \vee 2\sigma_5^2\right)\right]S^{\theta+1} + M\beta\left(c_1+c_4\right)I \\ &\quad - \frac{1}{4}\left[\gamma - \frac{\theta}{2}\left(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2 \vee 2\sigma_5^2\right)\right]\left(I^{\theta+1}+R^{\theta+1}+C^{\theta+1}\right) \\ &\quad + B + 4\gamma + 2\beta I + \alpha + \delta + \eta + \frac{\sigma_1^2+\sigma_2^2+\sigma_3^2+\sigma_4^2}{2} + \frac{\sigma_5^2}{2}\frac{I^2+S^2}{\phi^2(I)} \\ &\leqslant -\frac{1}{4}\left[\gamma - \frac{\theta}{2}\left(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2 \vee 2\sigma_5^2\right)\right]S^{\theta+1} + H \\ &\leqslant -\frac{1}{4}\left[\gamma - \frac{\theta}{2}\left(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2 \vee 2\sigma_5^2\right)\right]\frac{1}{\varepsilon_1^{\theta+1}} + H \leqslant -1,\end{aligned}$$

which is obtained from (20). Thereby, $\mathcal{L}\mathcal{V} \leqslant -1$ for any $X \in \mathcal{D}_5$.

Case 6: For all $X \in \mathcal{D}_6$,

$$\begin{aligned}\mathcal{L}\mathcal{V} &\leqslant -\frac{1}{4}\left[\gamma - \frac{\theta}{2}\left(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2 \vee 2\sigma_5^2\right)\right]I^{\theta+1} + M\beta\left(c_1+c_4\right)I \\ &\quad - \frac{1}{4}\left[\gamma - \frac{\theta}{2}\left(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2 \vee 2\sigma_5^2\right)\right]\left(S^{\theta+1}+R^{\theta+1}+C^{\theta+1}\right) \\ &\quad + B + 4\gamma + 2\beta I + \alpha + \delta + \eta + \frac{\sigma_1^2+\sigma_2^2+\sigma_3^2+\sigma_4^2}{2} + \frac{\sigma_5^2}{2}\frac{I^2+S^2}{\phi^2(I)}\end{aligned}$$

$$\leqslant -\frac{1}{4}\left[\gamma - \frac{\theta}{2}\left(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2 \vee 2\sigma_5^2\right)\right] I^{\theta+1} + J$$
$$\leqslant -\frac{1}{4}\left[\gamma - \frac{\theta}{2}\left(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2 \vee 2\sigma_5^2\right)\right] \frac{1}{\varepsilon_2^{\theta+1}} + J \leqslant -1,$$

which is obtained from (21). Therefore, $\mathcal{LV} \leqslant -1$ for any $X \in \mathcal{D}_6$.

Case 7: For all $X \in \mathcal{D}_7$,

$$\begin{aligned}\mathcal{LV} &\leqslant -\frac{1}{4}\left[\gamma - \frac{\theta}{2}\left(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2 \vee 2\sigma_5^2\right)\right] R^{\theta+1} + M\beta\left(c_1 + c_4\right) I \\ &\quad - \frac{1}{4}\left[\gamma - \frac{\theta}{2}\left(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2 \vee 2\sigma_5^2\right)\right] \left(S^{\theta+1} + I^{\theta+1} + C^{\theta+1}\right) \\ &\quad + B + 4\gamma + 2\beta I + \alpha + \delta + \eta + \frac{\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \sigma_4^2}{2} + \frac{\sigma_5^2}{2}\frac{I^2 + S^2}{\phi^2(I)} \\ &\leqslant -\frac{1}{4}\left[\gamma - \frac{\theta}{2}\left(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2 \vee 2\sigma_5^2\right)\right] R^{\theta+1} + K \\ &\leqslant -\frac{1}{4}\left[\gamma - \frac{\theta}{2}\left(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2 \vee 2\sigma_5^2\right)\right] \frac{1}{\varepsilon_3^{\theta+1}} + K \leqslant -1,\end{aligned}$$

which is obtained from (22). Then, $\mathcal{LV} \leqslant -1$ for any $X \in \mathcal{D}_7$.

Case 8: For all $X \in \mathcal{D}_8$,

$$\begin{aligned}\mathcal{LV} &\leqslant -\frac{1}{4}\left[\gamma - \frac{\theta}{2}\left(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2 \vee 2\sigma_5^2\right)\right] C^{\theta+1} + M\beta\left(c_1 + c_4\right) I \\ &\quad - \frac{1}{4}\left[\gamma - \frac{\theta}{2}\left(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2 \vee 2\sigma_5^2\right)\right] \left(S^{\theta+1} + I^{\theta+1} + R^{\theta+1}\right) \\ &\quad + B + 4\gamma + 2\beta I + \alpha + \delta + \eta + \frac{\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \sigma_4^2}{2} + \frac{\sigma_5^2}{2}\frac{I^2 + S^2}{\phi^2(I)} \\ &\leqslant -\frac{1}{4}\left[\gamma - \frac{\theta}{2}\left(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2 \vee 2\sigma_5^2\right)\right] C^{\theta+1} + L \\ &\leqslant -\frac{1}{4}\left[\gamma - \frac{\theta}{2}\left(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2 \vee 2\sigma_5^2\right)\right] \frac{1}{\varepsilon_4^{\theta+1}} + L \leqslant -1,\end{aligned}$$

which is obtained from (23). So, $\mathcal{LV} \leqslant -1$ for any $X \in \mathcal{D}_8$.

Obviously, the second condition of Lemma 1 is satisfied. The system (1) is then ergodic and has a unique stationary distribution. ■

5. Numerical results and discussion

This section is devoted to numerical simulations of the considered stochastic SIRC epidemic model in order to use the theoretical results obtained above and to give an insight in the understanding of the real-world dynamics. The numerical solution of the system (1) is simulated by using the Euler–Maruyama method [19, 20]. By discretizing the time interval into 100 equidistant time steps, we simulate the system (1) with specified parameters, initial conditions and that incidence function $\Phi(I) = 1 + I^2$.

The sample paths (solid lines) of Figure 1 present the dynamics of susceptible $S(t)$, infected $I(t)$, recovered $R(t)$ and cross-immune $C(t)$, which show that if the threshold $R_E < 1$, then the epidemic disease will go to extinct, and if the value $R_s > 1$, then the stochastic model has a unique ergodic stationary distribution in Figure 2. The corresponding mean of each group (dashed lines) is obtained by using Monte Carlo method with 10000 solutions of the considered stochastic SIRC in the case of disease extinction and stationary distribution.

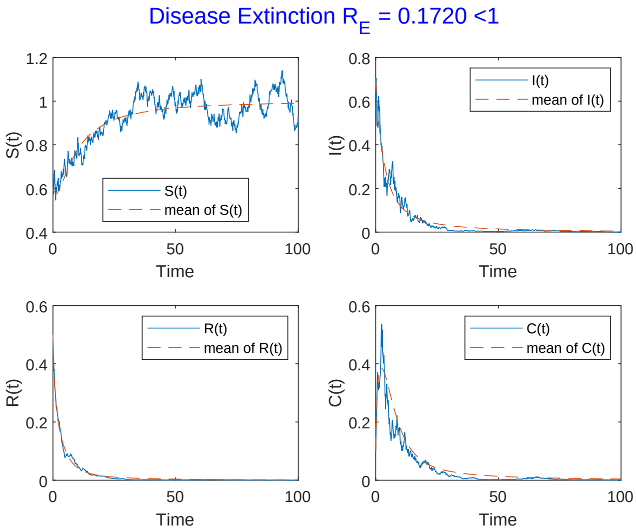

Fig. 1. Path solution and the corresponding mean of 10000 solutions of the stochastic SIRC model in the case of disease extinction.

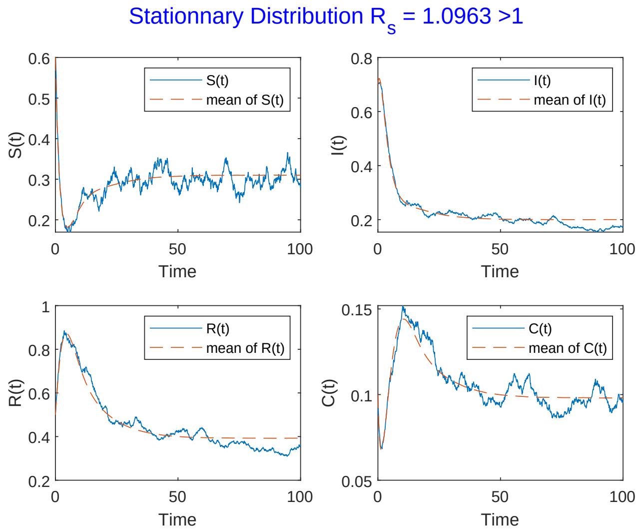

Fig. 2. Path solution and the corresponding mean of 10000 solutions of the stochastic SIRC model in the case of stationary distribution.

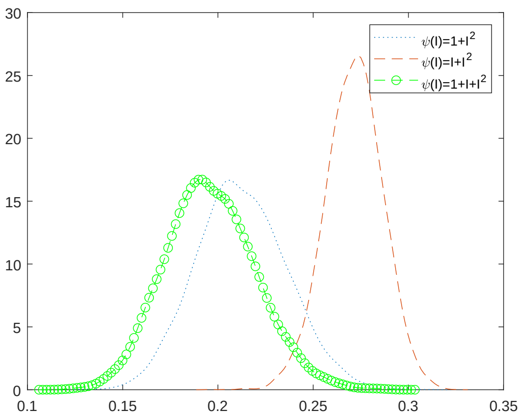

Fig. 3. The Kdensity based on 10000 simulations for Infected population at time $t = 100$ with different incidence functions.

In Figure 3, we compare the Kdensity of infected individuals $I(t)$ by using three different nonlinear functions.

6. Conclusion

Based on the deterministic analysis of SIRC epidemic model, the stochastic version is improved by incorporating white noise type of stochastic perturbation factors. We also combined time delay and nonlinear incidence function $\frac{\beta S(t)\,I(t-\varkappa)}{\phi(I(t))}$ to describe the mechanisms of transmission of the disease.

The considered stochastic SIRC epidemic model with time delay and nonlinear incidence is investigated theoretically and numerically. The Lyaponov analysis method is used to prove the existence and uniqueness of the solution. Moreover, the sufficient conditions for extinction and the existence of stationary distribution are obtained. Numerical procedures are also elaborated allowing to study the stochastic dynamic SIRC behavior. The effects of time delay, nonlinear incidence and intensity of withe noise can be analyzed.

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Вплив часової затримки та нелінійної захворюваності на стохастичну модель епідемії SIRC

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У цій статті подано теоретичні та чисельні дослідження стохастичної моделі епідемії SIRC із часовою затримкою та нелінійною частотою. Доведено існування та єдиність глобального додатного розв'язку. Використано метод аналізу Ляпунова для отримання достатніх умов існування стаціонарного розподілу та зникнення хвороби за певних припущень. Також здійснено чисельне моделювання для розглянутої стохастичної моделі з метою підтвердження теоретичних висновків.

Ключові слова: *модель SIRC; стохастичний; часова затримка; функція Ляпунова; нелінійна захворюваність; стаціонарний розподіл.*