УДК 519.17

Z. Kokosiński
Cracow University of Technology, Department of Automatic Control and Information Technology,
Faculty of Electrical and Computer Engineering, Kraków, Poland

PARALLEL METAHEURISTICS IN GRAPH COLORING

In this survey paper applications of parallel metaheuristics to solving graph coloring problems are described. The Graph Coloring Problem (GCP), Graph Coloring Sum Problem (GCSP) and Robust Graph Coloring Problem (RGCP) are known to be NP-complete. They do not have any polynomial algorithms. Therefore, a number of approximation, iterative and hybrid algorithms was developed for their solving. Recently a number of parallel algorithms was proposed for GCP and related coloring problems, including parallel metaheuristics like Parallel Genetic Algorithm (PGA), Parallel Tabu Search (PTS), Parallel Simulated Annealing (PSA) etc. DIMACS benchmarks as well as random graphs were used for their experimental verification. The results obtained for GCSP contributed to finding better lower and upper bounds on chromatic sum and chromatic sum number for many DIMACS graph instances, outperforming results known from the literature. The reported data support a conclusion, that parallel metaheuristics can be used efficiently for approximate solving of many graph coloring problems and for finding better upper bounds of many hard-to-compute graph parameters.

Key words: graph coloring, graph coloring sum, robust graph coloring, parallel metaheuristics, parallel iterative algorithms, chromatic sum, chromatic sum number.

Наведено огляд застосувань паралельних метаевристик для вирішення проблем колоризації графів. Проблеми колоризації графів (GCP), сумарної колоризації графів (GCSP) та робастної колоризації графів (RGCP) є NP-повними і не мають поліноміальних алгоритмів. З цієї причини для різних варіантів основної проблеми колоризації графів розроблено багато наближених алгоритмів, ітераційних і гібридних. Останнім часом для задачі колоризації графів і подібних їй проблем були розроблені паралельні алгоритми, зокрема паралельні метаевристики, зокрема паралельний алгоритм табу пошуку (PTS), паралельний генетичний алгоритм (PGA) і паралельний алгоритм імітації відпалу (PSA). В експериментальній перевірці алгоритмів використано графи зі сховищем DIMACS, а також випадкові графи. Дослідження застосування PGA для задач сумарної колоризації спричинило визначення нових верхніх і нижніх оцінок хроматичної суми і числа хроматичної суми для класу тестів з бази DIMACS, які є точнішими від відомих теоретичних оцінок. Отримані результати підтверджують думку, що паралельні метаевристики можуть стати потужним інструментом для наближеного розв'язування задач колоризації графів у практичних застосуваннях, а також для експериментального визначення верхньої оцінки обраних параметрів важко обчислювальних графів.

Ключові слова: колоризація графів, сумарна колоризація, робастна колоризація, паралельна метаевристика, паралельний ітераційний алгоритм, хроматична сума, число хроматичної суми.

Introduction

Graph coloring is a popular mathematical model for solving combinatorial optimization problems. In this survey paper applications of parallel iterative algorithms to graph coloring problems are described. Graph *k*-colorability problem (GCP) belongs to the class of NP-hard combinatorial problems [17]. The graph coloring was the subjects of the Second DIMACS Implementation Challenge in 1993 [23] and Computational Symposium on Graph Coloring and Generalizations in 2002. Numerous classes of graph colorings were defined and characterized in [22, 28], among them classical vertex and edge colorings, distance coloring, on-line coloring, equitable coloring, sum coloring, T-coloring, harmonious coloring, circular coloring, list coloring, total coloring etc. Most graph coloring problems are NP-hard. Graph coloring is strongly related to scheduling problems [18, 36].

The sequential approximation algorithms for GCP are given in [28]. Parallel approximation algorithms for GCP adapted efficiently for modern multi-core multithreaded architectures are shown in [4].

Metaheuristics are versatile iterative algorithms capable of solving a wide class of optimization problems. Several known iterative algorithms were also applied to graph coloring problem [2, 5, 16, 20]. Hybrid algorithms combining known metaheuristics were also developed [37, 38, 40].

Graph benchmarks for GCP include:

- DIMACS benchmark suite – graphs recommended for DIMACS computing challenges; at present almost all graphs have known chromatic numbers $\chi(G)$
- random graphs (various generation schemes) – usually with unknown value of $\chi(G)$;
- R-MAT graphs – a new class of scalable parametrized random graph instances, defined recursively (virtually any possible sizes are available) – no $\chi(G)$ is known.

Parallel metaheuristics are complex methods combining at least two metaheuristics of the same type that search for a solution either independently or in cooperation (with information exchange). The first book devoted solely to parallel metaheuristics was published in 2005 by Alba [1]. At that time the only research paper on application of a parallel metaheuristic to graph coloring was [25]. Since then, many new results in that area were reported in the literature. Parallel metaheuristics used for several graph coloring problems are listed and characterized in the following sections. The new original results include time-efficient optimal or suboptimal coloring algorithms for the benchmark graphs and also new bounds found experimentally for such hard-to-compute graph parameters like chromatic number, chromatic sum and chromatic sum number. The present survey reports the latest results from this fast growing research area.

1. Vertex coloring

GCP is defined for an undirected graph G(V,E) as an assignment of available colors $\{1, . . . , k\}$ to graph vertices providing that adjacent vertices receive different colors and the number of colors k is minimal. The resulting coloring is called conflict–free and k is called the graph chromatic number $\chi(G)$.

Models and properties of parallel genetic algorithms were described in [3]. The first population-based metaheuristic, i.e. PGA for solving the classical Graph Coloring Problem (GCP) in migration models was published by Kokosiński at all. [25]. Migration models of PGA consist of a finite set of disjoint populations that co-evolve and occasionally exchange genetic information under control of a migration operator. Populations are built of individuals of the same type and are ruled by the one adaptation function. In the paper new recombination operators: Sum-Product Partition Crossover (SPPX) and Conflict Elimination Crossover (CEX) were introduced and compared with UISX and GPX operators. In computer experiments DIMACS benchmark graphs were used [41–43]. An extended version of the paper was published in [26]. Another version of PGA for GCP was developed by Domagała [10]. In diffusion model of PGA the overlapping subpopulations are placed in the nodes of a mesh. The population size, the mesh size and architecture and other factors have to be considered in that model. The optimal tuning of the algorithm for the given problems can be achieved. An MPI implementation of PGA for GCP was described in [20].

In the following years many other parallel metaheuristics for GCP and related coloring problems were developed.

Three Parallel Tabu Search (PTS) algorithms (master-sleve search, independent search, cooperative search) on the basis of TABUCOL for GCP (Herz, de Werra) was proposed by Dąbrowski [8]. They were implemented on two different high cluster architectures. DIMACS graphs and random graphs with various node densities are used for experiments. PTS for GCP outperforms popular greedy DSATUR heuristic. The obtained experimental results showed the limited applicability of PTS for GCP due to problem complexity. Another PTS algorithm for GCP was developed by Kirsz [24].

The first Parallel Simulated Annealing Algorithm (PSA) for GCP was proposed by Łukasik et all. [32]. In synchronous master-slave model with periodic solution update two basic techniques were used:

1. multiple threads for computing independent chains of solutions and exchanging the obtained results on a regular basis;
2. parallel moves, where single Markov chain is being evaluated by multiple processing units calculating possible moves from one state to another.

The paper contains recommendations for optimal parameters settings. A comparison of PSA to PGA metaheuristic was provided which is not favorable for any of those two methods.

A Parallel Immune System (PIS) for graph coloring was proposed by Dąbrowski [9]. The algorithm is based on the mechanism of a clonal selection. Every processor operates on its own pool of antibodies and a migration operator is used to allow processors to exchange information.

2. Minimum sum coloring

GCS is defined for an undirected graph G(V,E) as an assignment of available colors *{1, . . ., h}* to graph vertices providing that sum of all color numbers in a conflict–free coloring must be minimal. The minimum number of colors *h* in a minimum–sum coloring is called chromatic sum number *s(G)*, $s(G) \geq \chi(G)$.

The problem was introduced by Kubicka and Schwenk [29]. Theoretical properties and bounds can be find in [32]. New theoretical lower bounds were given in [27].

The best lower bounds on $\sum(G)$ are as follows:

$$\lceil \sqrt{8m} \rceil \leq \sum(G) \tag{1}$$

$$n + \chi(G)(\chi(G)-1)/2 \leq \sum(G) \tag{2}$$

The best upper bounds on $\sum(G)$ are the following:

$$\sum(G) \leq n + m \tag{3}$$

$$\sum(G) \leq n(\chi(G)+1)/2 \tag{4}$$

Table 1

MSCP – research results (part I)

G(V,E)	*n*	*m*	*χ(G)*	*s(G)*	*Σ(G)*								
					theoretical bounds				*experimental bounds*				
					L.B.	rule	U.B.	rule	L.B.	source	U.B.	source	gap [%]
anna	138	493	11	**11**	193	2	631	3	**272**	[35]	**277**	[35]	1,81
david	87	406	11	**11**	142	2	494	3	**234**	[35]	**237**	[40]	1,27
huck	74	301	11	**11**	129	2	375	3	**243**	[35]	**243**	[27]	**0**
jean	80	254	10	**10**	125	2	334	3	**216**	[35]	**217**	[30]	0,46
queen5.5	25	160	5	**5**	36	1	75	4	**75**	[35]	**75**	[27]	**0**
queen6.6	36	290	7	7-8	57	2	144	4	**126**	[35]	**138**	[27]	8,70
queen7.7	49	476	7	**7**	70	2	196	4	**196**	[35]	**196**	[27]	**0**
queen8.8	64	728	9	**9**	100	2	320	4	**288**	[35]	**291**	[40]	1,03
games120	120	638	9	**9**	156	2	600	4	**442**	[35]	**443**	[40]	0,23
miles250	128	387	8	**8**	156	2	515	3	**316**	[35]	**328**	[40]	3,66
miles500	128	1170	20	**20**	318	2	1298	3	**677**	[35]	**709**	[40]	4,51
myciel3	11	20	4	**4**	**17**	2	27	4	*16*	[35]	**21**	[27]	**19,05**
myciel4	23	71	5	**5**	33	2	69	4	**34**	[35]	**45**	[27]	24,44
myciel5	47	236	6	**6**	62	2	164	4	**70**	[35]	**93**	[27]	24,73
myciel6	95	755	7	**7**	116	2	380	4	**142**	[35]	**189**	[27]	24,86
myciel7	191	2360	8	**8**	219	2	859	4	**286**	[35]	**381**	[30]	24,93

$$gap = [(\min(UB_{th}, UB_{exp}) - \max(LB_{th}, LB_{exp})]*100\% / (\min(UB_{th}, UB_{exp}))$$

An application of PGA to Graph Coloring Sum Problem (GCSP) was shown in Kokosiński and Kwarciany [27]. That paper initialized the intensive international research on experimental finding of better upper bounds on chromatic sum and chromatic sum number for a class of DIMACS graph instances.

The recent methods and results are given in [11–15, 30, 34, 35]. Local search heuristic for GCSP was described in [21]. In [40] a new hybrid algorithm EXSCOL is presented which incorporates a TS metaheuristic for finding a vertex partition into large independent sets. EXSCOL was tested on a set of 52 DIMACS graphs. For 17 out of 28 previously examined problem instances better colorings and bounds were found (cf. Tables 1 and 2).

Table 2

MSCP – research results (part II)

G(V,E)	*n*	*m*	$\chi(G)$	*s(G)*	*Σ(G)*								
					theoretical bounds				*experimental bounds*				
					L.B	rule	U.B.	rule	L.B.	source	U.B.	source	gap[%]
fpsol2.i.1	496	11654	65	**65**	2576	2	12150	4	**2590**	[14]	**3405**	[10]	23,9
inithx.i.1	864	18707	54	**54**	2295	2	19571	4	**2801**	[14]	**3679**	[10]	23,8
mug88-1	88	146	4	**4**	94	2	220	4	**163**	[14]	**190**	[10]	14,2
mug88-25	88	146	4	**4**	94	2	220	4	**162**	[14]	**187**	[10]	13,3
mug100-1	100	166	4	**4**	106	2	250	4	**187**	[14]	**211**	[10]	11,3
mug100-25	100	166	4	**4**	106	2	250	4	**185**	[14]	**214**	[10]	13,5
2-Inser 3	37	72	4	**4**	43	2	92	4	**55**	[14]	**63**	[10]	12,7
3-Inser 3	56	110	4	**4**	62	2	140	4	**84**	[14]	**92**	[10]	8,70
zeroin.i.2	211	3541	30	**30**	646	2	3270	4	**1003**	[14]	**1013**	[10]	0,99
zeroin.i.3	206	3540	30	**30**	641	2	3193	4	**997**	[14]	**1007**	[10]	0,99

$$gap = [(\min(UB_{th}, UB_{\exp}) - \max(LB_{th}, LB_{\exp})] * 100\% / (\min(UB_{th}, UB_{\exp})$$

The research on MSCP brought the following results :

- The theoretical gap between L.B. and U.B. for MSCP was reduced;
- DIMACS benchmarks for GCP has been accepted for examination of MSCP;
- Parallel metaheuristic can be used for finding unknown hard-to-compute parameters of benchmark graphs like $\chi(G)$ and *s(G)*;
- The gap between L.B. and U.B. for most examined graph problems was significantly reduced.

3. Robust graph coloring

RGCP is defined for undirected graph G(V,E) as an assignment of available colors *{1, . . ., k}* to graph vertices, providing that

$$\forall (u,v) \in E \ : (p_{uv} = 1) \Rightarrow \ c(u) \neq c(v), \tag{5}$$

and

$$\sum_{((u,v)\in E)\wedge(c(u)\neq c(v))\wedge(p_{uv}<1)\wedge(u<v))} p_{uv} \geq T \ , \tag{6}$$

where p_{uv} – is an edge e(u,v) weight in [0, 1]; p_{uv} may be considered as a probability of an edge existence, in the classical vertex coloring $p_{uv} \in \{0,1\}$; T – is an assumed threshold of robustness.

The robust graph coloring problem (RGCP) was introduced in [39]. Some applications of basic metaheuristics for RGCP was reported in [6,31]. They include uncertainty management, crew assignment problem and robust energy supply. Metaheuristics for RGCP [31] were tested for simple random graphs.

The first parallel metaheuristic – PGA for RGCP – was developed by Chrząszcz [6]. In the algorithm the efficient Best Crossover (BCX) recombination operator [36] was used in addition to those tested in [25, 26]. For computer experiments parametrized problem instances were generated by a random modification of selected DIMACS graphs. A given number of $p_{uv}=1$ in adjacency matrix representation was replaced with p_{uv} in (0,1). No other parallel metaheuristic for RGCP is available for comparison.

Conclusions

PGA, PSA, PTS and PIS are well known parallel iterative algorithms that are often applied for solving approximate solution of NPO problems. Efficient use of parallel metaheuristics requires careful design of details, proper tuning of parallel scheme, hybridization etc. Parallel metaheuristics are well suited for solving graph coloring problems and new algorithms can still be designed for graph colorings, f.i. PACO. Parallel metaheuristic can also be used for computing unknown hard-to-compute parameters of benchmark graphs.

Few classes of graph colorings has been carefully tested with computers but research in this area is prospective. It seems that for many coloring problems DIMACS graphs will be gradually replaced by random problem instances, in particular R-MAT graphs [4]. On the other hand DIMACS benchmarks shall be of the first choice when we look for the test graphs with known parameters. They can be modified with a little effort according to the particular needs.

There are many open questions concerning parallelization of computations using new techniques and modern multi-core computers. In particular, parallel metaheuristics may be more difficult to implement on massively parallel architectures then dedicated approximation algorithms [4].

Acknowledgements

The paper is an extended version of the conference paper presented at the The First International Conference on Automatic Control and Information Technology *ICACIT'2011*, held in Lviv (Ukraine) on December 15-17, 2011. The work was supported in part by Cracow University of Technology, Kraków, Poland.

1. Alba E. (Ed.) Parallel metaheuristics: a new class of algorithms, John Wiley & Sons. – 2005. 2. Bouziri H., Jouini M. A tabu search approach for the sum coloring problem // Electronic Notes in Discrete Mathematics. – Vol. 36. – 2010. – P. 915–922. 3. Cantú-Paz E. Efficient and accurate parallel genetic algorithms, Kluwer. – 2000. 4. Catalyurek U., Feo J., Gebremedhin A., Halappanavar M., Pothen A. Graph coloring algorithms for multi-core and massively multithreaded architectures // Parallel Computing. – 2012. (accepted) 5. Costa D., Hertz A. Ants can color graphs // J. of the Operational Research Society. – Vol. 48. – 1997. – P. 295–305. 6. Chrząszcz G. Parallel evolutionary algorithm for the design of robust scheduling in power systems, M.Sc. thesis, Politechnika Krakowska. – 2009. (in Polish). 7. Crainic T.G., Nourredine, H. Parallel meta-heuristics applications, [in:] Alba E. (Ed.): Parallel metaheuristics: a new class of algorithms, John Wiley & Sons. – 2005. 8. Dąbrowski J. Parallelization techniques for tabu search, Proc. 8th Int. Conference on Applied parallel computing: state of the art in scientific computing. – 2007. 9. Dąbrowski J. Parallel immune system for graph coloring // Lecture Notes in Computer Science. – Vol. 4910. – 2008 . – P. 497–505. 10. Domagała P. : A paralel genetic algorithm for graph coloring in a diffusion model, M.Sc. thesis, Politechnika Krakowska. – 2006. (in Polish). 11. Douiri M., El Bernoussi S. New algorithm for the sum coloring problem // Int. J. Contemp. Math. Sciences. – Vol. 6. – 2011. – No.10. – P.453 – 463. 12. Douiri M., El Bernoussi S. A new heuristic for the sum coloring problem // Applied Math. Sciences. – Vol. 5. – 2011. – No.63 – P. 3121–3129. 13. Douiri M., El Bernoussi S. A hybrid algorithm for the sum coloring problem, Proc. 2011 International Conference on Multimedia Computing and Systems (ICMCS), 7–9 April 2011, Ouarzazate, Marocco, P. 1–4. 14. Douiri S.M., El Bernoussi S. An ant colony optimization for the sum coloring problem // International Journal of Applied Mathematics & Statistics. – Vol. 27. – 2011. – No.3. – P. 102–110. 15. Douiri S.M., El Bernoussi S. An new ant colony optimization algorithm for the lower bound of sum coloring problem // Int. J. Math. Modeling and Algorithms, Springer. (DOI: 10.1007/s10852-012-9172-x). 16. Erfani M., Ghanizadeh A., Abarghouei A.A., Sinaie S., Shamsuddin S.M. A Modified PSO method enhanced with fuzzy inference system for solving the planar graph coloring problem, WORLDCOMP'10, Proceedings of the 2010 International Conference on Artificial Intelligence, IC-AI, July 12-15, Las Vegas, Nevada, CSREA Press. –

2010. – P.160 – 165. 17. Garey R., Johnson D.S. Computers and intractability. A guide to the theory of NP-completeness, Freeman, San Francisco. – 1979. 18. Giaro K. Task scheduling by graph coloring // Gdańsk University of Technology, Monographs. – Vol. 37. – 2003. (in Polish). 19. Hadjikyriacou E., Samaras N., Margaritis K. An experimental evaluation of a parallel genetic algorithm using MPI, Proc. 13th Panhellenic Conference on Informatics, Corfu, Greece, Computer Society Press. – 2009. – P. 75–79. 20. Han L., Han Z. A novel bi-objective genetic algorithm for the graph coloring problem, Int. Conf. on Computer Modeling and Simulation ICCMS'2010, January 22–24, Sanya, China, 2010, P. 3–6. 21. Helmar A., Chiarandini M. Local search heuristic for chromatic sum, Proc. of The Metaheuristics Int Conference, MIC 2011, July 25–28, Udine, Italy. 22. Jensen T.R., Toft B. Graph coloring problems, Wiley-Interscience. – 1995. 23. Johnson D.S., Trick M.A. Cliques, coloring and satisfiability: Second DIMACS Implementation Challenge, DIMACS Series in Discr. Math. and Theor. Comp. Sc. – Vol. 26. – 1996. 24. Kirsz K. Parallel tabu search algorithm in application to graph coloring problem, M.Sc. thesis, Politechnika Krakowska. – 2011. (in Polish) 25. Kokosiński Z., Kołodziej M., Kwarciany K. Parallel genetic algorithm for graph coloring problem // Lecture Notes in Computer Science. – Vol. 3036. – 2004. – P. 217–224. 26. Kokosiński Z., Kołodziej M., Kwarciany K. Efficient graph coloring with parallel genetic algorithms // Computing and Informatics. – Vol. 24. – 2005. – P. 123–147. 27. Kokosiński Z., Kwarciany K. On sum coloring of graphs with parallel genetic algorithms // Lecture Notes in Computer Science. – Vol. 4431. – 2007. – P. 211–219. 28. Kubale M. (Ed.) Discrete optimization. Models and methods for graph coloring, WNT, Warszawa. – 2002. (in Polish). 29. Kubicka E., Schwenk A.J. An introduction to chromatic sums, Proc. 17th Annual ACM Computer Science Conf. – 1989. – P. 39–45. 30. Li Y., Lucet C., Moukrim A., Sghiouer K. Greedy algorithms for the minimum sum coloring problem, Proc. Int. Workshop "Logistique et transports", March 22-24, Sousse, Tunisia, – 2009. 31. Lim A., Wang F. Meta-heuristic for robust graph coloring problem, IEEE Int. Conference on Tools with Artificial Intelligence, ICTAI. – 2004. 32. Łukasik S., Kokosiński Z., Świętoń G. Parallel simulated annealing algorithm for graph coloring problem, Lecture Notes in Computer Science. – Vol. 4967. – 2008. – P. 229–238. 33. Małafiejski M. Sum coloring of graphs, [in:] Kubale M. (Ed.): Graph colorings. American Math. Society.Contemporary Mathematics. – Vol. 352. – 2004. – P.55–65. 34. Mohamed D.S., El Bernoussi S. Max-Min ant system for the sum coloring problem, Proc. International Conference on Communications, Computing and Control Applications, CCCA 2011, March 3–5, Hammamet, Tunisia. – P. 1–4. 35. Moukrim A., Sghiouer K., Lucet C., Li Y. Lower bounds for the minimal sum coloring problem // Electronic Notes in Discrete Mathematics. – Vol. 36. – 2010. – P. 663–670. 36. Myszkowski P.B.: Solving scheduling problems by evolutionary algorithms for graph coloring problem, [in:] Xhafa F., Abraham A. (Eds.) Metaheuristics for Scheduling in Industrial and Manufacturing Applications. Studies in Computational Intelligence. – Vol.128. – 2008. – P. 145– 167. 37. Pahlavani A., Eshghi K. A hybrid algorithm of simulated annealing and tabu search for graph colouring problem // International Journal of Operational Research. – Vol. 11. – 2011. – No.2. – P. 136–159. 38. Salari E., Ehgshi K. An ACO algorithm for the graph coloring problem // Int. J. Contemp. Math. Sciences. – Vol. 3. – 2008. – No.6. – P. 293–304. 39. Yáñez J., Ramirez J. The robust coloring problem // European Journal of Operational Research. – Vol. 148. – 2003. – No. 3. – P. 546–558. 40. Wu Q., Hao J.-K. An effective heuristic algorithm for sum coloring of graphs // Computers and Operations Research. – Vol. 39. – 2012. – No.7. 41. COLOR web site, http://mat.gsia.cmu.edu/COLOR/instances.html. 42. DIMACS ftp site, ftp://dimacs.rutgers.edu/pub/challenge/graph/benchmarks/. 43. COLORING 3 web site, http://mat.gsia.cmu.edu/COLORING03/.