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A Lévy process approach coupled to the stochastic Leslie–Gower model

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This paper focuses on a two-dimensional Leslie–Grower continuous-time stochastic predator–prey system with Lévy jumps. Firstly, we prove that there exists a unique positive solution of the system with a positive initial value. Then, we establish sufficient conditions for the mean stability and extinction of the considered system. Numerical algorithms of higher order are elaborated. The obtained results show that Lévy jumps significantly change the properties of population systems.

Keywords: *Leslie–Gower; SDEs; Itô's formula; extinction; stochastic predator–prey model; Taylor method.*

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1. Introduction

The predator–prey interaction is one of the basic relationships in ecological models and is also a basic building block of the more complex food chain, food web and biochemical web structure [1]. In 1926, Volterra [2] proposed a differential equation model to explain the oscillatory levels of some fish catches in the Adriatic. Lotka [3] also derived the model to describe a hypothetical chemical reaction in which chemical concentrations oscillate in 1925. Inspired by the Lotka–Volterra equations, several ecologists and mathematicians elaborated important models in [4, 5], Leslie introduced a predator–prey model where the carrying capacity of the predator environment is proportional to the number of prey. Leslie points out that there are upper bounds on the rates of increase of prey and predators that are not recognized in the Lotka–Volterra model and presented the following predator–prey model:

$$\begin{cases} \dfrac{dH_t}{dt} = H_t\left(r_1 - a_1 P - b_1 H_t\right), \\ \dfrac{dP_t}{dt} = P_t\left(r_2 - \dfrac{a_2 P_t}{H_t}\right), \end{cases} \tag{1}$$

where H_t and P_t represent the densities of prey and predator populations at time t respectively, r_1 is the intrinsic growth rate of the prey, b_1 represents the effect of interspecific prey competition in the absence of a predator, a_1HP is the functional response of the predator to the prey, r_2 is the intrinsic growth rate of the predator, a_2 is a measure of the amount of food the prey provides for conversion to a predator birth. The term $\frac{a_2P_t}{H_t}$ called Leslie-Gowere term.

The deterministic model (1) is studied by Korobeinikov [6]. In [5], Leslie and Gower considered a stochastic model of the prey–predator system and examined its properties in a constrained environment where the assumption of sufficient food supply for the prey is taken into account. In the article [7] Lahrouz et al. studied the random extension of the predator–prey model (1). This paper focuses on the stochastic effects of noise associated with a population environment acting on prey and predator crossover rates, respectively. These perturbations are modeled using independent white Gaussian noise. Lahrouz et al. [7] studied the system (1) with an impulse, taking into account the white noise, by considering the following stochastic model

$$\begin{cases} dH_t = H_t\,(r_1 - a_1 P_t - b_1 H_t)\,dt + \sigma_1 H_t\,dB_1(t), \\ dP_t = P_t\left(r_2 - \dfrac{a_2 P_t}{H_t}\right)dt + \sigma_2 P_t\,dB_2(t). \end{cases} \tag{2}$$

Where $(B_1(t), B_2(t))$ are mutually independent Brownian motions defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $\{\mathcal{F}_t\}_{t\in\mathbb{R}_+}$ satisfying the usual conditions (i.e., it is right continuous and increasing while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets), the parameters (σ_1, σ_2) represent the intensity of the perturbation. Actually, population systems may suffer sudden environmental perturbations, such as epidemics, earthquakes, hurricanes, etc. These phenomena cannot be modeled by the stochastic system (2). Bao et al. [8] suggested that these phenomena can be described by a Lévy jump process and they considered stochastic Lotka–Volterra population systems with jumps for the first time. Motivated by these, we consider in this paper the following more general stochastic model:

$$\begin{cases} dH_t = H_t\,(r_1 - a_1 P_t - b_1 H_t)\,dt + \sigma_1 H_t\,dB_1 + \displaystyle\int_{\mathbb{Z}} H_{t^-}\gamma_1(u)\,\tilde{N}(dt,du), \\ dP_t = P_t\left(r_2 - \dfrac{a_2 P_t}{H_t}\right)dt + \sigma_2 P_t\,dB_2 + \displaystyle\int_{\mathbb{Z}} P_{t^-}\gamma_2(u)\,\tilde{N}(dt,du) \end{cases} \tag{3}$$

where H_{t^-} and P_{t^-} are the left limit of H_t and P_t respectively, N is a Poisson counting measure with characteristic λ on a measurable subset $\mathbb{Z}$ of $(0, +\infty)$ with $\lambda(\mathbb{Z}) < +\infty$, $\tilde{N}(dt, du) = N(dt, du) - \lambda(du)dt$, $\gamma_i \colon \mathbb{Z} \times \Omega \to \mathbb{R}$ is bounded and continuous with respect to λ, and is $\mathfrak{B}(Z) \times \mathcal{F}_t$-measurable, $i = 1, 2$.

2. Main results

Throughout this paper, as a standing hypothesis we assume that N, $B_1(t)$ and $B_2(t)$ are independent, we also assume that $1 + \gamma_i(u) > 0$, $u \in \mathbb{Z}$, $i = 1, 2$ and there is a constant $c > 0$ such that

$$\int_{\mathbb{Z}} \left[\ln\left(1 + \gamma_i(u)\right)\right]^2 \lambda(du) < c. \tag{4}$$

For simplicity, we introduce the following notations,

$$\beta_i = 0.5\sigma_i^2 + \int_{\mathbb{Z}} \left[\gamma_i(u) - \ln\left(1 + \gamma_i(u)\right)\right]\lambda(du), \quad i = 1, 2$$

$$\mathbb{R}^2_+ = \left\{\xi \in R^2 \mid \xi_i > 0, i = 1, 2\right\},$$

$$k_i(t) = \int_0^t \int_{\mathbb{Z}} \ln\left(1 + \gamma_i(u)\right)\tilde{N}(ds, du), \quad i = 1, 2.$$

Let $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant 0}, \mathbb{P}\right)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$ verifying the usual conditions.

Firstly, let us recall two lemmas that will be used later

Lemma 1 (Ref. [1]). *Suppose that $z(t) \in C\left(\Omega \times [0, +\infty), \mathbb{R}_+\right)$,*

1. *If there exist three positive constants T, ρ and ρ_0 such that*

$$\ln z(t) \leqslant \rho\, t - \rho_0 \int_0^t z(s)\,ds + \sum_{i=1}^{2} \sigma_i B_i(t)$$

for all $t \geqslant T$, where both σ_1 and σ_2 are constants, then

$$\langle z\rangle^* = \limsup_{t\to+\infty} t^{-1} \int_0^t z(s)\,ds \leqslant \rho/\rho_0 \quad \text{a.s.}$$

2. *If there exist three positive constants T, ρ and ρ_0 such that*

$$\ln z(t) \geqslant \rho\, t - \rho_0 \int_0^t z(s)\,ds + \sum_{i=1}^{2} \sigma_i B_i(t)$$

for all $t \geqslant T$, then

$$\langle z\rangle_* = \liminf_{t\to+\infty} t^{-1}\int_0^t z(s)\,ds \geqslant \rho/\rho_0 \quad \text{a.s.}$$

Lemma 2 (Ref. [2]). *Suppose that $M(t)$, $t \geqslant 0$, is a local martingale cancelling at time zero then:*

$$\lim_{t\to+\infty} \rho_M(t) < +\infty \Rightarrow \lim_{t\to+\infty} \frac{M(t)}{t} = 0 \quad \text{a.s.},$$

where

$$\rho_M(t) = \int_0^t \frac{d\langle M,M\rangle(s)}{(1+s)^2}, \quad t \geqslant 0$$

and $\langle M,M\rangle$ is Meyer's angle bracket process (see [9,10]).

In order to study the properties of the solution (H_t,P_t), we must first show the global existence, uniqueness and positivity of the solution of the stochastic model (3).

Theorem 1. *For any initial value $(H_0,P_0)\in\mathbb{R}^2_+$, the model (3) has a unique solution (H_t,P_t) for $t\geqslant 0$ and the solution will remain in $\mathbb{R}^2_+$ a.s.*

Proof. We pose

$$X_t = \ln H_t \quad \text{and} \quad Y_t = \ln P_t. \tag{5}$$

According to Itô's formula:

$$\begin{aligned}
dX_t = d\ln H_t &= \frac{\partial \ln H_t}{\partial H_t}\left[H_t\left((r_1 - a_1P_t - b_1H_t)\,dt + \sigma_1 dB_1(t)\right)\right] \\
&\quad + \int_{\mathbb{Z}} \left[\ln\left(H_t + \gamma_1(u)H_t\right) - \ln H_t\right]\tilde{N}(dt,du) \\
&\quad + \int_{\mathbb{Z}} \left[\ln\left(H_t + \gamma_1(u)H_t\right) - \ln H_t - \frac{\partial \ln H_t}{\partial H_t}\gamma_1(u)H_t\right]\lambda(du)\,dt + \frac{1}{2}\frac{\partial^2 \ln H_t}{\partial H_t^2}\sigma_1^2 H_t^2\,dt \\
&= (r_1 - a_1P_t - b_1H_t)\,dt + \sigma_1 dB_1(t) + \int_{\mathbb{Z}}\left[\ln\left(1+\gamma_1(u)\right)\right]\tilde{N}(dt,du) \\
&\quad + \int_{\mathbb{Z}}\left[\ln\left(1+\gamma_1(u)\right) - \gamma_1(u)\right]\lambda(du)\,dt - \frac{1}{2}\sigma_1^2\,dt \\
&= (r_1 - a_1P_t - b_1H_t - \beta_1)\,dt + \sigma_1 dB_1(t) + \int_{\mathbb{Z}}\left[\ln\left(1+\gamma_1(u)\right)\right]\tilde{N}(dt,du)
\end{aligned}$$

From (5)

$$H_t = \exp X_t \quad \text{and} \quad P_t = \exp Y_t$$

Even for Y_t, the equation (3) becomes:

$$\begin{cases}
dX_t = \left(r_1 - \beta_1 - a_1 e^{Y_t} - b_1 e^{X_t}\right)dt + \sigma_1\,dB_1(t) + \displaystyle\int_{\mathbb{Z}} \ln\left(1+\gamma_1(u)\right)\tilde{N}(dt,du) \\
dY_t = \left(r_2 - \beta_2 - a_2 e^{Y_t - X_t}\right)dt + \sigma_2\,dB_2(t) + \displaystyle\int_{\mathbb{Z}} \ln\left(1+\gamma_2(u)\right)\tilde{N}(dt,du)
\end{cases} \tag{6}$$

with initial conditions $X_0 = \ln H_0$ and $Y_0 = \ln P_0$.

Clearly, the coefficients of system (6) satisfy the local Lipschitz condition, then there is a unique local solution (X_t,Y_t) on $[0,\tau_e)$, where τ_e is the explosion time (see Mao [11]). Therefore $H_t = e^{X_t}$, $P_t = e^{Y_t}$ is the unique positive local solution to system (3) with initial data (H_0,P_0).

To complete the proof, we have to show that $\tau_e = \infty$.

Consider the following two stochastic equations

$$dM_t = M_t\left(r_1 - b_1M_t\right)dt + \sigma_1 M_t\,dB_1(t) + \int_{\mathbb{Z}} M_{t^-}\gamma_1(u)\tilde{N}(dt,du), \tag{7}$$

$$d\psi(t) = \psi(t)\left(r_2 - a_2\psi(t)\right)dt + \sigma_2\psi(t)\,dB_2(t) + \int_{\mathbb{Z}}\psi\left(t^-\right)\gamma_2(u)\tilde{N}(dt,du), \tag{8}$$

$$dN_t = N_t\left(r_2 - \frac{a_2N_t}{M_t}\right)dt + \sigma_2 N_t\,dB_2(t) + \int_{\mathbb{Z}} N_{t^-}\gamma_2(u)\tilde{N}(dt,du) \tag{9}$$

with initial conditions $N_0 = H_0$, $\psi(0) = P_0$, $M_0 = P_0$.

The logistic equations (7) and (9) admit a single continuous positive solution for any initial value $N_0 = H_0 > 0$, $M_0 = P_0 > 0$.

By the comparison theorem for the stochastic equation [12], we get for $t \in [0, \tau_e)$

$$H_t \leqslant M_t \quad \text{and} \quad \psi(t) \leqslant P_t \leqslant N_t \quad \text{a.s.} \tag{10}$$

According to the Lemma:4 − 2 of [13] the equation (7) has the explicit formula:

$$M_t = \frac{\exp\left\{(r_1 - \beta_1)\, t + \sigma_1 B_1(t) + k_1(t)\right\}}{\frac{1}{H_0} + b_1 \int_0^t \exp\left\{(r_1 - \beta_1)\, s + \sigma_1 B_1(s) + k_1(s)\right\} ds}. \tag{11}$$

Similarly,

$$\psi(t) = \frac{\exp\left\{(r_2 - \beta_2)\, t + \sigma_2 B_2(t) + k_2(t)\right\}}{y_0^{-1} + a_2 \int_0^t \exp\left\{(r_2 - \beta_2)\, s + \sigma_2 B_2(s) + k_2(s)\right\} ds}, \tag{12}$$

$$N_t = \frac{\exp\left\{(r_2 - \beta_2)\, t + \sigma_2 B_2(t) + k_2(t)\right\}}{\frac{1}{P_0} + a_2 \int_0^t \frac{1}{M_s} \exp\left\{(r_2 - \beta_2)\, s + \sigma_2 B_2(s) + k_2(s)\right\} ds}, \tag{13}$$

where

$$k_i(t) = \int_0^t \int_{\mathbb{Z}} \ln\left(1 + \gamma_i(u)\right) \bar{N}(ds, du), \quad i = 1, 2.$$

Now, suppose that $\tau_e < \infty$, then there exists $T > 0$ such that $\mathbb{P}\left(\tau_e < T\right) > 0$ and let $w \in (\tau_e < T)$. According to the theorem A.2 of [14],

$$\lim_{t \to \tau_e} \sup \|(H_{\tau_e}, P_{\tau_e})\| = \infty$$

with (10) this implies that

$$\infty = \limsup_{t \to \tau_e} \|(H_{\tau_e}, P_{\tau_e})\| \leqslant \|(M_{\tau_e}, N_{\tau_e})\| < \infty,$$

which is a contradiction. Finally, we have $\tau_e = \infty$ a.s. ■

Theorem 2. *Let (H_t, P_t) be the solution of the SDE (3) of initial value $(H_0, P_0) \in \mathbb{R}^2_+$. Then for all $m > 0$, there exists $C(m) \in (0, \infty)$ such that*

$$\sup_{t \geqslant 0} \mathbb{E}\left(H_t^m + P_t^m\right) \leqslant C(m).$$

Proof. Since H_t, N_t are now defined on $[0, \infty)$, we can write from (10) that

$$H_t \leqslant N_t \quad \text{for all} \quad t \geqslant 0. \tag{14}$$

On the other hand, we know that the solutions of the stochastic logistic equation (7) verify for any $m > 0$

$$\sup_{t \geqslant 0} \mathbb{E} N_t^m < \infty. \tag{15}$$

From (14) and (15) we get

$$\sup_{t \geqslant 0} \mathbb{E} H_t^m \leqslant \sup_{t \geqslant 0} \mathbb{E} N_t^m < \infty.$$

Therefore, to complete the proof, it is sufficient to show that $\sup_{t \geqslant 0} \mathbb{E} P_t^m < \infty$.

Applying Itô's formula to P_t^m we get:

$$dP_t^m = \frac{\partial P_t^m}{\partial P_t}\left[P_t\left(\left(r_2 - \frac{a_2 P_t}{H_t}\right) + \sigma_2\, dB_2\right)\right] dt + \int_{\mathbb{Z}} \left[(P_t + \gamma_2(u) P_t)^m - P_t^m\right] \tilde{N}(dt, du)$$
$$+ \int_{\mathbb{Z}} \left((P_t + \gamma_2(u) P_t)^m - P_t^m - \frac{\partial P_t^m}{\partial P_t} \gamma_2(u) P_t\right) \nu\,(du)\, dt + \frac{1}{2} \frac{\partial^2 P_t^m}{\partial P_t^2} \sigma_2^2 P_t^2 dt,$$

which gives

$$dP_t^m = \left[m P_t^m \left(r_2 - \frac{a_2 P_t}{H_t}\right) + \frac{1}{2} m(m-1) \sigma_2^2 P_t^m\right] dt + m P_t^m \sigma_2\, dB_2$$

$$+ P_t^m \int_{\mathbb{Z}} [(1+\gamma_2(u))^m - 1]\, \tilde{N}(dt,du) + P_t^m \int_{\mathbb{Z}} ((1+\gamma_2(u))^m - 1 - m\gamma_2(u))\, \lambda\,(du)\, dt.$$

We pose

$$K(m) = \int_{\mathbb{Z}} ((1+\gamma_2(u))^m - 1 - m\gamma_2(u))\, \nu\,(du)$$

and deduce that

$$dP_t^m = P_t^m \left[\left(m\left(r_2 - \frac{a_2 P_t}{H_t}\right) + \frac{1}{2}m(m-1)\sigma_2^2 + K(m) \right) dt + m\,\sigma_2\, dB_2 \right]$$
$$+ P_t^m \int_{\mathbb{Z}} [(1+\gamma_2(u))^m - 1]\, \tilde{N}(dt,du).$$

By introducing the expectation of two terms of this equation one gets

$$\begin{aligned} \frac{d}{dt}\mathbb{E}P_t^m &= \mathbb{E}\left[m\left(r_2 + \frac{1}{2}(m-1)\sigma_2^2 + K(m)\right) P_t^m - m\frac{a_2}{H_t}P_t^{m+1} \right] \\ &= \mathbb{E}\left[\left(mr_2 + \frac{1}{2}m(m-1)\sigma_2^2 + K(m) + \varepsilon \right) P_t^m - m\frac{a_2}{H_t}P_t^{m+1} \right] - \varepsilon\mathbb{E}P_t^m, \end{aligned} \tag{16}$$

where $\varepsilon > 0$.

By studying the function $Ax^m - Bx^{m+1}$ where $A, B > 0$, it is easy to see that

$$\max_{x>0}\left(Ax^m - Bx^{m+1}\right) = \frac{m^m}{(m+1)^{m+1}}\frac{A^{m+1}}{B^m}.$$

Hence

$$\left(mr_2 + \frac{1}{2}m(m-1)\sigma_2^2 + K(m) + \varepsilon \right) P_t^m - m\frac{a_2}{H_t}P_t^{m+1}$$
$$\leqslant \frac{m^m}{(m+1)^{m+1}} \frac{\left(mr_2 + \frac{1}{2}m(m-1)\sigma_2^2 + K(m) + \varepsilon\right)^{m+1}}{(ma_2)^m} H_t^m.$$

Introducing the above inequality into equation (16) leads t:

$$\frac{d}{dt}\mathbb{E}P_t^m \leqslant \frac{\left(mr_2 + \frac{1}{2}m(m-1)\sigma_2^2 + K(m) + \varepsilon\right)^{m+1}}{a_2^m (m+1)^{m+1}} \mathbb{E}H_t^m - \varepsilon\mathbb{E}P_t^m.$$

Since $\sup_{t\geqslant 0} \mathbb{E}H_t^m < \infty$, we deduce that

$$\frac{d}{dt}\mathbb{E}P_t^m \leqslant \frac{\left(mr_2 + \frac{1}{2}m(m-1)\sigma_2^2 + K(m) + \varepsilon\right)^{m+1}}{a_2^m (m+1)^{m+1}} \sup_{t\geqslant 0}\mathbb{E}H_t^m - \varepsilon\mathbb{E}P_t^m.$$

From the previous results and the comparison theorem [12], we obtain for any $\varepsilon > 0$:

$$\limsup_{t\to\infty} \mathbb{E}P_t^m \leqslant \frac{\left(mr_2 + \frac{1}{2}m(m-1)\sigma_2^2 + K(m) + \varepsilon\right)^{m+1}}{a_2^m (m+1)^{m+1}\varepsilon} \sup_{t\geqslant 0}\mathbb{E}H_t^m$$

So,

$$\limsup_{t\to\infty} \mathbb{E}P_t^m \leqslant \frac{\sup_{t\geqslant 0}\mathbb{E}H_t^m}{a_2^m (m+1)^{m+1}} \inf_{\varepsilon>0} \frac{\left(mr_2 + \frac{1}{2}m(m-1)\sigma_2^2 + K(m) + \varepsilon\right)^{m+1}}{\varepsilon}.$$

It is easy to demonstrate that

$$\inf_{\varepsilon>0} \frac{\left(mr_2 + \frac{1}{2}m(m-1)\sigma_2^2 + K(m) + \varepsilon\right)^{m+1}}{\varepsilon} = (m+1)\left(mr_2 + \frac{1}{2}m(m-1)\sigma_2^2 + K(m)\right).$$

This implies that

$$\limsup_{t\to\infty} \mathbb{E}P_t^m \leqslant \frac{m\left(r_2 + \frac{1}{2}(m-1)\sigma_2^2 + K(m)\right)}{a_2^m (m+1)^m} \sup_{t\geqslant 0}\mathbb{E}H_t^m.$$

Hence the result. ■

3. Numerical simulation

3.1. Strong order 1.0 Taylor scheme

Let us consider the following system:

$$\begin{cases} dH_t = a(t,H_t)\,dt + b(t,H_t)\,dB_1 + \int_{\mathbb{Z}} H_{t-}\gamma_1(u)\tilde{N}(dt,du), \\ dP_t = e(t,P_t)\,dt + f(t,P_t)\,dB_2 + \int_{\mathbb{Z}} P_{t-}\gamma_2(u)\tilde{N}(dt,du) \end{cases} \tag{17}$$

where $a(t,H_t) = H_t(r_1 - a_1P_t - b_1H_t)$, $b(t,H_t) = \sigma_1 H_t$, $e(t,P_t) = P_t\big(r_2 - \frac{a_2P_t}{H_t}\big)$ and $f(t,P_t) = \sigma_1 P_t$, $\tilde{N}(dt,du) = N(dt,du) - \lambda(du)\,dt$, $\gamma_i \colon \mathbb{Z}\times\Omega \to \mathbb{R}$, with $(H_0,P_0) \in \mathbb{R}^2_+$, and $B = \{B_1(t),B_2(t)\}$, $t\in[0,T]$, are Brownian motion $\mathcal{F}$-adapted.

The first equation of system (17) can be written in an integral form as

$$\begin{aligned} H_t &= H_0 + \int_0^t a(s,H_s)\,ds + \int_0^t b(s,H_s)\,dB_1(s) + \int_0^t\int_{\mathbb{Z}} H_{t-}\gamma_1\tilde{N}(dt,du) \\ &= H_0 + \int_0^t a\,(s,H_s)\,ds + \int_0^t b\,(s,H_s)\,dB_1(s) + \sum_{j=1}^{\tilde{N}} H_{\tau_j-}\gamma_1(u) \end{aligned} \tag{18}$$

where $\{\tau_j, j\in\{1,2\ldots,\tilde{N}\}\}$ is the jump time.

Therefore, we also consider a one-dimensional SDE in integral form, of the form

$$H_t = H_0 + \int_0^t a\,(s,H_s)\,ds + \int_0^t b\,(s,H_s)\,dB_1(s) + \sum_{j=1}^{\tilde{N}} H_{\tau_j-}\gamma_1(u). \tag{19}$$

When accuracy and efficiency are required, it is important to build numerical methods with higher order of convergence. This can be achieved with the Taylor scheme (see e.g. [15]).

The Taylor scheme of order 1.0, which in the one-dimensional case, $d=m=1$, is given by

$$\begin{aligned} H_{n+1} = H_n &+ a\Delta_n + b\Delta B_n + \int_{t_n}^{t_{n+1}}\int_{\mathbb{Z}} c(v)\,\tilde{N}(dz,du) + b\,b'\int_{t_n}^{t_{n+1}}\int_{t_n}^{z_2} dB_1(z_1)\,dB_1(z_2) \\ &+ \int_{t_n}^{t_{n+1}}\int_{\mathbb{Z}}\int_{t_n}^{z_2} b\,c'(v)\,dB_1(z_1)\,\tilde{N}(dz_2,du) \\ &+ \int_{t_n}^{t_{n+1}}\int_{t_n}^{z_2}\int_{\mathbb{Z}} \big\{b(t_n,H_n+c(v)) - b\big\}\tilde{N}(du,dz_1)\,dB_1(z_2) \\ &+ \int_{t_n}^{t_{n+1}}\int_{\mathbb{Z}}\int_{t_n}^{z_2}\int_{\mathbb{Z}} \big\{c(t_n,H_n+c(v_1),v_2) - c(v_2)\big\}\,\tilde{N}(du_1,dz_1)\,\tilde{N}(du_2,dz_2), \end{aligned} \tag{20}$$

where $b' = b'(t,x) = \frac{\partial b(t,x)}{\partial x}$ and $c'(v) = c'(t,x,v) = \frac{\partial c(t,x,v)}{\partial x}$.

The scheme (20) achieves strong order $\gamma = 1.0$, as we will see later. It represents a generalization of the Milstein scheme, see Milstein (1974), to the case of jump diffusions.

By applying the Itô formula for jump diffusion processes, and the integration by parts formula, we can simplify four double stochastic integrals appearing in (20) and rewrite the strong order 1.0 Taylor scheme (20) as follows

$$\begin{aligned} H_{n+1} = Y_n &+ a\Delta_n + b\Delta B_n + \sum_{i=\tilde{N}(t_n)+1}^{\tilde{N}(t_{n+1})} c(v) + \frac{b\,b'}{2}\left((\Delta B_n)^2 - \Delta_n\right) \\ &+ b\sum_{i=\tilde{N}(t_n)+1}^{\tilde{N}(t_{n+1})} c'(v)\big(B_1(\tau_i) - B_1(t_n)\big) \\ &+ \sum_{i=\tilde{N}(t_n)+1}^{\tilde{N}(t_{n+1})} \big\{b(H_n+c(v)) - b\big\}\big(B_1(t_{n+1}) - B_1(\tau_i)\big) \end{aligned}$$

$$+\sum_{j=\tilde{N}(t_n)+1}^{\tilde{N}(t_{n+1})}\sum_{i=\tilde{N}(t_n)+1}^{\tilde{N}(\tau_j)}\big\{c(H_n+c(v))-c(v)\big\}. \tag{21}$$

In the particular case of an independent jump coefficient, $c(t,x,v)=c(t,x)$, the Taylor scheme of order 1.0 reduces to

$$\begin{aligned}H_{n+1}=H_n+a\Delta_n+b\Delta B_n+c\Delta p_n+b\,b'I_{(1,1)}+b\,c'I_{(1,-1)}\\+\big\{b(t_n,H_n+c)-b\big\}I_{(-1,1)}+\big\{c(t_n,H_n+c)-c\big\}I_{(-1,-1)},\end{aligned}$$

with the following multiple stochastic integrals

$$\begin{aligned}I_{(1,1)}&=\int_{t_n}^{t_{n+1}}\int_{t_n}^{s_2}dB_1(s_1)\,dB_1(s_2)=\frac{1}{2}\left\{(\Delta B_n)^2-\Delta_n\right\},\\ I_{(1,-1)}&=\int_{t_n}^{t_{n+1}}\int_{\mathbb{Z}}\int_{t_n}^{s_2}dB_1(s_1)\,\tilde{N}(du,ds_2)=\sum_{i=\tilde{N}(t_n)+1}^{\tilde{N}(t_{n+1})}B_1(\tau_i)-\Delta p_nB_1(t_n),\\ I_{(-1,1)}&=\int_{t_n}^{t_{n+1}}\int_{t_n}^{s_2}\int_{\mathbb{Z}}\tilde{N}(du,ds_1)\,dB_1(s_2)=\Delta p_n\Delta B_n-I_{(1,-1)},\\ I_{(-1,-1)}&=\int_{t_n}^{t_{n+1}}\int_{\mathcal{E}}\int_{t_n}^{s_2}\int_{\mathbb{Z}}\tilde{N}(du_1,ds_1)\,\tilde{N}(du_2,ds_2)=\frac{1}{2}\left\{(\Delta p_n)^2-\Delta p_n\right\}.\end{aligned}$$

The same procedure is followed to obtain the Taylor scheme for P_t.

3.2. Euler scheme of order 1

The simplest scheme is again the well-known Euler scheme, which, in the one-dimensional case $d=m=1$, is given by the algorithm

$$\begin{aligned}H_{n+1}&=H_n+a\Delta_n+b\Delta B_n+\int_{t_n}^{t_{n+1}}\int_{\mathbb{Z}}H_{t^-}\gamma_i\,\tilde{N}(dt,du)\\&=H_n+a\Delta_n+b\Delta B_n+\sum_{j=\tilde{N}(t_n)+1}^{\tilde{N}(t_{n+1})}c(v),\end{aligned} \tag{22}$$

where $\tilde{N}(dt,du)=N(dt,du)-\lambda(du)\,dt$ with N is again a Poisson measure $\mathcal{F}$-adapted.

Note that $a=a(t_n,X_n)$, $b=b(t_n,X_n)$, $c(v)=X_{t^-}\gamma_i$ and $c'(v)=c'(t,x,v)=\frac{\partial c(t,x,v)}{\partial x}$

$$\Delta_n=t_{n+1}-t_n=I_{0,n} \tag{23}$$

is the length of the time interval $[t_n,t_{n+1}]$ and

$$\Delta B_n=B_{t_{n+1}}-B_{t_n} \tag{24}$$

is the nth Gaussian $N(0,\Delta_n)$ distributed increment of the Brownian motion B, $n\in\{0,1,\dots,N-1\}$. Furthermore,

$$\tilde{N}(t)=\tilde{N}([0,t]) \tag{25}$$

represents the total number of jumps of the Poisson random measure up to time t, which is a Poisson distribution of mean λt.

The Euler scheme reduces to

$$H_{n+1}=H_n+a\Delta_n+b\Delta B_n+c\Delta p_n, \tag{26}$$

where

$$\Delta p_n=\tilde{N}(t_{n+1})-\tilde{N}(t_n) \tag{27}$$

follows a Poisson distribution of mean $\lambda\Delta_n$.

4. Examples and numerical simulations

In this section, we will use the Taylor method of order 1 and Euler scheme to illustrate the analytical results. In numerical results presented in the figure below, we choose $a_1 = 0.9$, $r_1 = 2$, $r_2 = 0.8$, $b_1 = 0.7$, $a_2 = 1.6$, $\lambda = 1$, $Z = (0, +\infty)$.

4.1. Taylor's method simulation

The only difference between the conditions in these Figures is that the values of γ_1, γ_2 are different.

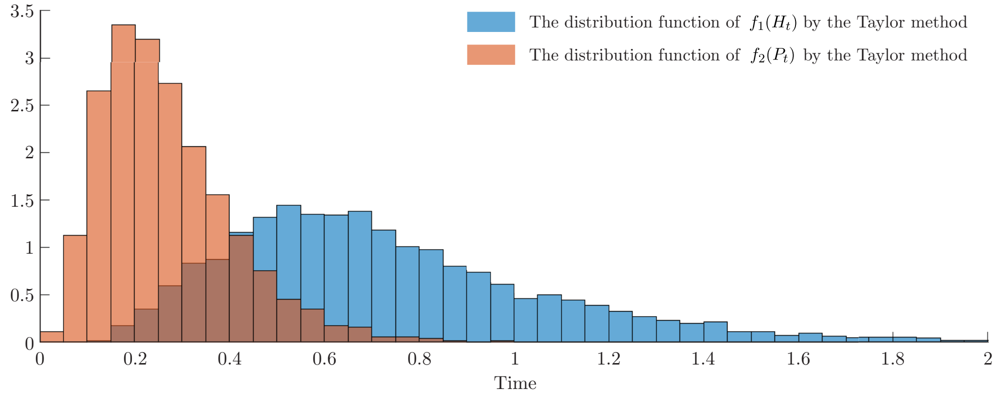

Fig. 1. Distribution function for stationary state $f_1(H_t)$ and $f_2(P_t)$ of system (3) using the Taylor method order 1 for $a_1 = 0.9$, $r_1 = 0.9$, $r_2 = 0.8$, $b_1 = 0.7$, $a_2 = 1.6$, $\sigma_1 = 0.5$, $\sigma_1 = 0.5$, $\lambda = 1$, $\gamma_1 = 0$, $\gamma_2 = 0$, $t = 7$, initial value $H_0 = 1.5$, $P_0 = 1.5$.

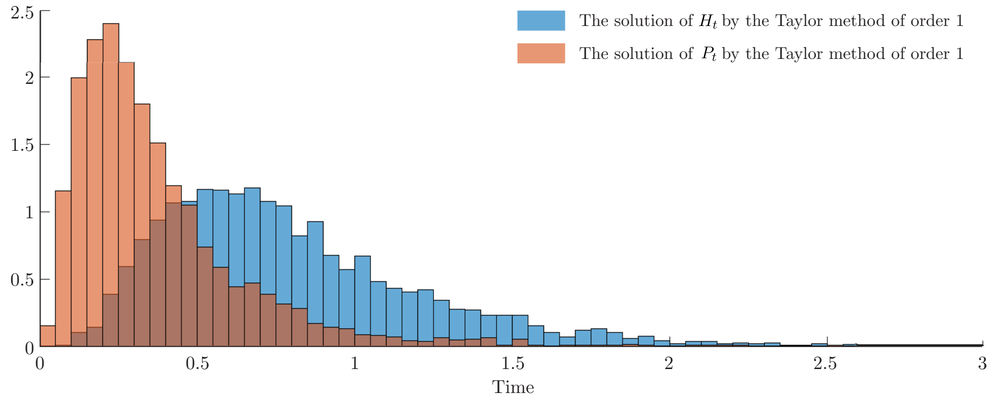

Fig. 2. Distribution function for stationary state $f_1(H_t)$ and $f_2(P_t)$ of system (3) using the Euler and Taylor method order 1 for $a_1 = 0.9$, $r_1 = 0.9$, $r_2 = 0.8$, $b_1 = 0.7$, $a_2 = 1.6$, $\sigma_1 = 0.5$, $\sigma_1 = 0.5$, $\lambda = 1$, $\gamma_1 = 0.3$, $\gamma_2 = 0.3$, $t = 7$, initial value $H_0 = 1.5$, $P_0 = 1.5$.

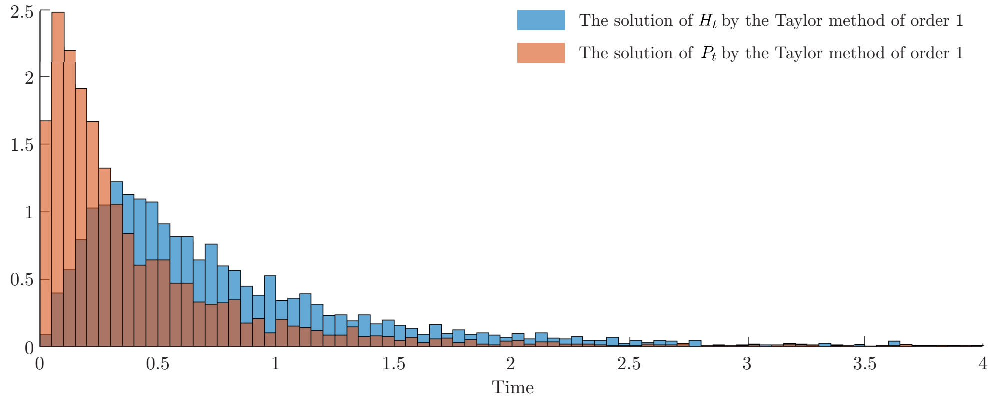

Fig. 3. Distribution function for stationary state $f_1(H_t)$ and $f_2(P_t)$ of system (3) using the Euler and Taylor method order 1 for $a_1 = 0.9$, $r_1 = 0.9$, $r_2 = 0.8$, $b_1 = 0.7$, $a_2 = 1.6$, $\sigma_1 = 0.5$, $\sigma_1 = 0.5$, $\lambda = 1$, $\gamma_1 = 0.8$, $\gamma_2 = 0.7$, $t = 7$, initial value $H_0 = 1.5$, $P_0 = 1.5$.

Figure 1 confirms that when Lévy noise is not taken into account, the system is stable in mean. In contrast to Figures 2 and 3, we can clearly see the effect of Lévy noise, as it force a population to die.

4.2. Euler method's simulation

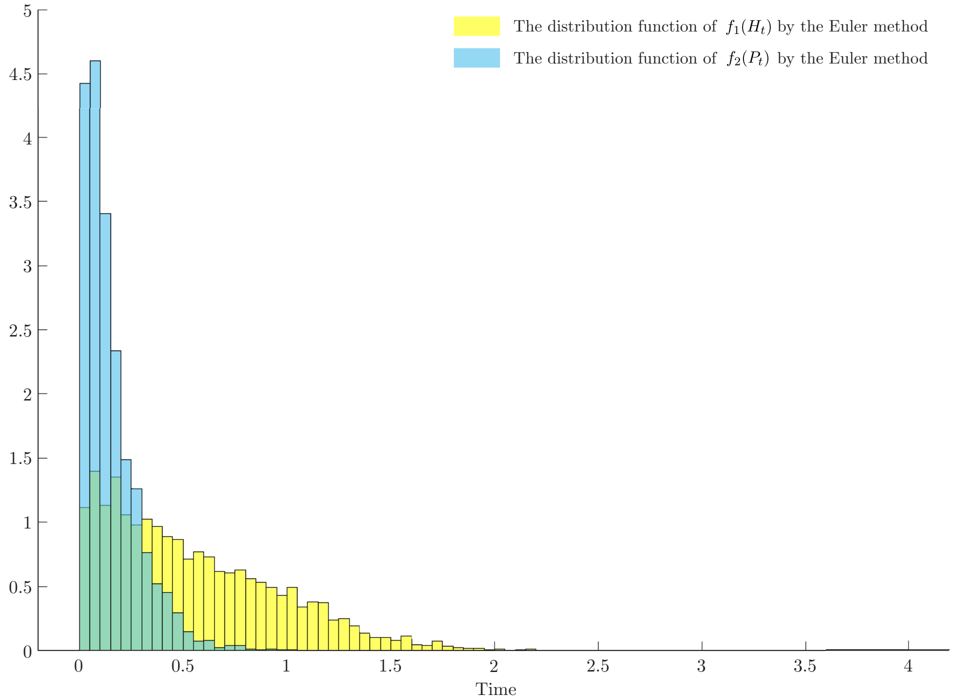

Fig. 4. Distribution function for stationary state $f_1(H_t)$ and $f_2(P_t)$ of system (3) using the Euler method for $a_1 = 0.9$, $r_1 = 0.9$, $r_2 = 0.8$, $b_1 = 0.7$, $a_2 = 1.6$, $\sigma_1 = 0.5$, $\sigma_1 = 0.5$, $\lambda = 1$, $\gamma_1 = 0$, $\gamma_2 = 0$, $t = 7$, initial value $H_0 = 1.5$, $P_0 = 1.5$.

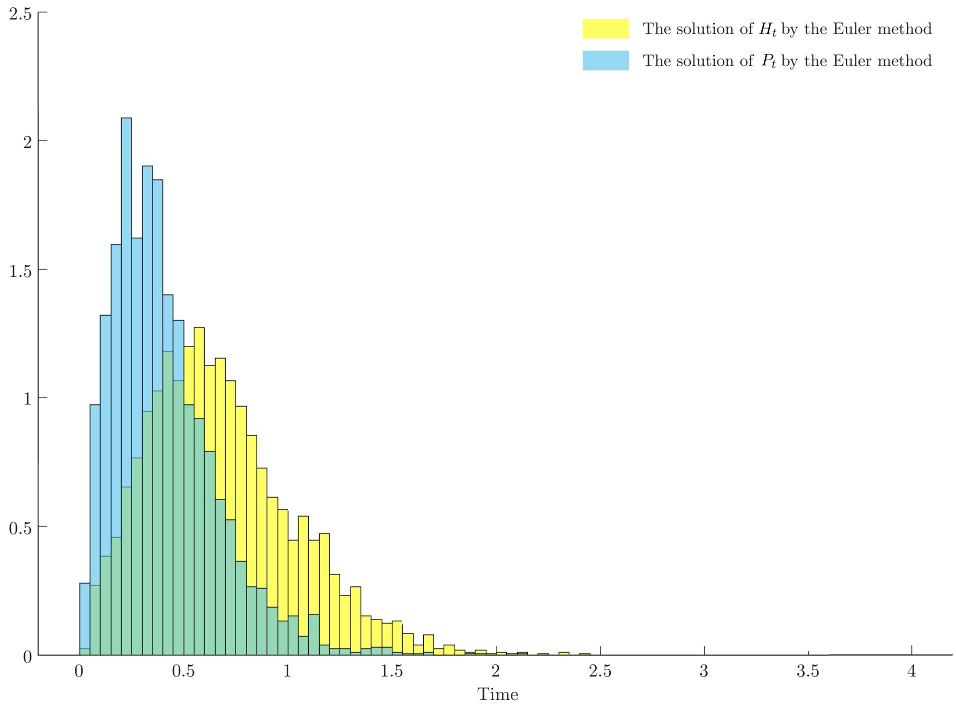

Fig. 5. Distribution function for stationary state $f_1(H_t)$ and $f_2(P_t)$ of system (3) using the Euler method for $a_1 = 0.9$, $r_1 = 0.9$, $r_2 = 0.8$, $b_1 = 0.7$, $a_2 = 1.6$, $\sigma_1 = 0.5$, $\sigma_1 = 0.5$, $\lambda = 1$, $\gamma_1 = 0.3$, $\gamma_2 = 0.3$, $t = 7$, initial value $H_0 = 1.5$, $P_0 = 1.5$.

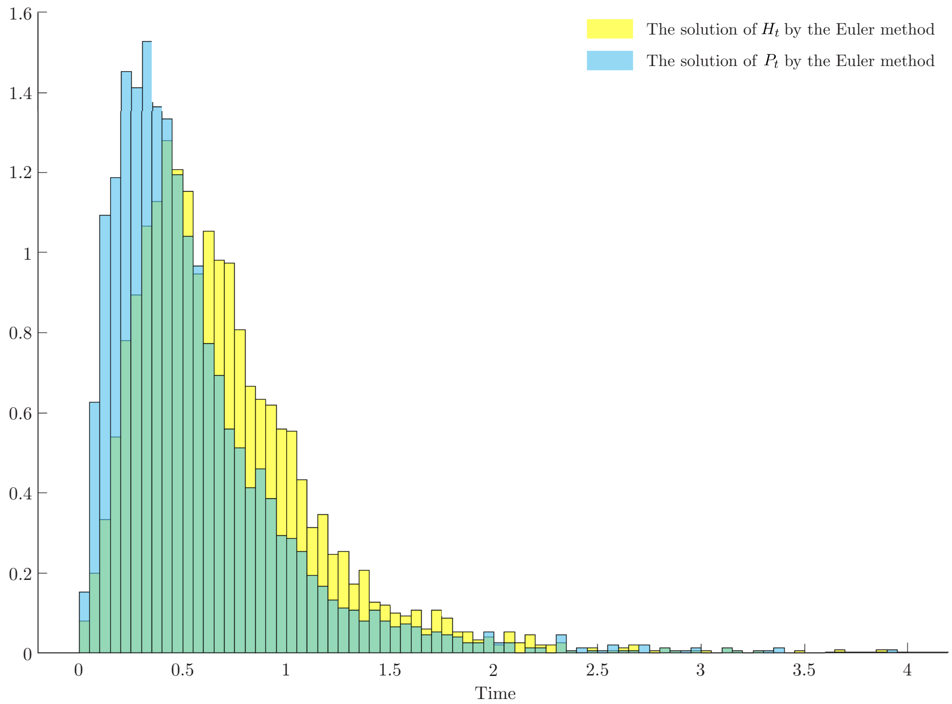

Fig. 6. Distribution function for stationary state $f_1(H_t)$ and $f_2(P_t)$ of system (3) using the Euler method for $a_1 = 0.9$, $r_1 = 0.9$, $r_2 = 0.8$, $b_1 = 0.7$, $a_2 = 1.6$, $\sigma_1 = 0.5$, $\sigma_1 = 0.5$, $\lambda = 1$, $\gamma_1 = 0.8$, $\gamma_2 = 0.7$, $t = 7$, initial value $H_0 = 1.5$, $P_0 = 1.5$.

It is demonstrated in these figures that Lévy noise has an interesting and important property that can force a population to die. From a biological perspective, this is reasonable. Lévy noise is a sudden and severe environmental disturbance. When these disturbances occur, the most sensitive factor to population dynamics is growth rate, because the young stage is the most sensitive stage of the life cycle.

5. Conclusion

In this paper, we studied the stochastic differential equations excited by Lévy processes. The Leslie–Grower model with environmental perturbations that are modeled by a Lévy process was considered. We demonstrate that the model has a unique positive solution, establish sufficient and necessary conditions for the stability in mean and extinction of each population. Numerical algorithms of higher order are elaborated for response simulation the result reveals that the Lévy noise can change the properties of the population systems significantly and it can force the population to die out. This is not the case for the deterministic model, which persists in the positive steady state for all parameter values.

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Підхід на основі процесу Леві в поєднанні зі стохастичною моделлю Леслі–Гауера

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Стаття присвячена двовимірній стохастичній системі "хижак–жертва" Леслі–Гроуера з неперервним часом і стрибками Леві. Спершу доведено, що існує єдиний додатний розв'язок системи з додатним початковим значенням. Потім встановлено достатні умови середньої стійкості та загасання розглянутої системи. Розроблено чисельні алгоритми вищого порядку. Отримані результати показують, що стрибки Леві істотно змінюють властивості популяційних систем.

Ключові слова: *Леслі–Гауер; SDEs; формула Іто; загасання; стохастична модель "хижак–жертва"; метод Тейлора.*