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Dynamics of an ecological prey–predator model based on the generalized Hattaf fractional derivative

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In this paper, we propose and analyze a fractional prey–predator model with generalized Hattaf fractional (GHF) derivative. We prove that our proposed model is ecologically and mathematically well-posed. Furthermore, we show that our model has three equilibrium points. Finally, we establish the stability of these equilibria.

Keywords: *ecology; mathematical modeling; prey–predator; Hattaf fractional derivative; Lyapunov function; stability.*

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1. Introduction

Prey–predator models are widely used in ecological biology studies to understand the interactions between two populations in an ecosystem: predators and their prey. In such models, the predator population is dependent on the prey population for food while the prey population is kept in check by the predator population. In addition, the models help to understand the dynamics of the prey–predator relationship and how changes in one population can affect the other. In the literature, there are various predator–prey models. For instance, Lotka [1] and Volterra [2] who first introduced the classic model of the prey–predator interaction. In [3], Kar studied a Lotka Volterra type predator prey model with Michaelis–Menten type functional response. In [4], Garain et al. investigated a model that explores the dynamics of prey–predator interaction and incorporates a density-dependent death rate for predators and a functional response of Beddington–DeAngelis type.

Memory must be a fundamental aspect within the interaction of prey and predators in the natural world as the growth rates of both hang not only on their current local state but also on the complete history of variables from all preceding times. Recently, the utilization of the fractional derivative that is the extension of the traditional integer derivative has been employed to study the influence of memory on the dynamics of diverse systems in various fields, including epidemiology [5], ecology [6], economics [7], viral immunology [8, 9], cancerology [10] and viscoelastic fluid flows [11] as well as adaptive control engineering [12]. Some interesting articles about the fractional prey–predator models who many researchers discussed the stability of the models are included in [13–15]. In [13], Ahmed et al. proved the existence and uniqueness of the solution of a fractional-order predator–prey model and they studied the local asymptotic stability of the equilibrium points. In [14], Javidi and Nyamoradi investigated the local stability of a fractional-order prey–predator model based on Caputo fractional derivative with singular kernel and in [15] Ghaziani et al. have been extensively studied the dynamical behavior of a fractional order Leslie–Gower prey–predator model.

Over the past few years, the definition of fractional derivative has garnered the interest of numerous researchers. In 2020, a new generalized definition of the fractional derivative with a nonsingular and nonlocal kernel for Caputo and Riemann–Liouville types was defined by Hattaf in [16] in order to study the influence of memory on the dynamics of certain dynamical systems in the fields of epidemiology and virology. This definition includes the widely renowned fractional derivatives with nonsingular kernels found in existing literature, as the fractional derivative of Caputo–Fabrizio [17], the fractional derivative of Atangana–Baleanu [18] and the weighted fractional derivative of Atangana–Baleanu [19].

With inspiration drawn from the above mentioned points, the aim of this paper is to study the predator–prey model of fractional order by utilizing the generalized Hattaf fractional (GHF) derivative. The subsequent sections of this paper are structured as follows: the next section presents the model formulation and preliminaries. Section 3 explores the existence and uniqueness of solutions for our model. Section 4 discusses the boundedness and non-negativity of solutions. Section 5 focuses on the determination of equilibria and their local and global stability. Finally, a brief conclusion is given in Section 6.

2. Model formulation and preliminaries

Within this section, we introduce the definition and present essential results concerning the GHF derivative with a nonsingular kernel. These findings will be indispensable for the subsequent discussions. After, we present the formulation of the fractional prey–predator model that we will study.

Definition 1 (see [16]). *Let $\alpha \in [0,1)$, $\beta,\gamma > 0$ and $f \in H^1(a,b)$. The GHF derivative of order α in Caputo sense of the function $f(t)$ with respect to the weight function $w(t)$ is defined as follows:*

$$ {}^{C}D^{\alpha,\beta,w}_{a,t,w} f(t) = \frac{1}{w(t)} \frac{N(\alpha)}{1-\alpha} \int_a^t E_\beta[-(t-x)^\gamma \mu_\alpha] \frac{d}{dx}(wf)(x)\,dx, $$

where $w \in C^1(a,b)$, $w, w' > 0$ on $[a,b]$, $\mu_\alpha = \frac{\alpha}{1-\alpha}$, $N(\alpha)$ is a normalization function such that $N(0) = N(1) = 1$ and $E_\beta(t) = \sum_{k=0}^{+\infty} \frac{t^k}{\Gamma(\beta k+1)}$ is the Mittag–Leffler function of parameter β.

In the above definition, $H^1(a,b)$ is the Sobolev space of order one defined as follows:

$$ H^1(a,b) = \big\{u \in L^2(a,b)\colon u' \in L^2(a,b)\big\}. $$

By virtue of Lemma 1 in [16], we can readily derive the subsequent result.

Theorem 1 (Ref. [16]). *The expression for the Laplace transform of ${}^{C}D^{\alpha,\beta,\gamma}_{a,t,w}$ is as follows:*

$$ \mathcal{L}\big\{w(t)\,{}^{C}D^{\alpha,\beta,\gamma}_{0,t,w} f(t)\big\}(s) = \frac{N(\alpha)\,[s\,\mathcal{L}\{w(t)f(t)\}(s) - w(0)f(0)]}{(1-\alpha)s} \sum_{k=0}^{+\infty} \left(\frac{-\mu_\alpha}{s^\gamma}\right)^k \frac{\Gamma(\gamma k+1)}{\Gamma(\beta k+1)}. $$

When $\beta = \gamma$, we have

$$ \mathcal{L}\big\{w(t)\,{}^{C}D^{\alpha,\beta,\beta}_{0,t,w} f(t)\big\}(s) = \frac{N(\alpha)}{1-\alpha} \frac{s^\beta \mathcal{L}\{w(t)f(t)\}(s) - s^{\beta-1}w(0)f(0)}{s^\beta + \mu_\alpha}. $$

In addition, the Laplace transform of ${}^{R}D^{\alpha,\beta,\gamma}_{a,t,w}$ is as follows:

$$ \mathcal{L}\big\{w(t)\,{}^{R}D^{\alpha,\beta,\gamma}_{0,t,w} f(t)\big\}(s) = \frac{N(\alpha)}{1-\alpha} \mathcal{L}\{w(t)f(t)\}(s) \sum_{k=0}^{+\infty} \left(\frac{-\mu_\alpha}{s^\gamma}\right)^k \frac{\Gamma(\gamma k+1)}{\Gamma(\beta k+1)}. $$

When $\beta = \gamma$, we have

$$ \mathcal{L}\big\{w(t)\,{}^{R}D^{\alpha,\beta,\beta}_{0,t,w} f(t)\big\}(s) = \frac{N(\alpha)}{1-\alpha} \frac{s^\beta \mathcal{L}\{w(t)f(t)\}(s)}{s^\beta + \mu_\alpha}. $$

Clearly, we have the subsequent remark.

Remark 1. When $\beta = \alpha = \gamma$ and $w(t) = 1$, we get the Laplace transform of the Atangana–Baleanu fractional derivatives in the sense of Riemann–Liouville and Caputo determinate in [18].

Corollary 1 (Ref. [20]). *Let $\lambda \in (0,+\infty)$ and $g(t)$ be a function verifying the subsequent inequality: $D^{\alpha,\beta}_{0,w} g(t) \leqslant -\lambda\, g(t)$. Then $g(t) \leqslant g(0)\, E_\beta\big(\frac{-\lambda \alpha t^\beta}{N(\alpha)+\lambda(1-\alpha)}\big)$.*

For simplicity, denote ${}^{C}D^{\alpha,\beta,\beta}_{a,t,w}$ by $D^{\alpha,\beta}_{a,w}$. Now, we extend the model given in [4] by including the generalized Hattaf fractional derivative. Hence, the model becomes:

$$\begin{cases} D^{\alpha,\beta}_{0,w}x = r\,x\left(1-\dfrac{x}{k}\right) - \dfrac{a\,x\,y}{\alpha_0+\alpha_1 x+\alpha_2 y}, \\ D^{\alpha,\beta}_{0,w}y = \dfrac{a\,b\,x\,y}{\alpha_0+\alpha_1 x+\alpha_2 y} - c\,y - d\,y^2, \end{cases} \tag{1}$$

where $x(t)$ represents the prey density at time t and $y(t)$ denote the predator density at time t. The parameter r denotes the rate at which the prey population grows naturally, k is the maximum population size that can be sustained by the environment for the prey, b signifies the rate at which prey are converted into predators, c represents the rate at which predators experience mortality and d indicates the rate of competition between predators. The relationship between the prey and the predator is described using a specific form of the Hattaf–Yousfi functional response given by $\frac{a\,x\,y}{\alpha_0+\alpha_1 x+\alpha_2 y+\alpha_3 x\,y}$, by taking $\alpha_3 = 0$ where a is the rate at which predators capture prey also known as the consumption rate, the saturation factors $\alpha_0, \alpha_1, \alpha_2 \geqslant 0$ measuring the inhibitory or psychological effects.

3. Existence and uniqueness

Within this section, we investigate that the solution of the system (1) exists and unique in $\Omega \times [0,T]$, where $\Omega = \{(x,y) \in \mathbb{R}^2_+ / \max\{|x|,|y|\} \leqslant M\}$, $T < +\infty$. Consider a mapping $F(Z) = (F_1(Z), F_2(Z))$ with the following

$$F_1(Z) = r\,x\left(1-\frac{x}{k}\right) - \frac{a\,x\,y}{\alpha_0+\alpha_1 x+\alpha_2 y}, \tag{2}$$

$$F_2(Z) = \frac{a\,b\,x\,y}{\alpha_0+\alpha_1 x+\alpha_2 y} - c\,y - d\,y^2. \tag{3}$$

For any $Z = (x,y)$, $\bar{Z} = (\bar{x},\bar{y})$, $Z, \bar{Z} \in \Omega$, it can be deduced from equations (2) and (3) that the following holds:

$$\begin{aligned}
\|F(Z)-F(\bar{Z})\| &= |F_1(Z)-F_1(\bar{Z})| + |F_2(Z)-F_2(\bar{Z})| \\
&= \left| r(x-\bar{x}) - \frac{r}{k}(x^2-\bar{x}^2) - a\left(\frac{x\,y}{\alpha_0+\alpha_1 x+\alpha_2 y} - \frac{\bar{x}\bar{y}}{\alpha_0+\alpha_1\bar{x}+\alpha_2\bar{y}}\right)\right| \\
&\quad + \left| -c(y-\bar{y}) - d(y^2-\bar{y}^2) + a\,b\left(\frac{x\,y}{\alpha_0+\alpha_1 x+\alpha_2 y} - \frac{\bar{x}\,\bar{y}}{\alpha_0+\alpha_1\bar{x}+\alpha_2\bar{y}}\right)\right| \\
&\leqslant r|x-\bar{x}| + \frac{r}{k}\left|x^2-\bar{x}^2\right| + a\left|\frac{x\,y(\alpha_0+\alpha_1\bar{x}+\alpha_2\bar{y}) - \bar{x}\,\bar{y}(\alpha_0+\alpha_1 x+\alpha_2 y)}{(\alpha_0+\alpha_1 x+\alpha_2 y)(\alpha_0+\alpha_1\bar{x}+\alpha_2\bar{y})}\right| \\
&\quad + c|y-\bar{y}| + d\left|y^2-\bar{y}^2\right| + a\,b\left|\frac{x\,y(\alpha_0+\alpha_1\bar{x}+\alpha_2\bar{y}) - \bar{x}\,\bar{y}(\alpha_0+\alpha_1 x+\alpha_2 y)}{(\alpha_0+\alpha_1 x+\alpha_2 y)(\alpha_0+\alpha_1\bar{x}+\alpha_2\bar{y})}\right| \\
&\leqslant r|x-\bar{x}| + \frac{r}{k}|x-\bar{x}|\,|x+\bar{x}| + a\left|\frac{(\alpha_0 x+\alpha_1 x\,\bar{x})(y-\bar{y}) + (\alpha_0\bar{y}+\alpha_2 y\,\bar{y})(x-\bar{x})}{(\alpha_0+\alpha_1 x+\alpha_2 y)(\alpha_0+\alpha_1\bar{x}+\alpha_2\bar{y})}\right| \\
&\quad + c|y-\bar{y}| + d|y-\bar{y}|\,|y+\bar{y}| + a\,b\left|\frac{(\alpha_0 x+\alpha_1 x\,\bar{x})(y-\bar{y}) + (\alpha_0\bar{y}+\alpha_2 y\,\bar{y})(x-\bar{x})}{(\alpha_0+\alpha_1 x+\alpha_2 y)(\alpha_0+\alpha_1\bar{x}+\alpha_2\bar{y})}\right| \\
&\leqslant r|x-\bar{x}| + \frac{r}{k}|x-\bar{x}|\,|x+\bar{x}| + a\big|(\alpha_0 x+\alpha_1 x\,\bar{x})(y-\bar{y})\big| + a\big|(\alpha_0\bar{y}+\alpha_2 y\,\bar{y})(x-\bar{x})\big| \\
&\quad + c|y-\bar{y}| + d|y-\bar{y}|\,|y+\bar{y}| + a\,b\big|(\alpha_0 x+\alpha_1 x\,\bar{x})(y-\bar{y})\big| + a\,b\big|(\alpha_0\bar{y}+\alpha_2 y\,\bar{y})(x-\bar{x})\big| \\
&\leqslant r|x-\bar{x}| + 2\frac{r}{k}M|x-\bar{x}| + a|\alpha_0 x+\alpha_1 x\,\bar{x}|\,|y-\bar{y}| + a|\alpha_0\bar{y}+\alpha_2 y\,\bar{y}|\,|x-\bar{x}| \\
&\quad + c|y-\bar{y}| + d|y-\bar{y}|\,2M + a\,b|(\alpha_0\bar{y}+\alpha_2 y\,\bar{y})(x-\bar{x})| + a\,b|(\alpha_0 x+\alpha_1 x\,\bar{x})(y-\bar{y})|.
\end{aligned}$$

Then

$$\begin{aligned}
\|F(Z)-F(\bar{Z})\| &\leqslant r|x-\bar{x}| + 2\frac{r}{k}M|x-\bar{x}| + \big(a\,\alpha_0 M + a\,\alpha_1 M^2\big)|y-\bar{y}| + \big(a\,\alpha_0 M + a\,\alpha_2 M^2\big)|x-\bar{x}| \\
&\quad + c|y-\bar{y}| + 2d\,M|y-\bar{y}| + \big(a\,b\,\alpha_0 M + a\,b\,\alpha_2 M^2\big)|x-\bar{x}|
\end{aligned}$$

$$
\begin{aligned}
&\quad + \big(a\,b\,\alpha_0 M + a\,b\,\alpha_1 M^2\big)|y-\bar{y}| \\
&= L_1|x-\bar{x}| + L_2|y-\bar{y}| \\
&\leqslant L\|Z-\bar{Z}\|.
\end{aligned}
$$

Consequently,

$$\|F(Z)-F(\bar{Z})\| \leqslant L\|Z-\bar{Z}\|,$$

where

$$
\begin{aligned}
L_1 &= r + 2\frac{r}{k}M + \big(a\,\alpha_0 M + a\,\alpha_2 M^2\big) + \big(a\,b\,\alpha_0 M + a\,b\,\alpha_2 M^2\big), \\
L_2 &= \big(a\,\alpha_0 M + a\,\alpha_1 M^2\big) + \big(a\,b\,\alpha_0 M + a\,b\,\alpha_1 M^2\big) + c + 2d\,M, \\
L &= \max(L_1, L_2).
\end{aligned}
$$

Since the Lipschitz condition is satisfied by $F(Z)$ with respect to Z, we can deduce based on Theorem 4.3 [21] that there is only one solution $Z(t)$ of system (1) with initial condition $Z(t_0) = (x(t_0), y(t_0))$ if $L\big(\frac{1-\alpha}{N(\alpha)} + \frac{\alpha T^\beta}{N(\alpha)\Gamma(\beta+1)}\big) < 1$.

Consequently, we can state the following theorem.

Theorem 2. *Suppose that* $L\big(\frac{1-\alpha}{N(\alpha)} + \frac{\alpha T^\beta}{N(\alpha)\Gamma(\beta+1)}\big) < 1$, *where*

$$
\begin{aligned}
L &= \max(L_1, L_2), \\
L_1 &= r + 2\frac{r}{k}M + \big(a\,\alpha_0 M + a\,\alpha_2 M^2\big) + \big(a\,b\,\alpha_0 M + a\,b\,\alpha_2 M^2\big), \\
L_2 &= \big(a\,\alpha_0 M + a\,\alpha_1 M^2\big) + \big(a\,b\,\alpha_0 M + a\,b\,\alpha_1 M^2\big) + c + d\,2M.
\end{aligned}
$$

For each $Z(t_0) = (x(t_0), y(t_0)) \in \Omega$, *in such a case, the initial value problem associated with system* (1) *possesses one and only one solution* $Z(t) \in \Omega$ *that is defined for all* $t > t_0$.

4. Non-negativity and boundedness

In this section, we demonstrate the biological soundness of system (1) by proving that its unique solution is both non-negative and bounded.

Lemma 1. *Suppose that* $f(t) \in C[a,b]$ *and the GHF derivative of order* α *in Caputo sense of the function* f *with respect to the weight function* $w(t)$ $D^{\alpha,\beta}_{0,w} f(t) \in C[a,b]$ *for* $0 < \alpha < 1$,

If $D^{\alpha,\beta}_{0,w} f(t) \leqslant 0$, $\forall t \in [a,b]$, *then* $w\,f(t)$ *is non-increasing for all* $t \in [a,b]$.

If $D^{\alpha,\beta}_{0,w} f(t) \geqslant 0$, $\forall t \in [a,b]$, *then* $w\,f(t)$ *is non-decreasing for all* $t \in [a,b]$.

Proof. The demonstration to Lemma 1 follows from Corollary 4 of [22]. ■

Theorem 3. *If* $x(t_0) \geqslant 0$ *and* $y(t_0) \geqslant 0$, *every solution of system* (1) *satisfies non-negativity.*

Proof. Let $W(t_0) = \begin{pmatrix} x(t_0) \\ y(t_0) \end{pmatrix} \in \mathbb{R}^2_+$ and assume that $W(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ for $t \geqslant t_0$ be the solutions of system (1).

Let us assume that the given assumption is not true.

Consequently, there exists $t^* > t_0$ such that $W(t) > 0$ for $t_0 \leqslant t < t^*$, $W(t^*) = 0$ and $W(t^*_+) < 0$, $\forall t^*_+ > t^*$.

From system (1),

$$D^{\alpha,\beta}_{a,w} W(t)\big|_{t=t^*} = 0.$$

Utilizing Lemma 1, we have $w\,W(t^*_+) = 0$, which is absurd. Hence, we can conclude that $W(t) \geqslant 0$, $\forall t \geqslant 0$. ■

Subsequently, we establish the boundedness of all solutions to system (1).

Lemma 2. *Consider a continuous function $u(t)$ defined on the interval $[t_0,+\infty)$ that satisfies the following condition:*

$$\begin{cases} D^{\alpha,\beta}_{0,\omega}u(t) \leqslant -\lambda u(t) - \mu, \\ u(t_0) = u_{t_0}, \end{cases} \tag{4}$$

where $(\lambda,\mu)\in\mathbb{R}^2$, $0<\alpha<1$, $\lambda\neq 0$ and $t_0\geqslant 0$ is the initial time. Then

$$u(t) \leqslant \left(u_{t_0} - \frac{\mu}{\lambda}\right) E_\beta\left(\frac{-\alpha\lambda t^\beta}{a_\alpha}\right) + \frac{\mu}{\lambda}. \tag{5}$$

Proof. Let $U(t) = u(t) - \frac{\mu}{\lambda}$. Then (4) is changed into

$$\begin{cases} D^{\alpha,\beta}_{0,w}U(t) \leqslant -\lambda U(t), \\ U(t_0) = u_{t_0} - \frac{\mu}{\lambda}. \end{cases}$$

Utilizing Corollary 1 from the publication referenced as [20], we get

$$U(t) \leqslant U(0)\, E_\beta\left(\frac{-\alpha\,\lambda\,t^\beta}{N(\alpha)+\lambda(1-\alpha)}\right).$$

Then

$$u(t) \leqslant \left(u_{t_0} - \frac{\mu}{\lambda}\right) E_\beta\left(\frac{-\alpha\lambda t^\beta}{N(\alpha)+\lambda(1-\alpha)}\right) + \frac{\mu}{\lambda}.$$ ∎

Theorem 4. *If $x(t_0)\geqslant 0$ and $y(t_0)\geqslant 0$, then all solutions of the system (1) are uniformly bounded.*

Proof. We can deduce the following from the first equation of the system (1):

$$\begin{aligned} D^{\alpha,\beta}_{a,\omega}x(t) + x(t) &= r\,x\left(1-\frac{x}{k}\right) - \frac{a\,x\,y}{\alpha_0+\alpha_1x+\alpha_2y} + x \\ &= -\frac{r}{k}x^2 + (1+r)x - \frac{a\,x\,y}{\alpha_0+\alpha_1x+\alpha_2y} \\ &= -\frac{r}{k}x^2 + x(1+r) - \frac{(1+r)^2}{4\frac{r}{k}} + \frac{(1+r)^2}{4\frac{r}{k}} - \frac{a\,x\,y}{\alpha_0+\alpha_1x+\alpha_2y} \\ &= -\frac{r}{k}\left(x^2 - x\frac{(1+r)}{\frac{r}{k}} + \left(\frac{(1+r)}{2\frac{r}{k}}\right)^2\right) + \frac{(1+r)^2}{4\frac{r}{k}} - \frac{a\,x\,y}{\alpha_0+\alpha_1x+\alpha_2y} \\ &= \frac{(1+r)^2}{4\frac{r}{k}} - \left(\frac{r}{k}\left(x - \frac{1+r}{2\frac{r}{k}}\right)^2 + \frac{a\,x\,y}{\alpha_0+\alpha_1x+\alpha_2y}\right) \\ &\leqslant \frac{(1+r)^2}{4\frac{r}{k}}. \end{aligned}$$

Then

$$D^{\alpha,\beta}_{a,w}x(t) \leqslant -x(t) + \frac{(1+r)^2}{4\frac{r}{k}}.$$

By applying Lemma 2, we have

$$x(t) \leqslant \left(x(t_0) - \frac{(1+r)^2}{4\frac{r}{k}}\right) E_\beta\left(\frac{-\alpha\,\lambda\,t^\beta}{N(\alpha)+\lambda(1-\alpha)}\right) + \frac{(1+r)^2}{4\frac{r}{k}} \to \frac{(1+r)^2}{4\frac{r}{k}}, \quad t\to\infty,$$

where E_β is the Mittag–Leffler function. As a result, the solutions of $x(t)$ with the initial condition $x(t_0)$ are restricted to the region Ω_1, where the following holds:

$$\Omega_1 = \left\{x(t) \leqslant \frac{(1+r)^2}{4\frac{r}{k}} + \varepsilon_1 = \delta_1, \varepsilon_1 > 0\right\}. \tag{6}$$

Based on the second equation of system (1),

$$D^{\alpha,\beta}_{0,w}y = \frac{a\,b\,x\,y}{\alpha_0+\alpha_1x+\alpha_2y} - c\,y - d\,y^2,$$

we have also $x(t)\leqslant\delta_1$ from (6), then the following obtains

$$
\begin{aligned}
D^{\alpha,\beta}_{a,w} y(t) + \frac{a\,b}{\alpha_1} y(t) &= \frac{a\,b\,x\,y}{\alpha_0 + \alpha_1 x + \alpha_2 y} - c\,y - d\,y^2 + \frac{a\,b}{\alpha_1} y \\
&\leqslant \frac{a\,b\,x\,y}{\alpha_1 x} - c\,y - d\,y^2 + \frac{a\,b}{\alpha_1} y \\
&\leqslant -d\,y^2 + 2\frac{a\,b}{\alpha_1} y - c\,y \\
&\leqslant -d\left(y^2 - 2\frac{a\,b}{d\,\alpha_1} y\right) - c\,y \\
&\leqslant -d\left(y^2 - 2\frac{a\,b}{d\,\alpha_1} y + \left(\frac{a\,b}{d\,\alpha_1}\right)^2\right) + d\left(\frac{a\,b}{d\,\alpha_1}\right)^2 - c\,y \\
&\leqslant -d\left(y - \frac{a\,b}{d\,\alpha_1}\right)^2 + d\left(\frac{a\,b}{d\,\alpha_1}\right)^2 - c\,y \\
&\leqslant \frac{(a\,b)^2}{d\,\alpha_1^2}.
\end{aligned}
$$

Again by using Lemma 2, we get

$$
y(t) \leqslant \left(y(t_0) - \frac{a\,b}{d\,\alpha_1}\right) E_\beta\left(\frac{-\alpha\lambda t^\beta}{N(\alpha) + \lambda(1-\alpha)}\right) + \frac{a\,b}{d\,\alpha_1} \to \frac{a\,b}{d\,\alpha_1}, \quad t \to \infty.
$$

Consequently, the solution $y(t)$ with $y(t_0)$ are confined within the region Ω_2, where

$$
\Omega_2 = \left\{ y(t) \leqslant \frac{a\,b}{d\,\alpha_1} \right\}. \qquad \blacksquare
$$

5. Equilibrium points and their stability

The aim of this section is on analytical findings, which encompass the identification of equilibria and an analysis of their stability.

5.1. Equilibrium points

Within this subsection, we prove the presence of equilibria in system (1). It is evident that the system (1) possesses two equilibria: $E^0(0,0)$ and $E^1(k,0)$. Biologically, $E^0(0,0)$ signifies the trivial equilibrium where both predator and prey are absent, while $E^1(k,0)$ corresponds to the predator–free axial equilibrium also known as the situation where the prey population attains its carrying capacity in the absence of predators. The remainder of the equilibrium points are governed by the subsequent equations:

$$
r\,x\left(1 - \frac{x}{k}\right) - \frac{a\,x\,y}{\alpha_0 + \alpha_1 x + \alpha_2 y} = 0, \tag{7}
$$

$$
\frac{a\,b\,x\,y}{\alpha_0 + \alpha_1 x + \alpha_2 y} - c\,y - d\,y^2 = 0. \tag{8}
$$

From (7),

$$
y = \frac{r(1 - \frac{x}{k})(\alpha_0 + \alpha_1 x)}{a - r\alpha_2(1 - \frac{x}{k})}.
$$

From (8),

$$
(c + d\,y)(\alpha_0 + \alpha_1 x + \alpha_2 y) = a\,b\,x. \tag{9}
$$

Substituting y into Eq. (9), we obtain the subsequent:

$$
A\,x^3 + B\,x^2 + C\,x + D = 0,
$$

where

$$\begin{aligned}
A &= \frac{a\,d\,k^2\alpha_1^2}{r^2\alpha_0^3} + \frac{a\,b\,k}{r\,\alpha_0^3}\alpha_2^2,\\
B &= -\frac{a\,c\,k\,\alpha_1\alpha_2}{r^2\alpha_0^3} - \frac{a\,d\,k\,\alpha_1}{r^2\alpha_0^3}(\alpha_1 k + 2\alpha_0) + \frac{2a\,b\,k}{r^2\alpha_0^3}(a\,\alpha_2 - r\,\alpha_2^2),\\
C &= -\frac{a\,c\,\alpha_2}{r\alpha_0^3} + \frac{a\,c\,k\,\alpha_1}{r^3\alpha_0^3}(r\alpha_2 - a) - \frac{a\,d}{r^2\alpha_0^2}(\alpha_0 + 2\alpha_1 k) + \frac{a\,b\,k}{r^3\alpha_0^3}(a^2 - r^2\alpha_2^2 - 2r\,a\,\alpha_2),\\
D &= \frac{a\,c}{r^3\alpha_0^2}(r\,\alpha_2 - a) - \frac{a\,d}{r^2\alpha_0}.
\end{aligned}$$

Let the subsequent function:

$$f(x) = A\,x^3 + B\,x^2 + C\,x + D.$$

Since $A > 0$, we have $\lim\limits_{x\to+\infty} f(x) = +\infty$. Furthermore, $f(0) = D < 0$, if $\alpha_2 \in (-\infty, \frac{a}{r})$. Consequently, there exists a $x^* \in (0, +\infty)$ such that $f(x^*) = 0$. This implies that system (1) possesses one and only one interior coexistence equilibrium $E^*(x^*, y^*)$, where $x^* \in (0, +\infty)$ and $y^* = \frac{r\left(1-\frac{x^*}{k}\right)(\alpha_0+\alpha_1 x^*)}{a-r\,\alpha_2\left(1-\frac{1}{k}x^*\right)}$.

5.2. Stability analysis

Within this subsection, we study the stability of the model (1).

Consider an arbitrary equilibria $(\bar{x}, \bar{y})$ of system (1). We put

$$\begin{aligned}
f_1(x,y) &= r\,x\left(1 - \frac{x}{k}\right) - \frac{a\,x\,y}{\alpha_0 + \alpha_1 x + \alpha_2 y},\\
f_2(x,y) &= \frac{a\,b\,x\,y}{\alpha_0 + \alpha_1 x + \alpha_2 y} - c\,y - d\,y^2.
\end{aligned}$$

Using the change of variable following $X = x - \bar{x}$ and applying Taylor's formula, we obtain

$$\begin{aligned}
D_{0,w}^{\alpha,\beta} f_1(x,y) &= f_1(X + \bar{x}, Y + \bar{y})\\
&= f_1(\bar{x},\bar{y}) + \frac{df_1}{dx}(\bar{x},\bar{y})X + \frac{df_1}{dy}(\bar{x},\bar{y})Y\\
&= \frac{df_1}{dx}(\bar{x},\bar{y})X + \frac{df_1}{dy}(\bar{x},\bar{y})Y.
\end{aligned}$$

And

$$\begin{aligned}
D_{0,w}^{\alpha,\beta} f_2(x,y) &= f_2(X + \bar{x}, Y + \bar{y})\\
&= f_2(\bar{x},\bar{y}) + \frac{df_2}{dx}(\bar{x},\bar{y})X + \frac{df_2}{dy}(\bar{x},\bar{y})Y\\
&= \frac{df_2}{dx}(\bar{x},\bar{y})X + \frac{df_2}{dy}(\bar{x},\bar{y})Y,
\end{aligned}$$

where

$$\begin{aligned}
\frac{df_1}{dx}(\bar{x},\bar{y}) &= r\left(1 - \frac{2\bar{x}}{k}\right) - \frac{a\,\bar{y}(\alpha_0 + \alpha_2\bar{y})}{(\alpha_0 + \alpha_1\bar{x} + \alpha_2\bar{y})^2},\\
\frac{df_1}{dy}(\bar{x},\bar{y}) &= -a\frac{(\alpha_0 + \alpha_1\bar{x})}{(\alpha_0 + \alpha_1\bar{x} + \alpha_2\bar{y})^2}\bar{x},\\
\frac{df_2}{dx}(\bar{x},\bar{y}) &= b\,r\,\frac{(1 - \frac{\bar{x}}{k})(\alpha_0 + \alpha_2\bar{y})}{\alpha_0 + \alpha_1\bar{x} + \alpha_2\bar{y}},\\
\frac{df_2}{dy}(\bar{x},\bar{y}) &= a\,b\,\frac{\alpha_0 + \alpha_1\bar{x}}{(\alpha_0 + \alpha_1\bar{x} + \alpha_2\bar{y})^2}\bar{x} - c - 2d\,\bar{y},\\
&= -\frac{a\,b\,\alpha_2\bar{x}\,\bar{y}}{(\alpha_0 + \alpha_1\bar{x}) + \alpha_2\bar{y})^2} - d\,\bar{y}.
\end{aligned}$$

The system (1) becomes

$$\begin{cases} D_{0,w}^{\alpha,\beta} x = A\,X + B\,Y, \\ D_{0,w}^{\alpha,\beta} y = C\,X + D\,Y, \end{cases} \tag{10}$$

where

$$A = \frac{df_1}{dx}(\bar{x},\bar{y}), \quad B = \frac{df_1}{dy}(\bar{x},\bar{y}), \quad C = \frac{df_2}{dx}(\bar{x},\bar{y}), \quad D = \frac{df_2}{dy}(\bar{x},\bar{y}).$$

According to the first equation of (10), we obtain

$$w(t)\,D_{0,w}^{\alpha,\beta} X(t) = A\,w(t)\,X(t) + B\,w(t)\,Y(t). \tag{11}$$

We put

$$\tilde{X}(S) = \mathcal{L}\{\omega(t)X(t)\}(S), \tag{12}$$

$$\tilde{Y}(S) = \mathcal{L}\{\omega(t)Y(t)\}(S). \tag{13}$$

By applying Laplace transform to (11),

$$\mathcal{L}\big\{w(t)\,D_{0,w}^{\alpha,\beta} X(t)\big\}(S) = \mathcal{L}\big\{A\,w(t)\,X(t) + B\,w(t)\,Y(t)\big\}(S).$$

Then

$$N(\alpha)\big(S^{\beta}\tilde{X}(S) - S^{\beta-1}w(0)X(0)\big) = \big(S^{\beta}+\mu_\alpha\big)(1-\alpha)A\tilde{X}(S) + (1-\alpha)\big(S^{\beta}+\mu_\alpha\big)B\tilde{Y}(S).$$

As a result

$$\big(N(\alpha)S^{\beta} - \big(S^{\beta}+\mu_\alpha\big)(1-\alpha)A\big)\tilde{X}(S) - \big(S^{\beta}+\mu_\alpha\big)(1-\alpha)B\,\tilde{Y}(S) = N(\alpha)\,\omega(t_0)\,X(t_0)\,S^{\beta-1}.$$

Similarly for the second equation of system (10),

$$N(\alpha)\big(S^{\beta}\tilde{Y}(S) - S^{\beta-1}w(0)\,Y(0)\big) = \big(S^{\beta}+\mu_\alpha\big)(1-\alpha)\big(C\,\tilde{X}(S) + D\,\tilde{Y}(S)\big).$$

Consequently,

$$-(1-\alpha)\big(S^{\beta}+\mu_\alpha\big)C\tilde{X}(S) + \big(S^{\beta}N(\alpha) - D\big(S^{\beta}+\mu_\alpha\big)(1-\alpha)\big)\tilde{Y}(S) = N(\alpha)\,S^{\beta-1}\omega(t_0)\,Y(t_0).$$

Then the system (10) becomes

$$\begin{cases} \big(N(\alpha)S^{\beta} - (1-\alpha)\big(S^{\beta}+\mu_\alpha\big)A\big)\tilde{X}(S) - (1-\alpha)\big(S^{\beta}+\mu_\alpha\big)B\tilde{Y}(S) = N(\alpha)\omega(t_0)X(t_0)S^{\beta-1}, \\ -(1-\alpha)\big(S^{\beta}+\mu_\alpha\big)C\tilde{X}(S) + \big(N(\alpha)S^{\beta} - D(1-\alpha)(S^{\beta}+\mu_\alpha)\big)\tilde{Y}(S) = N(\alpha)S^{\beta-1}\omega(t_0)Y(t_0). \end{cases}$$

This implies that

$$\Delta(S)\,\tilde{Z}(s) = p,$$

where

$$\Delta(S) = \begin{pmatrix} N(\alpha)S^{\beta} - (1-\alpha)\big(S^{\beta}+\mu_\alpha\big)A & -(1-\alpha)\big(S^{\beta}+\mu_\alpha\big)B \\ -(1-\alpha)\big(S^{\beta}+\mu_\alpha\big)C & N(\alpha)S^{\beta} - D(1-\alpha)\big(S^{\beta}+\mu_\alpha\big) \end{pmatrix},$$

$$\tilde{Z}(S) = \begin{pmatrix} \tilde{X}(S) \\ \tilde{Y}(S) \end{pmatrix}, \qquad p = N(\alpha)\,S^{\beta-1}\omega(t_0)\begin{pmatrix} X(t_0) \\ Y(t_0) \end{pmatrix}.$$

Theorem 5. *The equilibrium $E^0(0,0)$ is unstable if $N(\alpha) - (1-\alpha)r > 0$.*

Proof. At $E^0(0,0)$, we have

$$A = \frac{df_1}{dx}(0,0) = r, \quad B = \frac{df_1}{dy}(0,0) = 0, \quad C = \frac{df_2}{dx}(0,0) = 0, \quad D = \frac{df_2}{dy}(0,0) = -c.$$

And

$$\begin{aligned} \det(\Delta(S)) = 0 &\Longleftrightarrow \begin{vmatrix} N(\alpha)S^{\beta} - (1-\alpha)\big(S^{\beta}+\mu_\alpha\big)r & 0 \\ 0 & N(\alpha)S^{\beta} + c(1-\alpha)\big(S^{\beta}+\mu_\alpha\big) \end{vmatrix} = 0, \\ &\Longleftrightarrow \big(N(\alpha)S^{\beta} - (1-\alpha)\big(S^{\beta}+\mu_\alpha\big)r\big)\big(N(\alpha)S^{\beta} + c(1-\alpha)\big(S^{\beta}+\mu_\alpha\big)\big) = 0, \\ &\Longleftrightarrow \big(N(\alpha)\lambda - (1-\alpha)(\lambda+\mu_\alpha)r\big)\big(N(\alpha)\lambda + c(1-\alpha)(\lambda+\mu_\alpha)\big) = 0 \quad (\text{with } \lambda = S^{\beta}). \end{aligned} \tag{14}$$

This implies that

$$\big(N(\alpha)\lambda-(1-\alpha)(\lambda+\mu_\alpha)r\big)=0 \quad \text{or} \quad N(\alpha)\lambda+c(1-\alpha)(\lambda+\mu_\alpha)=0,$$

where the roots of Eq. (14) are $\lambda_1=\frac{\alpha r}{N(\alpha)-(1-\alpha)r}$ and $\lambda_2=-\frac{c\,\mu_\alpha(1-\alpha)}{N(\alpha)+c(1-\alpha)}<0$. Assume that $N(\alpha)-(1-\alpha)r>0$, then $\lambda_1>0$.

This implies that $E^0(0,0)$ is unstable if $N(\alpha)-(1-\alpha)r>0$. ■

Theorem 6. *Let $\mathcal{R}_0=\frac{a\,b\,k}{c(\alpha_0+\alpha_1 k)}$. The equilibrium $E^1(k,0)$ is locally asymptotically stable if $\mathcal{R}_0<1$, and becomes unstable if $\mathcal{R}_0>1$.*

Proof. At $E^1(k,0)$, we have

$$A=\frac{df_1}{dx}(k,0)=-r, \quad B=\frac{df_1}{dy}(k,0)=-\frac{c}{b}\mathcal{R}_0, \quad C=\frac{df_2}{dx}(k,0)=0, \quad D=\frac{df_2}{dx}(k,0)=\frac{c}{k}(\mathcal{R}_0-1).$$

And

$$\det(\Delta(S))=0 \Longleftrightarrow \begin{vmatrix} N(\alpha)S^\beta+(1-\alpha)\big(S^\beta+\mu_\alpha\big)r & \frac{c}{b}\mathcal{R}_0(1-\alpha)(\lambda+\mu_\alpha) \\ 0 & N(\alpha)\lambda-\frac{c}{k}(\mathcal{R}_0-1)(\lambda+\mu_\alpha) \end{vmatrix}=0,$$

$$\Longleftrightarrow \big(N(\alpha)S^\beta+(1-\alpha)\big(S^\beta+\mu_\alpha\big)r\big)\big(N(\alpha)\lambda-\frac{c}{k}(\mathcal{R}_0-1)(\lambda+\mu_\alpha)\big)=0,$$

$$\Longleftrightarrow \big(N(\alpha)\lambda+(1-\alpha)(\lambda+\mu_\alpha)r\big)\Big(N(\alpha)\lambda-\frac{c}{k}(\mathcal{R}_0-1)(\lambda+\mu_\alpha)\Big)=0 \quad (\text{with } \lambda=S^\beta), \tag{15}$$

which leads to

$$\big(N(\alpha)\lambda+(1-\alpha)(\lambda+\mu_\alpha)r\big)=0 \quad \text{or} \quad \Big(N(\alpha)\lambda-\frac{c}{k}(\mathcal{R}_0-1)(\lambda+\mu_\alpha)\Big)=0,$$

where the roots of Eq. (15) are

$$\lambda_1=-\frac{(1-\alpha)\mu_\alpha r}{N(\alpha)+(1-\alpha)r}<0 \quad \text{and} \quad \lambda_2=\frac{\frac{c}{k}(\mathcal{R}_0-1)}{N(\alpha)-\frac{c}{k}(\mathcal{R}_0-1)}.$$

Assume that $\mathcal{R}_0<1$, we have $\lambda_2<0$. Then $E^1(k,0)$ is locally asymptotically stable. ■

For the equilibrium point $E^*(x^*,y^*)$, we can express the characteristic equation as follows:

$$a_0\lambda^2+a_1\lambda+a_2=0, \tag{16}$$

where

$$\begin{aligned} a_0&=N^2(\alpha)-N(\alpha)(1-\alpha)(D+A)+(1-\alpha)^2(AD-CB),\\ a_1&=-N(\alpha)(1-\alpha)\mu_\alpha(D+A)+2\mu_\alpha(1-\alpha)^2(AD-CB),\\ a_2&=(AD-CB)\,\mu_\alpha^2(1-\alpha)^2. \end{aligned}$$

And

$$\begin{aligned} D+A&=\frac{a\,\bar{x}\,\bar{y}(\alpha_1-b\,\alpha_2)}{(\alpha_0+\alpha_1\bar{x}+\alpha_2\bar{y})^2}-d\,\bar{y}-\frac{r}{k}\bar{x},\\ AD-CB&=\frac{r\,d}{k}\,\bar{x}\,\bar{y}+\frac{r\,b\,\alpha_2\bar{x}-k\,d\,\alpha_1\bar{y}}{k(\alpha_0+\alpha_1\bar{x}+\alpha_2\bar{y})^2}\,a\,\bar{x}\,\bar{y}+\frac{a\,b\,\alpha_0}{(\alpha_0+\alpha_1\bar{x}+\alpha_2\bar{y})^3}\,a\,\bar{x}\,\bar{y}. \end{aligned}$$

We assume that $a_0\neq 0$, Eq. (16) reduces to

$$\lambda^2+\beta_1\lambda+\beta_2=0, \tag{17}$$

where $\beta_1=\frac{a_1}{a_0}$ and $\beta_2=\frac{a_2}{a_0}$.

Now, according to the Routh–Hurwitz criterion, the equation (17) will have all its roots with negative real parts if and only if

$$\beta_1>0, \quad \beta_2>0. \tag{18}$$

Therefore, we have the following result.

Theorem 7. *The equilibria $E^*(x^*,y^*)$ is locally asymptotically stable if the condition* (18) *satisfied.*

Let $\Omega=\big\{(x,y)\in\mathbb{R}_+^2/x\leqslant k\big\}$. Assume that $\mathcal{R}_0<1$ and take into consideration the subsequent Lyapunov function:

$$L(x,y)=y(t).$$

Clearly, L is a candidate Lyapunov function. In fact, $L(E^1(k,0))=0$ and $L(x,y)>0$, $\forall (x,y)\in \Omega\backslash\{E^1(k,0)\}$. Additionally,

$$\begin{aligned}
D_{0,w}^{\alpha,\beta}L(x,y) &= D_{0,w}^{\alpha,\beta}y(t)\\
&= \frac{a\,b\,x(t)\,y(t)}{\alpha_0+\alpha_1x(t)+\alpha_2y(t)} - c\,y(t) - d\,y^2(t)\\
&\leqslant \frac{a\,b\,x(t)\,y(t)}{\alpha_0+\alpha_1x(t)+\alpha_2y(t)} - c\,y(t)\\
&\leqslant c\left(\frac{a\,b\,x(t)\,y(t)}{c(\alpha_0+\alpha_1x(t)+\alpha_2y(t))}-1\right)y(t)\\
&\leqslant c\left(\frac{a\,b\,k}{c(\alpha_0+\alpha_1k)}-1\right)L(x,y)\\
&\leqslant c(\mathcal{R}_0-1)\,L(x,y).
\end{aligned}$$

Then $D_{0,w}^{\alpha,\beta}L(x,y)\leqslant 0$. By applying Theorem 5 in [23], we conclude that the equilibria $E^1(k,0)$ of (1) is globally stable in Ω when $\mathcal{R}_0\leqslant 1$.

6. Conclusion

Within this article, we have introduced a predator–prey model with fractional-order that utilizing the GHF derivative. Firstly, we have rigorously established the model's mathematical and ecological validity, confirming its well-posed nature. Moreover, we have managed to show that the system under study has one and only one solution. Additionally, we have identified three equilibrium points in the proposed model. These include the trivial equilibrium denoting the lack of prey and predator, denoted as $E^0(0,0)$. Another equilibrium point is the predator-free axial equilibrium, which occurs when the prey population attains its carrying capacity when predators do not exist, denoted as $E^1(k,0)$. The third equilibrium, denoted as $E^*(x^*,y^*)$, represents the interior coexistence equilibrium. We have found that the predator–free axial equilibrium $E^1(k,0)$ exhibits local asymptotic stability when the threshold parameter $\mathcal{R}_0<1$. However, it becomes unstable when $\mathcal{R}_0>1$. Additionally, we have shown that the interior equilibrium point E^* is locally asymptotically stable under certain criteria. Finally, by employing the Lyapunov theorem as referenced in [23], we have demonstrated the global stability of the predator–free axial equilibrium under the condition $\mathcal{R}_0\leqslant 1$.

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Динаміка екологічної моделі "жертва–хижак" на основі узагальненої дробової похідної Хаттафа

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У цій статті запропоновано та проаналізовано дробову модель "жертва-хижак" з узагальненою дробовою похідною Хаттафа (GHF). Доведено, що запропонована модель є екологічно та математично правильною. Крім того, показано, що ця модель має три точки рівноваги. Накінець, встановлено стійкість цих рівноваг.

Ключові слова: *екологія; математичне моделювання; здобич–хижак; дробова похідна Хаттафа; функція Ляпунова; стійкість.*