MATHEMATICAL MODELING AND COMPUTING, Vol. 11, No. 1, pp. 203–210 (2024)

Viable control of COVID-19 spread with vaccination

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(Received 27 June 2023; Revised 26 February 2024; Accepted 28 February 2024)

The rapid and widespread transmission of COVID-19 has necessitated the development and implementation of effective control measures. Vaccination has emerged as a key tool in combating the pandemic. This article introduces a novel approach to modeling the dynamics of COVID-19 transmission by integrating vaccination strategies into the susceptible-infected-recovered (SIR) framework using viability theory. We have defined a set of constraints including a guaranteed level of vaccination, we analyze the impact of different vaccination rates on curbing the spread of the virus. Our findings reveal the significant role of vaccination in reducing transmission and offer valuable insights into optimizing vaccination campaigns. The viability-based SIR model provides a comprehensive framework for policymakers and healthcare professionals to devise targeted strategies and allocate resources effectively in the battle against COVID-19.

Keywords: *control theory; viability theory; epidemiology; SIR model; COVID-19; vaccination control.*

2010 MSC: 93C95, 92D30 **DOI:** 10.23939/mmc2024.01.203

1. Introduction

The ongoing COVID-19 pandemic has posed unprecedented challenges to global health systems, economies, and societies. As the world continues to grapple with the devastating impact of the virus, identifying effective control measures remains a top priority. Vaccination has emerged as a critical intervention to curb the transmission of the virus and prevent severe illness and mortality. However, optimizing vaccination strategies is crucial to ensure the maximum impact and efficient allocation of limited resources. In this article, we present an innovative approach to modeling COVID-19 transmission control, integrating vaccination strategies into the classical susceptible-infected-recovered (SIR) framework. Building upon the concept of viability theory, we consider the dynamic interactions between the virus, the vaccinated population, and the overall population's viability. By exploring different vaccination rates and their effects on controlling the spread of COVID-19, this study aims to provide valuable insights and support evidence-based decision-making for policymakers and healthcare professionals.

In the fight against communicable diseases, it is important to understand how infectious agents spread in the population to assess the impact of the epidemics they generate, anticipate their evolution and the effectiveness of control measures. Mathematical models are a useful tool to describe these complex and multifactorial epidemic dynamics, and to interpret often limited epidemiological data. Since the start of the COVID-19 pandemic, the Mathematical Modeling of Infectious Diseases has used these approaches to study the spread of the SARS-CoV-2 virus. These analyzes make it possible, for example, to estimate the risks associated with SARS-CoV-2 infection, to assess the impact of control measures, to monitor the spread of the SARS-CoV-2 virus, for example by estimating the proportion of the population that has been infected with SARS-CoV-2, to produce projections to anticipate the short-term hospital needs of COVID-19 patients, to determine the impact of variants and vaccination on the dynamics of epidemic and to estimate the level of vaccination coverage allowing a return to normal life.

In this paper, we focus on COVID-19 pandemic control by posing and solving a more realistic control problem that takes into account vaccine supply and treatment. The peculiarity of our work is that we used the viability approach to investigate the controlled dynamic of COVID-19.

The viability approach [1] pertains to dynamic systems that operate within specific state and control constraints. Its objective is to assess the harmony between a system's dynamics and the imposed limitations on states or controls, with the aim of identifying the range of permissible states and decisions that prevent the system from violating these constraints. When applied to sustainability, viability analysis incorporates economic and environmental restrictions, making it a multi-criteria approach that accounts for multiple factors simultaneously.

An important tool in the viability approach is provided by the mathematical concept of the *viability kernel*. This kernel is the set of initial states from which viable trajectories begin. We will see that the viability kernel provides some a posteriori constraints to satisfy. These constraints are stronger than the initial viability constraints. They require that the constraints be satisfied not only now, but also in the future. Such a characteristic of the approach makes it compatible with the definition of sustainable development.

Mathematically speaking, the viability kernel can be regarded as a set that exhibits weak invariance in the dynamical system, which is why it is referred to as weakly invariant.

2. Mathematical formulation and control problem

2.1. Mathematical formulation and description of the problem

The COVID-19 model into account in the present work describes the dynamics of a population splitted into three categories:

- Susceptible (x_1): individuals who could have the infection.
- Infected (x_2): individuals who might spread the virus.
- Individuals to recover (x_3): those who have recovered, or are immune.

We will assume that the whole population: $N = x_1 + x_2 + x_3$.

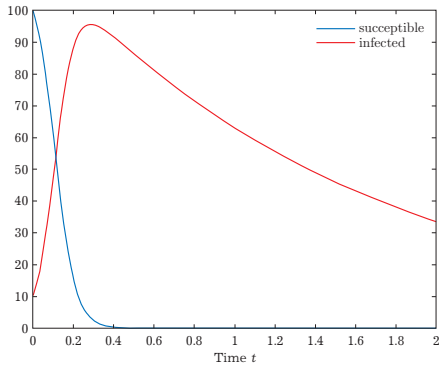

Fig. 1. The dynamic of the spread of COVID-19 without control.

The model will have the following nonlinear system of differential equations:

$$\begin{aligned} \frac{\mathrm{d}x_1}{\mathrm{d}t} &= -a\,x_1x_2 - b\,x_1, \\ \frac{\mathrm{d}x_2}{\mathrm{d}t} &= a\,x_1x_2 - (c+d+b)\,x_2, \\ \frac{\mathrm{d}x_3}{\mathrm{d}t} &= c\,x_2 - b\,x_3 \end{aligned} \tag{1}$$

with initial conditions

$$x_1(0) = x_{1_0} \geqslant 0, \quad x_2(0) = x_{2_0} \geqslant 0, \quad x_3(0) = x_{3_0} \geqslant 0.$$

With, a is contact rate, b is natural death rate, c is the healing rate, d is death rate due to the disease.

2.2. Control problem under constraints

We present two control tactics, α_1 vaccine and α_2 wareness due to the media coverage strategies, to eradicate the infection from a specific population. This controled problem is solved within the framework of the viability approach, as indicated in Section 3, and the following modified dynamics of the SIR model (1) with vaccination and wareness are proposed. We focus our study in the following controlled dynamic system:

$$\begin{aligned} \frac{\mathrm{d}x_{1_t}}{\mathrm{d}t} &= -a\,x_{1_t}x_{2_t} - b\,x_{1_t} - \alpha_1 x_{1_t}, \\ \frac{\mathrm{d}x_{2_t}}{\mathrm{d}t} &= a\,x_{1_t}x_{2_t} - (c+d+b)\,x_{2_t} - \alpha_2 x_{2_t} \end{aligned} \tag{2}$$

under control constraint:

$$0 \leqslant \alpha_i \leqslant \bar{\alpha_i} \quad \text{for} \quad i = 1, 2.$$

The control in system (2) is represented by $\alpha = (\alpha_1, \alpha_2)' \in \mathcal{A}$.

$$\mathcal{A} = \prod_{i=1}^{2} [0, \bar{\alpha_i}] \quad \text{where} \quad \bar{\alpha_i} \leqslant 1.$$

Let $0 < \bar{x_2} \leqslant N$.

The main purpose is to determine the control α for each $(x_{1_{t_0}}, x_{2_{t_0}})$ such that keeps the number of infectifs below the $\bar{x_2}$ boundary, where the viability constraint expresses the case of a controlled number of infection of a community. As long as the viability constraint is achieved:

$$x_2(t) < \bar{x_2}, \quad \forall t \geqslant t_0. \tag{3}$$

The presence of control essentially relies upon initial state. We will currently concentrate on these initial states, likewise called the viability kernel [1].

3. The viability approach

Here we will relax the idea of optimality by using the viability approach [1], also called weak invariance [2]. This approach does not rely on an optimality criterion, but focuses on feasible paths. It requires the definition of a set of constraints, called viability constraints, representing certain desired properties. This will allow us to see if the dynamics (2) are compatible with the viability constraint (3) at any given time t.

We will write the dynamic system (2) in the form:

$$\dot{X}(t) = F(X(t), \alpha(t)), \quad \alpha(t) \in \mathcal{A}, \quad X(t) \in \Omega, \tag{4}$$

$$F(X(t), \alpha(t)) = \begin{pmatrix} -a\, x_{1_t} x_{2_t} - b\, x_{1_t} - \alpha_1 x_{1_t} \\ a\, x_{1_t} x_{2_t} - (c + d + b)\, x_{2_t} - \alpha_2 x_{2_t} \end{pmatrix},$$

where $X \in \mathbb{R}^2_+$ is the system state and $\alpha \in \mathbb{R}^2_+$ represents the control. The set $\mathcal{A}$ is the domain of admissible controls. The set Ω corresponds to the domain of admissible states, with,

$$\Omega = \{X = (x_1, x_2) \mid x_1, x_2 \geqslant 0,\ x_2 < \bar{x_2},\ x_1 + x_2 < N\}. \tag{5}$$

Viability kernel is the set of initial states X_0 from which a feasible path $(X(\cdot); \alpha(\cdot))$ respecting the constraints (stay in Ω) at any time:

$$\text{Viab}(F, \mathcal{A}, \Omega) = \{X_0 \in \Omega \setminus \exists X(t) \text{ verifying (4) } \forall t \geqslant 0,\ X(0) = X_0\}.$$

3.1. Viable or weakly invariant set

If a set Ω satisfies the condition where the viability kernel $\text{Viab}(F, \Omega, \mathcal{A})$ is equivalent to the set of initial constraints $\mathbb{A}$ for the dynamic (F, α), it is considered viable. This implies that starting from any state within Ω, there exists a feasible control that results in a trajectory remaining within Ω. Such a favorable scenario occurs at a state $X \in \Omega$ when a control α leads to velocities $\dot{X} = F(X, \alpha)$ that are either tangent to or point inward the domain Ω. For closed sets Ω, under appropriate assumptions on the dynamics (e.g., if $\mathcal{A}$ is convex, closed, and bounded), this condition can be expressed using a Hamiltonian formulation. Let us consider the Hamiltonian:

$$\mathcal{H}(X, p, \alpha) = \sum_{i=1}^{n} p_i\, F_i(X, \alpha).$$

In this case, the following statements are equivalent,

i. Ω is viable for (F, Ω);
ii. $\text{Viab}(F, \Omega, \mathcal{A}) = \Omega$;
iii. $\inf\limits_{\alpha \in \Omega(X)} \mathcal{H}(X, p, \alpha) \leqslant 0,\ \forall X \in \Omega\ \forall p \in \mathcal{N}_{\Omega(X)}$;

where $\mathcal{N}_{\Omega(X)}$ is the normal cone to the set Ω at point X. In our case the set of constraints $\Omega = \{X \in \mathbb{R}^2_+, g_j(X) \leqslant 0, j = 1, 2\}$ with $g_1(X) = x_2 - \bar{x}_2$ and $g_2(X) = x_1 + x_2 - N$.

$\mathcal{N}_{\Omega(X)}$ is given by

$$\mathcal{N}_{\Omega}(X) = \left\{ p \Big| \sum_{i=1}^{2} p_i \frac{\partial g_i}{\partial x_i}(X) \geqslant 0 \text{ and } q_i g_j(X) = 0,\ j = 1, 2 \right\}. \tag{6}$$

3.2. Viable control problem and viability kernel

The viability kernel is formally defined as follows:

Definition 1. *The viability kernel* $\mathrm{Viab}(\bar{x}_2)$ *is a set of initial states* $(x_{1_{t_0}}, x_{2_{t_0}})$ *for which a control strategy* $t \mapsto \alpha_t \in \mathcal{A}$ *exists so that the dynamic system (2) solution meets the viability constraint (3),*

$$\mathrm{Viab}(\bar{X}_2) = \left\{ (x_{1_0}, x_{2_0}) \mid \begin{array}{l} \text{there exist a control } \alpha(\cdot) \text{ so that the solution to (2)} \\ \text{that starts from } (x_{1_0}, x_{2_0}) \text{ satisfies the constraint (3)} \end{array} \right\}. \tag{7}$$

Note that the positively invariant set $\{(x_1, x_2) \mid 0 \leqslant x_1, 0 \leqslant x_2, x_1 + x_2 \leqslant N\}$ is our unconstrained domain of research. Due to the necessity of fulfilling the viability constraint (3) at the initial point, the viability kernel $\mathrm{Viab}(\bar{x}_2)$ must be included in the rectangle $[0, N] \times [0, \bar{x}_2[$.

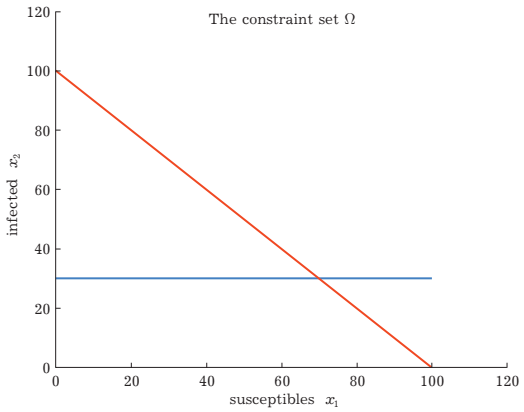

Fig. 2. The constraint set Ω.

The constraint set Ω (Figure 2) is the intersection of the unconstrained domain of study and the rectangle $[0, N] \times [0, \bar{x}_2[$. That is $\mathrm{Viab}(\bar{X}_2) \subset \Omega$.

Definition 2. *The vaccination barrier*

$$\mathcal{V} := \{(x_1, \phi(x_1)) \mid \bar{x}_1 \leqslant x_1 \leqslant N\} \tag{8}$$

with $\bar{x}_1 := \frac{N}{\mathcal{R}_0} = \frac{b+c+d}{a}$.

The basic reproductive number $\mathcal{R}_0$ *is*

$$\mathcal{R}_0 := \frac{a}{b+c+d} N. \tag{9}$$

With $\phi(x_1))$ *is is the set of applications such as* x_1 *the solution* $x_1 \in [\bar{x}_1, N] \mapsto \phi(x_1)$ *to the differential equation:*

$$\begin{aligned} x_1(a\,\phi(x_1) + b)\,\phi'(x_1) + a\,\phi(x_1)\,(x_1 - \bar{x}_1) = 0, \\ \phi\,(\bar{x}_1) = \bar{x}_2. \end{aligned} \tag{10}$$

We introduce and give a geometric description of viability domains for the system (2), which play a pivotal role in characterizing the viability kernel. The viability kernel is linked to the viability domains in the following way.

Theorem 1 (Ref. [1]). *The viability kernel is the constraint set is largest viability domain.*

We associate the vector field $(u_{x_1}; u_{x_2})$ produced by two components:

$$\begin{pmatrix} u_{x_1} \\ u_{x_2} \end{pmatrix} = \begin{pmatrix} -a\,x_{1_t} x_{2_t} - b\,x_{1_t} - \alpha_1 x_{1_t} \\ a\,x_{1_t} x_{2_t} - (c+d+b)\,x_{2_t} - \alpha_2 x_{2_t} \end{pmatrix}$$

with system (2).

The system (2) is equivalent to

$$\begin{aligned} \dot{x}_1 &= u_{x_1}(x_1(t), x_2(t), \alpha_1(t)), \\ \dot{x}_2 &= u_{x_2}(x_1(t), x_2(t), \alpha_2(t)). \end{aligned} \tag{11}$$

Using the vector field u, we present a geometric description of the system's viability domains with control. To begin, we must state that the system is Marchaud.

Theorem 2. *For a Marchaud controlled system [1], a closed subset S is considered viable when the control varies, the tangent cone at any point within S includes at least one vector from the family generated by the vector field.*

Corollary 1. *Consider a closed subset S of $[0;N]$. If there is a control $\alpha \in \mathcal{A}$ such that (u_{x_1}, u_{x_2}) is an inward-pointing vector, then the set S is a viability domain for the system (2) whenever $(x_1;x_2)$ varies along the frontier ∂S of the set S.*

For a closed subset S with a piecewise smooth boundary δS, the scalar product between the vector (u_{x_1}, u_{x_2}) and any outward-pointing normal vector (with respect to S) must be less than or equal to zero for S to qualify as a viability domain for the system (2).

3.3. Characterization of the viability kernel

If $\bar{x}_1 + \bar{x}_2 < N$, then

$$\mathrm{Viab}\,(\bar{x}_2) = \Omega \cap \{(x_1, x_2) \mid \bar{x}_1 \leqslant x_1 \leqslant N \text{ and } x_2 < \phi(x_1)\}\,. \tag{12}$$

We assume that $\bar{x}_2 < \frac{\mathcal{R}_0 - 1}{\mathcal{R}_0} N$.

Lemma 1. *There is a unique solution $x_1 \in [\bar{x}_1, N] \mapsto \phi(x_1)$ of the differential equation (10). The solution ϕ is decreasing and strictly positive.*

Proof. We have $\bar{x}_1 + \bar{x}_2 < N \Rightarrow \bar{x}_2 < N$ in the vicinity of $\bar{x}_2 > 0$ and $\phi(\bar{x}_1) = \bar{x}_2 > 0$, the expression $(a\,\phi(x_1) + b + d)x_1$ is strictly positive, so we can write (10)

$$\phi'(x_1) = \frac{\phi(x_1)(x_1 - \bar{x}_1)}{(\phi(x_1) + b/a)x_1} < 0. \tag{13}$$

By Cauchy Lipschitz theorem, there exists a local solution ϕ of (10) and (13) in the neighborhood of $\bar{x}_1 > 0$.

We will now show that $\phi(x_1) > 0$, we suppose that there is $x_{1_0} \leqslant \bar{x}_1$ such that $\phi(x_{1_0}) = 0$, in the vicinity of $x_{1_0} > 0$, the expression $(a\,\phi(x_1) + b)x_1 > 0$ so (10) and (13) are equivalent.

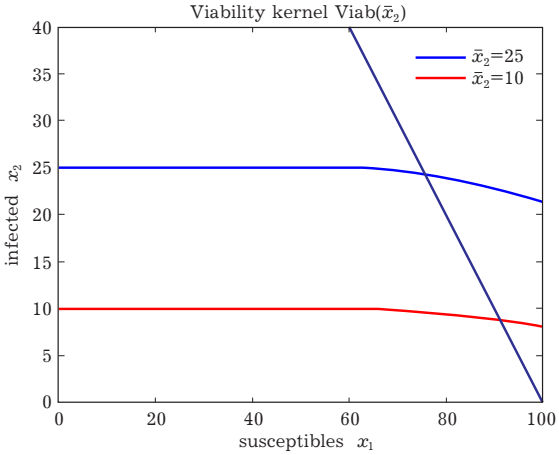

Fig. 3. Viability kernels for $\bar{x}_2 = 10$ and for $\bar{x}_2 = 25$.

We have two solutions $S \to \phi(x_1)$ and $x_1 \to 0$, by the Cauchy–Lipschitz theorem this cannot happen by uniqueness, and therefore there is no $x_{1_0} \geqslant \bar{x}_1$ hence $\phi(x_{1_0}) > 0$ for all x_1 and hence the solution is well defined.

We have $\phi'(x_1) < 0$ so ϕ is decreasing.

We conclude that ϕ is a unique and decreasing solution and lowered by 0, so it is defined for all $x_1 \geqslant \bar{x}_1$. ∎

Lemma 2. *The set $\mathrm{Viab}(\bar{x}_2) = \Omega \cap \{(x_1, x_2) \mid \bar{x}_1 \leqslant x_1 \leqslant N$ and $x_2 < \phi(x_1)\}$ is a viable set.*

Proof. Let $\varepsilon > 0$,

$$\begin{aligned} -\varepsilon\,\phi_\varepsilon(S) &= x_1(a\,\phi_\varepsilon(x_1) + b)\,\phi'_\varepsilon(x_1) + a\,\phi_\varepsilon(x_1)\,(x_1 - \bar{x}_1)\,, \\ \phi_\varepsilon\,(\bar{x}_1 - \varepsilon) &= \bar{x}_2 - \varepsilon \end{aligned} \tag{14}$$

by lemma 1 we show that the differential equation (14) has a unique solution $x_1 \in [\bar{x}_1 - \varepsilon, N] \mapsto \phi_\varepsilon(x_1)$, strictly positive and decreasing.

Let $\mathrm{Viab}(\bar{x}_2)_\varepsilon = \Omega \cap \{(x_1, x_2) \mid \bar{x}_1 - \varepsilon \leqslant x_1 \leqslant N$ and $x_2 < \phi_\varepsilon(x_1)\}$. by the comparison theorem, we have $\phi_\varepsilon < \phi$, so $\mathrm{Viab}(\bar{x}_2)_\varepsilon \subset \mathrm{Viab}(\bar{x}_2)$, by continuity $\mathrm{Viab}(\bar{x}_2)$ is the union of $\mathrm{Viab}(\bar{x}_2)_\varepsilon$ for all $\varepsilon > 0$.

We now show that $\mathrm{Viab}(\bar{x}_2)_\varepsilon$ is a viable set.

Since Ω is an invariant set, we can focus on the boundary line

$$\{(x_1, \bar{x}_2 - \varepsilon) \mid 0 \leqslant x_1 < \bar{x}_1 - \varepsilon\}$$

and the boundary curve $\{(x_1, \phi_\varepsilon(x_1)) \mid \bar{x}_1 - \varepsilon \leqslant x_1 \leqslant N\}$.

We examine the scalar product of u with the vector normal to these two borders, we will show that there is a control α such that the vector u is inside the domain $\mathrm{Viab}(\bar{x}_2)_\varepsilon$.

On the segment $\{(x_1, \phi_\varepsilon(x_1)) \mid \bar{x}_1 - \varepsilon \leqslant x_1 \leqslant N\}$ we have $\mathcal{H}(x_1, \bar{x}_2 - \varepsilon, u_{x_1}, u_{x_2}, \alpha) = (-a(\bar{x}_2 - \varepsilon)x_1 + \alpha_1 x_1 - b\,x_1)n_{x_1} + a(\bar{x}_2 - \varepsilon)\bar{x}_2(S - \bar{x}_1)n_{x_2}$ the vector $\binom{n_{x_1}}{n_{x_2}} = \binom{0}{n_{x_2}}$ construct the normal cone external to the segment with $n_{x_2} > 0$.

So,

$$\begin{aligned}\inf_{\alpha\in\mathcal{A}} \mathcal{H}(x_1, \bar{x}_2 - \varepsilon, 0, n_{x_1}, \alpha) &= a\,(\bar{x}_2 - \varepsilon)\,\bar{x}_2\,(x_1 - \bar{x}_1)\,n_{x_2}\\ &\leqslant -a\,\varepsilon\,(\bar{x}_2 - \varepsilon)\,n_{x_2} < 0.\end{aligned}$$

So, for the control $\alpha = 1$ u is inward from $\mathrm{Viab}(\bar{x}_2)_\varepsilon$.

On the border $\{(x_1, \phi_\varepsilon(x_1)) \mid \bar{x}_1 - \varepsilon \leqslant x_1 \leqslant N\}$ the normal vector: $\binom{n_{x_1}}{n_{x_2}} = \begin{pmatrix} -\phi'_\varepsilon(x_1) \\ 1 \end{pmatrix}$

$$\begin{aligned}\mathcal{H}(x_1, \phi_\varepsilon(x_1), -\phi'_\varepsilon(x_1), 1, \alpha) &= -\phi'_\varepsilon(x_1)(-a\,\phi_\varepsilon(x_1)\,x_1 + -\alpha_1 x_1 - b\,x_1)\\ &\quad + a\,\phi_\varepsilon(x_1)(x_1 - \bar{x}_1)\end{aligned}$$

we have $\phi'_\varepsilon(x_1) < 0$. So,

$$\begin{aligned}\inf_{\alpha\in\mathcal{A}} \mathcal{H}(x_1, \phi_\varepsilon(x_1), -\phi'_\varepsilon(x_1), 1, \alpha) &= -\phi'_\varepsilon(x_1)(-a\,\phi_\varepsilon(x_1)\,x_1 - b\,x_1) + a\,\phi_\varepsilon(x_1)(x_1 - \bar{x}_1)\\ &= -\varepsilon\,\phi'_\varepsilon(x_1) < 0\end{aligned}$$

and therefore the control $\alpha = 1$ on the boundary is such that u is within the domain $\mathrm{Viab}(\bar{x}_2)_\varepsilon$.

On the common end $(\bar{x}_1 - \varepsilon, \bar{x}_2 - \varepsilon)$ the normal cone: $\binom{n_{x_1}}{n_{x_2}} = \delta \begin{pmatrix} -\phi'_\varepsilon(\bar{x}_1 - \varepsilon) \\ 1 \end{pmatrix} + \beta \binom{0}{1}$, with $\delta \geqslant 0$ and $\beta \geqslant 0$ and $\delta + \beta > 0$. And therefore the Hamiltonian

$$\begin{aligned}\mathcal{H}(\bar{x}_1 - \varepsilon, \bar{x}_2 - \varepsilon, n_{x_1}, n_{x_2}, \alpha) &= -\alpha\,\phi'_\varepsilon(\bar{x}_1 - \varepsilon)(-a(\bar{x}_2 - \varepsilon)(\bar{x}_1 - \varepsilon)\\ &\quad + \alpha_1 - b\,(\bar{x}_1 - \varepsilon) - a\,\varepsilon\,(\delta + \beta)(\bar{x}_2 - \varepsilon).\end{aligned}$$

Since $\phi'_\varepsilon(x_1) < 0$ and $\delta \geqslant 0$, we have

$$\begin{aligned}\inf_{\alpha\in\mathcal{A}} \mathcal{H}(\bar{x}_1 - \varepsilon, \bar{x}_2 - \varepsilon, n_{x_1}, n_{x_2}, \alpha) &= -\delta\,\phi'_\varepsilon(\bar{x}_1 - \varepsilon)(-a(\bar{x}_2 - \varepsilon)(\bar{x}_1 - \varepsilon)\\ &\quad - b\,(\bar{x}_1 - \varepsilon) - a\,\varepsilon\,(\delta + \beta)(\bar{x}_2 - \varepsilon) < 0\end{aligned}$$

and therefore the control $\alpha = 1$ on the common end is such that u is towards the interior of the domain $\mathrm{Viab}(\bar{x}_2)_\varepsilon$.

On $(N, \phi_\varepsilon(N))$ the normal cone: $\binom{n_{x_1}}{n_{x_2}} = \delta \begin{pmatrix} -\phi'_\varepsilon(N) \\ 1 \end{pmatrix} + \beta \binom{1}{0}$ with $\delta \geqslant 0$ and $\beta \geqslant 0$

$$\begin{aligned}\inf_{\alpha\in\mathcal{A}} \mathcal{H}(N, \phi_\varepsilon(N), n_{x_1}, n_{x_2}, \alpha) &= (\beta\,\delta\,\phi'_\varepsilon(N))(-a\phi_\varepsilon(N)N - bN) + \delta\,a\,\phi_\varepsilon(N)(N - \bar{x}_1)\\ &= \beta\,(-a\,\phi_\varepsilon(N))N - bN) + \alpha(N(\beta\,\phi_\varepsilon(N) + b)\phi'_\varepsilon(N)\\ &\quad + a\,\phi_\varepsilon(N)(N - \bar{x}_1))\\ &= \beta\,(-a\,\phi_\varepsilon(N))N - bN) - \delta\,\varepsilon\,\phi_\varepsilon(N) < 0,\end{aligned}$$

$\phi_\varepsilon > 0$, for the control $\alpha = (1, 1)$, the vector u is within the domain $\mathrm{Viab}(\bar{x}_2)_\varepsilon$.

On conclusion $\mathrm{Viab}(\bar{x}_2)_\varepsilon$ is a viable set. ∎

4. Viable decisions

From any state of the viability kernel, it is possible to define at least one viable trajectory guaranteeing the satisfaction of the viability constraints at all times. However, the existence of a viable trajectory from a given state does not mean that all trajectories from this state will satisfy the constraints. An important question then is to determine the controls, or decisions, that generate viable trajectories. At each time t, these decisions are such that:

(1) the constraints are satisfied at time t;

(2) the trajectory generated by the decisions remains in the viability kernel.

Given a viable current state $(x_1, x_2) \in \text{Viab}(\bar{x}_2)$, viable controls (α_1, α_2) ensure that the velocity $(\dot{x}_1, \dot{x}_2)$ is tangent or inward to the viability kernel. Therefore, viable controls are not necessarily unique. From this point of view, this method provides not a single policy, but the set of all viable policies. We will see that this flexibility allows us to consider different viable trajectories based on different decision. But first, we describe viable controls.

Now we will compare our method with the more typical method.

For a fixed control α, the control reproduction number [3–5] is $\alpha\mathcal{R}_0$. We know that,

if $\alpha\mathcal{R}_0 < 1$, the equilibrium $(\alpha_1, 0)$ is globally asymptotically stable, and the epidemic stops asymptotically;

if $\alpha\mathcal{R}_0 > 1$, the susceptible and the infected approach constant levels [6] .

The asymptotic method operates like this [3,7]. For control α that is strictly bigger than the crucial fraction of the population that would be vaccinated,

$$\alpha_{x_2} := 1 - \frac{1}{\mathcal{R}_0}, \tag{15}$$

we have $\alpha\mathcal{R}_0 < 1$. Therefore, the stationary vaccination rate $\alpha_t \equiv \alpha > \alpha_{x_2}$ guarantees that as time progresses, the number of infected x_{2_t} will approach zero.

We are not seeking for a stationary vaccination rate policy that will lead to a stable state free of infected asymptotically. We first set a infected threshold $\bar{x}_2$. Next, explore non-stationary vaccination rate strategies that ensure the peak number of infected individuals is lower than $\bar{x}_2$.

Our analysis yields two types of data. Firstly, utilizing non-stationary control, we can ascertain whether it is possible to maintain the number of infected individuals $x_2(t)$ below the threshold for any initial state $(x_{1_{t_0}}, x_{2_{t_0}})$. Secondly, we gain insights into the appropriate vaccination approach to adopt.

When $(x_1(t), x_2(t))$ approaches the upper limit $[0, \bar{x}_1] \times \{\bar{x}_2\}$. The level of vaccination should reach the maximum level of vaccination, $\alpha = (\bar{\alpha}_1, \bar{\alpha}_2)$.

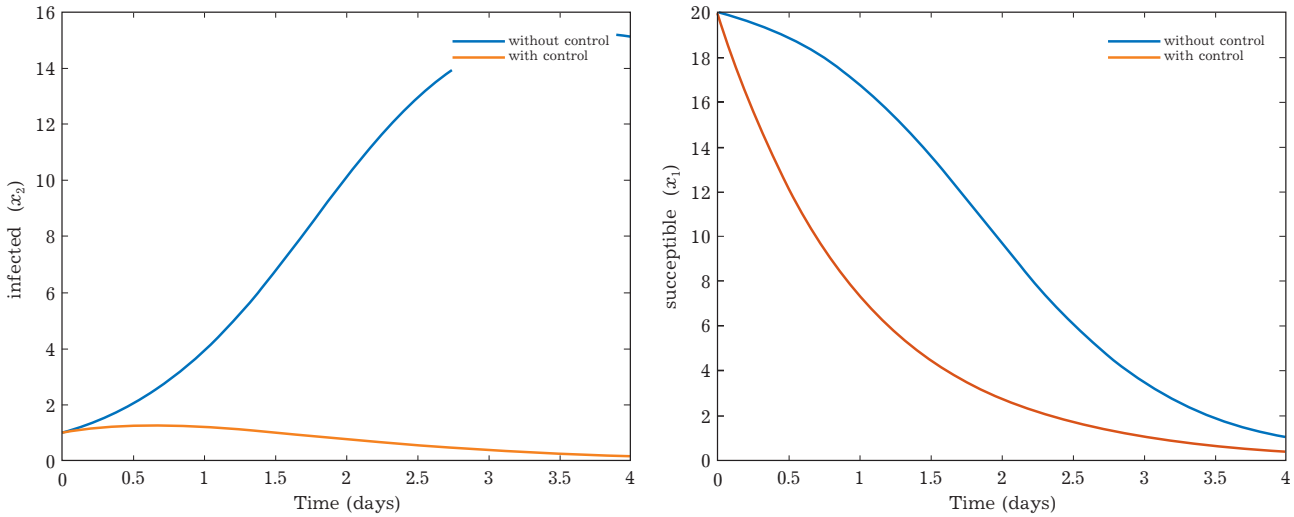

Fig. 4. Simulations of the COVID-19 model showing the effect of the controls $\alpha_1 \neq 0$ and $\alpha_2 \neq 0$.

5. Conclusion

Rather than aiming for equilibrium or optimization, our focus was on developing strategies that could consistently keep the number of infected individuals below a specific threshold. To achieve this, we employed time-dependent vaccination policies that aimed to minimize the susceptible population, thus ensuring the proportion of COVID-19 infected individuals remained below the desired threshold at all times. The viability kernel was defined as the set of initial conditions (including susceptible and infected individuals) for which a feasible control trajectory existed.

Initially, we determined a target infected level $\bar{x}_2$, and then utilized a non-stationary approach to identify all starting states where the maximum number of infected individuals at the peak could remain below $\bar{x}_2$. We presented potential solutions and provided examples of techniques to manage the peak number of infected individuals while also reducing the overall number of infections to zero.

Our approach relied on employing a SIR model of COVID-19, with vaccination serving as the control variable, and viability theory as a framework to address this epidemic. To make the model more realistic and practical, we imposed an upper constraint on the vaccination control, $\bar{\alpha} < 1$, which prevented full vaccination due to either infeasibility or high cost.

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Ефективне керування поширенням COVID-19 за допомогою вакцинації

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Швидке та широке поширення COVID-19 зумовило необхідність розроблення та впровадження ефективних заходів керування. Вакцинація стала ключовим інструментом у боротьбі з пандемією. У цій статті подано новий підхід до моделювання динаміки передачі COVID-19 шляхом інтеграції стратегій вакцинації в структуру "сприйнятливі–інфіковані–одужавші" (SIR) з використанням теорії життєздатності. Визначено набір обмежень, включаючи гарантований рівень вакцинації, проаналізовано вплив різних рівнів вакцинації на стримування поширення вірусу. Отримані результати показують значну роль вакцинації в зниженні передачі та дають цінну інформацію щодо оптимізації кампаній вакцинації. Модель SIR, яка базується на життєздатності, надає політикам і медичним працівникам комплексну основу для розробки цільових стратегій і ефективного розподілу ресурсів у боротьбі з COVID-19.

Ключові слова: *теорія керування; теорія життєздатності; епідеміологія; модель SIR; COVID-19; керування вакцинацією.*