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On fundamental solution of the Cauchy problem for ultra-parabolic equations in the Asian options models

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Paper studies ultra-parabolic equations with three groups of spatial variables appearing in Asian options problems. The class of these equations which satisfy some conditions was denoted by $\mathbf{E}_{22}^{B}$. This class is a generalization of the well-known class of degenerate parabolic Kolmogorov type equations $\mathbf{E}_{22}$. So called L-type fundamental solutions have been constructed for the equations from the class $\mathbf{E}_{22}^{B}$ previously, and some their properties have been established as well. The main feature of the research was the establishing of an one-to-one correspondence between the classes $\mathbf{E}_{22}^{B}$ and $\mathbf{E}_{22}$. The Cauchy problem classic fundamental solutions for the equations from the class $\mathbf{E}_{22}^{B}$ are considered. Special Hölder conditions with respect to spatial variables are applied to the coefficients of the equations.

Keywords: *Asian options; ultra-parabolic equation of Kolmogorov type; fundamental solution of the Cauchy problem.*

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1. Introduction

Let us consider the well-known Black–Scholes market model of financial derivatives (contracts) [1] and some of its generalizations.

Bonds and stocks are usually called *primary financial assets*. The simplest stochastic model of the price S_t of securities (stocks) on the asset market is the following one:

$$S_t = S_0 + \mu t + \sigma W_t, \tag{1}$$

where S_0 is the price of the asset at the time moment $t = 0$, the values μ and σ are called trend and volatility coefficients, and W_t is the Wiener process (or Brownian motion process).

Recall, if $(\Omega, \mathcal{F}, P)$ is a probability space then real-valued stochastic Wiener process $W_t = \{W_t, t \geqslant 0\}$ in this space satisfies the following conditions:

1) $W_0 = 0$;

2) the process W_t has independent increments;

3) increments $W_t - W_s$ for an arbitrary $0 \leqslant s \leqslant t$ have normal distribution law with zero mean and variance $t - s$ (that is, $W_t - W_s \sim N(0, t-s)$).

The model (1) has essential disadvantages: the price can have negative values, and the dispersion of S_t on the segment $[t, t+\Delta t]$ is equal to $\sigma^2 \Delta t$, thus, it does not depend on the values of S_t, that is contrary to reality.

These disadvantages of the model (1) are eliminated in the exponential model or in the model of geometric Brownian motion. We assume S_t to be a solution of the stochastic differential equation [2, p. 388]:

$$dS_t = \mu S_t \, dt + \sigma S_t \, dW_t. \tag{2}$$

It follows from (2) that the mathematical expectation $\mathbb{E}S_t = e^{\mu t}\mathbb{E}S_0$, then the value μ can be interpreted as the growth rate of the stock price or as the risk-free interest rate (that is, the percentage that the owner of the stock is guaranteed to receive). The volatility σ has the sense of a measure of stock "mobility". As a rule, it is measured as a percentage of change over an annual period. The volatility can also be understood as the dispersion of relative increments due to the approximate equality $\mathbb{D}\left(\frac{S_{t+\Delta t}-S_t}{S_t}\right) \approx \sigma^2 \Delta t$.

In the market of secondary securities (derivatives), we consider *payment obligations*. They are instruments with a time of execution T ("expiry time"), for which a certain reward is paid [3, p. 356]. When building models, payment obligations are equated with their payments.

From a point of view of stochastic modeling, the payment obligation C is a non-negative $\mathcal{F}_T$-dimensional stochastic variable in the probability space $(\Omega, \mathcal{F}_t, P)$, $t \geqslant 0$; it models the derivative of the primary assets, i.e. it determines the amount of payments that depends on these primary assets. The European payment obligation or *European option* is a derivative that depends only on the basic prices at the moment T, i.e. $C = f(S_T)$, f is called payout function of the option.

European call option with the strike price K and with an execution date T for a unit stock S_t is a contract that gives its buyer (the option holder) the right to buy a unit of the asset S_t at the time T by the agreed price K. The owner of the option decides whether to execute this right. As a rational decision, the owner executes the option if and only if S_t at the time T exceeds K. His profit in this case is the difference $S_T - K$. Indeed, the owner can make this profit selling the asset in the market at the current price immediately. In the general case, the profit of the owner is $C_{call} = (S_T - K)^+ := \max\{S_T - K, 0\}$.

Similarly, European put option gives its owner the right (but not the obligation) to sell the primary asset at time T by the strike price K. In this case, his profit (payout) is $C_{put} = (K - S_T)^+$.

Note that there is also another type of option — American option, the owner of which can execute his right at any time before the end date of the agreement.

In the basic paper [1] it was studied the problem of finding the fair price of the European option with the variable stock price according to the model (2). If we assume that μ in the model (2) coincides with the fixed risk-free interest rate ($\mu = r$), then in the case of "fair play" at the moment $t = T$ the reasonable value C of the $Call$-type European option (for purchase) should be equal to

$$V = e^{-rT}\mathbb{E}\max\{0, S_T - K\}.$$

By writing the mathematical expectation, we get

$$V = \frac{e^{-rT}}{\sqrt{2\pi T}} \int_{-\infty}^{\infty} f\left(S_0 e^{(r-\sigma^2/2)T+\sigma x}\right) \exp\left(-\tfrac{x^2}{2T}\right) dx, \quad f(x) = \max\{x - K, 0\}.$$

Calculating the integral leads to the so-called Black–Scholes or Black–Scholes–Merton formula [2, P. 423]:

$$V = S_0\Phi(d_1) - Ke^{-rT}\Phi(d_2),$$

where $d_1 = \frac{\ln(S_0/K)+(r+\sigma^2/2)T}{\sigma\sqrt{T}}$, $d_2 = \frac{\ln(S_0/K)+(r-\sigma^2/2)T}{\sigma\sqrt{T}}$, Φ is the distribution function of the normal Gaussian random variable.

In [1], Black and Scholes also proved that the option price as a function of the asset price and of time $V(S_t, t)$ under some assumptions for the financial market is a solution of the following partial differential equation

$$-rV(S,t) + \frac{\partial V(S,t)}{\partial t} + rS\frac{\partial V(S,t)}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V(S,t)}{\partial S^2} = 0, \quad (S,t) \in \mathbb{R}^+ \times (0,T), \tag{3}$$

with the final condition $V(S_T, T) = \max\{S_T - K, 0\}$.

2. Mathematical models of Asian options

Unlike the European option, the payout of *Asian derivative* depends on the entire trajectory of the price value, not the final value only. For instance, it may depend on the average price $A_T = \frac{1}{T}\int_0^T S_t dt$ of the asset during the time interval $[0, T]$. Such financial instruments are called dependent on the trajectory. Note that we will consider only so-called "European" type of Asian option, when the date of its execution is fixed: it coincides with the time moment T.

Asian options were introduced in currency and product markets to avoid the problems of European options, which can be manipulated by the price of the primary asset nearing maturity. They can be: price-averaged call options with a payout $(A_T - K)^+$, price-averaged put options with a payout $(K - A_T)^+$, a strike-averaged call options with a payout $(S_T - A_T)^+$, etc.

One of the methods of researching Asian options is to include dependent on the price trajectory variables in the state space. This way was done for the first time in the papers [4, 5].

Let A_t be an average arithmetic value of S_τ on the segment $[0, t]$:

$$A_t = \frac{1}{t}\int_0^t S_\tau d\tau, \quad t \leqslant T, \quad A_0 = S_0.$$

Then

$$dA_t = \frac{1}{t}(S_t - A_t)dt$$

and the option price at time t already depends on three values $V = V(S_t, A_t, t) = V(S, A, t)$. In this case, repeating the assumptions and actions for obtaining the equation (3), it can be obtained for V the following partial differential equations

$$\frac{\partial V(S,A,t)}{\partial t} + rS\frac{\partial V(S,A,t)}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V(S,A,t)}{\partial S^2} - rV(S,A,t) + \frac{1}{t}(S - A)\frac{\partial V(S,A,t)}{\partial A} = 0, \quad (S,A,t) \in \mathbb{R}^+ \times \mathbb{R}^+ \times (0,T), \quad (4)$$

with the final condition $V(S_T, A_T, T) = g(S_T, A_T)$, where g depends on the type of the option. For example, for the Asian call option with average price and payout K: $g(S_T, A_T) = (A_T - K)^+$.

The Asian option may depend on the geometric average value of the stock price $\exp\left\{\frac{1}{t}M_t\right\}$, where $M_t = \int_0^t \log(S_\tau)d\tau$, $t \leqslant T$. Also we state, for instance, $V(S_T, M_T, T) = \left(S_T - e^{\frac{M_T}{T}}\right)^+$. Then, reasoning similarly, the following equation [6] can be obtained for V:

$$\frac{\partial V(S,M,t)}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V(S,M,t)}{\partial S^2} + rS\frac{\partial V(S,M,t)}{\partial S} + (\log S)\frac{\partial V(S,M,t)}{\partial M} - rV(S,M,t) = 0, \quad (S,t) \in \mathbb{R}^+ \times (0,T), \quad M \in \mathbb{R}. \quad (5)$$

Note that the equation (5) by replacing the variables can be reduced (Ref. [7, p. 479]) to the following well-known equation

$$\partial_{x_1x_1}u + x_1\partial_{x_2}u - \partial_t u = 0, \quad (x_1, x_2, t) \in \mathbb{R}^3, \quad (6)$$

the fundamental solution of which was explicitly constructed by A. M. Kolmogorov describing the processes of diffusion with inertia in the paper [8].

Thus, as we can see, the expansion of the state space by including of dependent on the price trajectory variables transforms the path-dependent problem for the Asian option into an equivalent path-independent Markov problem. However, the increasing of the dimension usually leads to partial differential equations which are not uniformly parabolic. For example, the equations (4)–(6) contain the second order derivatives with respect to one of two the "spatial" variables only.

Early models which generalize the Black–Scholes model assumed that the volatility σ is a constant value. In [9] it was proposed a model for the European options with volatility depending on the difference between current and past price of the asset. Then the price of the Asian option, depending on time, on current price of the asset and on the geometric average value of the price for the period $[0, T]$ as $r = 0$ satisfies the equation

$$\frac{1}{2}\sigma^2(S,M)\left(\frac{\partial^2 V(S,M,t)}{\partial S^2} - \frac{\partial V(S,M,t)}{\partial S}\right) + (S - M)\frac{\partial V(S,M,t)}{\partial M} + \frac{\partial V(S,M,t)}{\partial t} = 0, \quad (S,t) \in \mathbb{R}^+ \times (0,T), \quad M \in \mathbb{R}. \quad (7)$$

In [10] authors used spectral analysis for calculation of option prices in the case when they depended on stochastic volatility.

Equations (4) and (7) are ultra-parabolic equations of the Kolmogorov type with Hölder-continuous coefficients.

In general, mathematical models of the options are reduced to a Markov–type financial model, the dynamics of which is determined by stochastic differential equation in the N-dimensional state space

$$dX_t = (BX_t + b(t, X_t))\, dt + \sigma(t, X_t)\, dW_t, \tag{8}$$

where W_t is the d-dimensional standard Wiener process, $d \leqslant N$, $\sigma = \sigma(t, x)$ is a matrix of dimension $N \times d$, $B = (b_{ij})$ is a constant matrix of dimension $N \times N$, the vector $b = (b_1, \ldots, b_N)$ is such that $b_{d+1} = \ldots = b_N = 0$.

Under certain assumptions on the matrix σ, B, b, in the paper [11] it was proved the existence and uniqueness of a weak solution of the equation (8), and in [12] it was proved that the transition probability density of this solution is a fundamental solution of the Cauchy problem (further we will denote it by FSCP) for the equation

$$L_1 u := \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(t,x)\partial_{x_i}\partial_{x_j} u(t,u) + \sum_{i,j=1}^{N} b_{ij}x_j\partial_{x_i}u(t,x) + \sum_{i=1}^{d} b_i(t,x)\partial_{x_i}u(t,x) + \partial_t u(t,x) = 0, \tag{9}$$

where the elements of the matrix $(a_{ij}(t,x))_{i,j=1}^{d}$ are determined by the elements of the matrix $\sigma(t,x)$. Note that the elements of the matrix σ and the vector b, in particular, must be bounded and satisfy the Hölder-continuity conditions, and the conditions on the matrix B are equivalent to the fact that for the operator L_1 with fixed in each point (t,x) coefficients, the hypoellipticity condition of L. Hermander is fulfilled.

Mathematical models of the options have been studied in many works. Equations of the type (9) are ultra-parabolic equations of the Kolmogorov type. In the more general form

$$L_2 u := \sum_{i,j=1}^{p_0} a_{ij}(t,x)\partial_{x_i}\partial_{x_j}u + \sum_{i=1}^{p_0} a_i(t,x)\partial_{x_i}u + c(t,x)u + \sum_{i,j=1}^{N} b_{ij}x_i\partial_{x_j}u - \partial_t u = 0, \tag{10}$$

where $1 \leqslant p_0 < N$, the matrix $A_0 := (a_{i,j})_{i,j=1}^{p_0}$ is symmetric and positive definite, and the matrix $B := (b_{i,j})_{i,j=1}^{N}$ with constant real elements has the form

$$\begin{pmatrix} * & B_1 & O & \ldots & O \\ * & * & B_2 & \ldots & O \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ * & * & * & \ldots & B_r \\ * & * & * & \ldots & * \end{pmatrix},$$

the equations were studied by a number of Italian mathematicians, in particular, in the works [13–15]. B_j are matrices of size $p_{j-1} \times p_j$ with the rank p_j, where p_0, p_1, …, p_r are integer positive numbers such that $p_0 \geqslant p_1 \geqslant \ldots \geqslant p_r \geqslant 1$, $p_0 + p_1 + \ldots + p_r = N$, O are zero-matrices of corresponding dimensions, and $*$-blocks are arbitrary.

In the equation (10), under the specified conditions on the matrix B, the operator L_2 is hypoelliptic, as well as it is invariant with respect to some group of extensions. With respect to this group, in the [13] it was introduced a special B-Hölder condition imposed on the coefficients a_{ij}, a_i, and c.

The main challenges in the study of the Asian options models while reducing them to ultra-parabolic equations of the Kolmogorov type are the following: the construction, researching of the existence, uniqueness and properties (for instance, such as non-negativity, normality, convolution formula) of the FSCP as the probability density of the transition between the states of the stochastic process, described by the corresponding stochastic differential equation of the form (8).

The following parts of this paper are devoted to this issue.

3. Equations with three groups of spatial variables

In the case of three groups of spatial variables, the equation (10) can be written in the following form:

$$(S_B - A(t, x, \partial_{x_1}))u(t,x) = 0, \quad (t,x) \in \Pi_{(0,T]}, \tag{11}$$

where n_1, n_2, n_3 are integer non-negative numbers such that $n_3 \leqslant n_2 \leqslant n_1$, $n := n_1 + n_2 + n_3$; $x := (x_1, x_2, x_3)$, $x_i := (x_{i1}, \dots, x_{in_i})$, $i \in \{1, 2, 3\}$; $\Pi_{(0,T]} := \{(t, x) | t \in (0, T], x \in \mathbb{R}^n\}$,

$$S_B := \partial_t - \sum_{j=1}^{n_2} \left(\sum_{s=1}^{n_1} b^1_{sj} x_{1s} \right) \partial_{x_{2j}} - \sum_{j=1}^{n_3} \left(\sum_{s=1}^{n_2} b^2_{sj} x_{2s} \right) \partial_{x_{3j}}, \tag{12}$$

$A(t, x, \partial_{x_1}) := \sum_{i,j=1}^{n_1} a_{ij}(t, x) \partial_{x_{1i}} \partial_{x_{1j}} + \sum_{i=1}^{n_1} a_i(t, x) \partial_{x_{1i}} + a_0(t, x)$.

The differential expression (12) is in matrix form

$$S_B = \partial_t - (x, BD_x),$$

where B is a $n \times n$-matrix:

$$B := \begin{pmatrix} O & B^1 & O \\ O & O & B^2 \\ O & O & O \end{pmatrix}, \tag{13}$$

where B^1, B^2 are matrices composed of real numbers b^1_{ij}, $i \in \{1, \dots, n_1\}$, $j \in \{1, \dots, n_2\}$, b^2_{ij}, $i \in \{1, \dots, n_2\}$, $j \in \{1, \dots, n_3\}$, O are null-matrices of corresponding dimensions, $D_x := \mathrm{col}(\partial_{x_{11}}, \dots, \partial_{x_{1n_1}}, \partial_{x_{21}}, \dots, \partial_{x_{2n_2}}, \partial_{x_{31}}, \dots, \partial_{x_{3n_3}})$, $(\cdot, \cdot)$ is a scalar product in $\mathbb{R}^n$.

We will use the following conditions:

$\mathbf{A}_1$. In the matrix (13), the blocks B^1 and B^2 are written in the form $\begin{pmatrix} B^1_1 \\ B^1_2 \end{pmatrix}$ and $\begin{pmatrix} B^2_1 \\ B^2_2 \end{pmatrix}$ respectively, where dimensions of matrices B^1_1, B^1_2, B^2_1 and B^2_2 are $n_2 \times n_2$, $(n_1 - n_2) \times n_2$, $n_3 \times n_3$ and $(n_2 - n_3) \times n_3$ respectively, and they satisfy the condition $\det B^i_1 \neq 0$, $i \in \{1, 2\}$;

$\mathbf{A}_2$. There exists a constant $\delta > 0$ such that for any point $(t, x) \in \Pi_{[0,T]}$ and $\sigma_1 \in \mathbb{R}^{n_1}$ the inequality

$$\mathrm{Re} \sum_{i,j=1}^{n_1} a_{ij}(t, x) \sigma_{1i} \sigma_{1j} \geqslant \delta \sum_{i=1}^{n_1} \sigma^2_{1i}$$

holds.

S. D. Ivasyshen and V. V. Laiuk in theirs works (for example, in [16]) denoted by $\mathbf{E}^B_{22}$ the class of equations (11) satisfying the conditions $\mathbf{A}_1$ and $\mathbf{A}_2$. This class generalizes the class of ultra-parabolic equations of the Kolmogorov type $\mathbf{E}_{22}$ introduced in the monograph [17]. The main feature of the research of the mentioned authors is the establishing of an one-to-one correspondence between the classes $\mathbf{E}^B_{22}$ and $\mathbf{E}_{22}$.

We will use expressions connecting spatial variables and the elements of the matrix B:

$$X(h) := (X_1(h), X_2(h), X_3(h)), \quad X_i(h) := (X_{i1}(h), \dots, X_{in_i}(h)), \quad i \in \{1, 2, 3\}, \tag{14}$$

$$X_{1j}(h) := x_{1j}, \quad j \in \{1, \dots, n_1\}, \quad X_{2j}(h) := x_{2j} + h \sum_{i=1}^{n_1} b^1_{ij} x_{1i}, \quad j \in \{1, \dots, n_2\},$$

$$X_{3j}(h) := x_{3j} + h \sum_{i=1}^{n_2} b^2_{ij} x_{2i} + \frac{h^2}{2} \sum_{i=1}^{n_1} \sum_{s=1}^{n_2} b^2_{sj} b^1_{is} x_{1i}, \quad j \in \{1, \dots, n_3\}, \quad h \in \mathbb{R}.$$

Statement 1. *Under the condition $\mathbf{A}_1$ the substitution of the spatial variables*

$$\hat{x}_{1j} = \begin{cases} \sum_{i=1}^{n_1} \sum_{s=1}^{n_2} b^2_{sj} b^1_{is} x_{1i}, & j \in \{1, \dots, n_3\}, \\ \sum_{i=1}^{n_1} b^1_{ij} x_{1i}, & j \in \{n_3 + 1, \dots, n_2\}, \\ x_{1j}, & j \in \{n_2 + 1, \dots, n_1\}; \end{cases} \quad \hat{x}_{2j} = \begin{cases} \sum_{i=1}^{n_2} b^2_{ij} x_{2i}, & j \in \{1, \dots, n_3\}, \\ x_{2j}, & j \in \{n_3 + 1, \dots, n_2\}; \end{cases}$$

$$\hat{x}_{3j} = x_{3j}, \quad j \in \{1, \dots, n_3\}$$

is non-degenerate.

The transformation of the variables from the Statement 1 can be written in the matrix form

$$\hat{x}' = U x', \tag{15}$$

where the matrix U is a block-diagonal one:

$$U := \begin{pmatrix} U_1 & O & O \\ O & U_2 & O \\ O & O & U_3 \end{pmatrix},$$

$$U_1 := \begin{pmatrix} \sum_{s=1}^{n_2} b^2_{s1} b^1_{1s} & \dots & \sum_{s=1}^{n_2} b^2_{s1} b^1_{n_2 s} & \sum_{s=1}^{n_2} b^2_{s1} b^1_{n_2+1,s} & \dots & \sum_{s=1}^{n_2} b^2_{s1} b^1_{n_1 s} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \sum_{s=1}^{n_2} b^2_{sn_3} b^1_{1s} & \dots & \sum_{s=1}^{n_2} b^2_{sn_3} b^1_{n_2 s} & \sum_{s=1}^{n_2} b^2_{sn_3} b^1_{n_2+1,s} & \dots & \sum_{s=1}^{n_2} b^2_{sn_3} b^1_{n_1 s} \\ b^1_{1,n_3+1} & \dots & b^1_{n_2,n_3+1} & b^1_{n_2+1,n_3+1} & \dots & b^1_{n_1,n_3+1} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ b^1_{1n_2} & \dots & b^1_{n_2 n_2} & b^1_{n_2+1,n_2} & \dots & b^1_{n_1 n_2} \\ 0 & \dots & 0 & 1 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & \dots & 1 \end{pmatrix},$$

$$U_2 := \begin{pmatrix} b^2_{11} & \dots & b^2_{n_3 1} & b^2_{n_3+1,1} & \dots & b^2_{n_2 1} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ b^2_{1n_3} & \dots & b^2_{n_3 n_3} & b^2_{n_3+1,n_3} & \dots & b^2_{n_2 n_3} \\ 0 & \dots & 0 & 1 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & \dots & 1 \end{pmatrix}, \quad U_3 = I_{n_3} := \begin{pmatrix} 1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 1 \end{pmatrix}.$$

In the formula (15) and below, the dash means the matrix transposition.

By direct calculation, we make sure that determinant of the matrix U is

$$|U| = |U_1| \cdot |U_2| \cdot |U_3| = |B^1_1| \cdot |B^2_1|^2 \neq 0.$$

It proves the Statement 1.

Structure and non-degeneracy of the variable substitution allow us to prove the next statement.

Statement 2. *Under the condition $\mathbf{A}_1$ the substitution of spatial variables (15) reduces the equation (11) to the equation*

$$(S_{\hat{B}} - \hat{A}(t, \hat{x}, \partial_{\hat{x}_1}))\hat{u}(t, \hat{x}) = 0, \quad (t, \hat{x}) \in \Pi_{(0,T]}, \tag{16}$$

where

$$\hat{B} := \begin{pmatrix} O & \hat{B}^1 & O \\ O & O & \hat{B}^2 \\ O & O & O \end{pmatrix}, \quad \hat{B}^1 := \begin{pmatrix} I_{n_2} \\ O \end{pmatrix}, \quad \hat{B}^2 := \begin{pmatrix} I_{n_3} \\ O \end{pmatrix},$$

I_{n_2} and I_{n_3} are unit matrices of dimensions n_2 and n_3 respectively, O are zero-matrices of corresponding dimensions, the differential expression $\hat{A}(t, \hat{x}, \partial_{\hat{x}_1})$ has the same form as the expression $A(t, x, \partial_{x_1})$, its coefficients $\hat{a}_{ij}$, $\hat{a}_i$, and $\hat{a}_0$ are expressed in terms of the coefficients a_{ij}, a_i, and a_0 with respect to new variables $\hat{x}$ and of the elements of the matrices B^1 and B^2.

Also, from the condition $\mathbf{A}_2$ for the equation (11) the condition $\hat{A}_2$ for the equation (16) implies. This new condition $\hat{A}_2$ actually does not differ from the condition $\mathbf{A}_2$.

4. Equations with coefficients depending on the time variable only

First, consider the case when the coefficients of the equation (11) do not depend on the spatial variables. That is, let us consider the equation

$$(S_B - A_0(t, \partial_{x_1}))u(t, x) = 0, \quad (t, x) \in \Pi_{(0,T]}, \tag{17}$$

where $A_0(t, \partial_{x_1}) := \sum_{i,j=1}^{n_1} a_{ij}(t)\partial_{x_{1i}}\partial_{x_{1j}} + \sum_{i=1}^{n_1} a_i(t)\partial_{x_{1i}} + a_0(t)$, with condition $\mathbf{A}_2^0$. The coefficients of the expression $A(t, \partial_{x_1})$ are continuous functions on $[0, T]$ and there exists a constant $\delta > 0$ such that for all $t \in [0, T]$ and $\sigma_1 \in \mathbb{R}^{n_1}$ the inequality

$$\mathrm{Re} \sum_{i,j=1}^{n_1} a_{ij}(t)\sigma_{1i}\sigma_{1j} \geqslant \delta \sum_{i=1}^{n_1} \sigma_{1i}^2$$

holds.

We denote by

$$M := \sum_{l=1}^{3}(l-1/2)n_l, \quad M_{kl} := \sum_{r=1}^{3}(2(r-1)+1)(|k_r|+|l_r|)/2, \quad \text{if} \quad \{k,l\} \subset \mathbb{Z}_+^n, \quad k := (k_1,k_2,k_3),$$
$$l := (l_1,l_2,l_3), \quad k_r := (k_{r1},\ldots,k_{rn_r}), \quad l_r := (l_{r1},\ldots,l_{rn_r}), \quad r \in \{1,2,3\};$$
$$x_t := (t^{-1/2}x_1, t^{-3/2}x_2, \quad t^{-5/2}x_3), \quad x := (x_1,x_2,x_3), \quad x_r := (x_{r1},\ldots,x_{rn_r}), \quad r \in \{1,2,3\};$$
$$E_c(t,x;\tau,\xi) := \exp\left\{-c\sum_{l=1}^{3}(t-\tau)^{1-2l}|X_l(t-\tau)-\xi_l|^2\right\}, \quad t>\tau, \quad \{x,\xi\}\subset\mathbb{R}^n,$$

where the expressions for X_l, $l \in \{1,2,3\}$, are given in (14).

Theorem 1. *If the conditions $\mathbf{A}_1$ and $\mathbf{A}_2^0$ are fulfilled for the equation* (17)*, then*

1) the equation (17) *has an unique FSCP G;*

2) the function G and its derivatives have extensions into the complex space $\mathbb{C}^n$ and the following formulas are correct for these extensions

$$\partial_x^k\partial_\xi^l G(t,x+iy;\tau,\xi+i\eta) = (t-\tau)^{-M-M_{kl}}\Omega_{kl}(t,\tau,z)|_{z=(X(t-\tau)-\xi)_{t-\tau}+i(Y(t-\tau)-\eta)_{t-\tau}},$$
$$0\leqslant\tau<t\leqslant T, \quad \{x,y,\xi,\eta\}\subset\mathbb{R}^n, \quad \{k,l\}\subset\mathbb{Z}_+^n,$$

where $\Omega_{kl}(t,\tau,z)$, $z := (z_1,\ldots,z_n) \in \mathbb{C}^n$ with fixed t and τ are entire functions with respect to $z_1,\ldots,z_n$ with increasing order $q=2$ and with the same decreasing order at $z=x\in\mathbb{R}^n$;

3) the estimates

$$|\partial_x^k\partial_\xi^l G(t,x;\tau,\xi)| \leqslant C_{kl}(t-\tau)^{-M-M_{kl}}E_c(t,x;\tau,\xi), \quad 0\leqslant\tau<t\leqslant T, \quad \{x,\xi\}\subset\mathbb{R}^n, \quad \{k,l\}\subset\mathbb{Z}_+^n,$$

hold, where C_{kl} and c are positive constants depending only on the following: numbers n_1, n_2, n_3, T, the maxima of the modules of the coefficients of the differential expression A_0, the constant δ from the condition $\mathbf{A}_2^0$ and the coefficients of the matrices B^1 and B^2;

4) the formula

$$\int_{\mathbb{R}^n} G(t,x,\tau,\xi)\,d\xi = \exp\left\{(t-\tau)\int_0^1 a_0(\tau+(t-\tau)\beta)\,d\beta\right\}, \quad 0\leqslant\tau<t\leqslant T, \quad x\in\mathbb{R}^n$$

is correct;

5) for $0\leqslant\tau<t\leqslant T$ and $x\in\mathbb{R}^n$

$$\partial_x^k\int_{\mathbb{R}^n} G(t,x,\tau,\xi)\,d\xi = 0, \quad k\in\mathbb{Z}_+^n\setminus\{0\};$$
$$\partial_{x_2}^{k_2}\partial_{x_3}^{k_3}\int_{\mathbb{R}^n} G(t,x,\tau,\xi)\,d\xi_2\,d\xi_3 = 0, \quad (k_2,k_3)\in\mathbb{Z}_+^{n_2+n_3}\setminus\{0\};$$
$$\partial_{x_3}^{k_3}\int_{\mathbb{R}^n} G(t,x,\tau,\xi)\,d\xi_3 = 0, \quad k_3\in\mathbb{Z}_+^{n_3}\setminus\{0\}.$$

The statements of the Theorem 1 are substantiated in [18] using the substitution of variables (15) and similar results from the work [17] for the corresponding equation (16) with the differential expression $\hat{A}_0(t,\partial_{\hat{x}_1})$ with the constant coefficients.

5. *L*-solutions

In [12], the following definitions were introduced for equations from the class $\mathbf{E}_{22}^B$.

Definition 1. *The function u is called differentiable by Lie at the point (t,x) with respect to the vector field which are given by the differential expression (12) if there is the finite limit*

$$(S_B^L u)(t,x) := \lim_{h\to0}\frac{1}{h}\big(u(\gamma(t,x,h)) - u(\gamma(t,x,0))\big),$$

where $\gamma(t,x,h) := \big(t-h,(e^{hB'}x')'\big)$, $h\in\mathbb{R}$, is the integral curve of the given vector field that passes through the point (t,x). The limit $(S_B^L u)(t,x)$ is called the Lie derivative of the function u at the point (t,x) with respect to the given vector field.

Considering the structure of the matrix B, one can see that the matrix exponent $e^{hB'}$ decomposes into a finite sum, and we can receive that
$$(e^{hB'}x')' = X(h), \quad \gamma(t,x,h) = (t-h, X(h)),$$
where the expression $X(h)$ is given by the formula (14).

Note if there exist the derivatives $\partial_t u$, $\partial_{x_{2j}} u$ and $\partial_{x_{3j}} u$ at the point (t,x) then $(S_B^L u)(t,x) = (S_B u)(t,x)$.

Definition 2. *We will call the function u by the L-solution of the equation (11) in $\Pi_{(0,T]}$ if there exist in $\Pi_{(0,T]}$ continuous Lie derivative $S_B^L u$ and ordinary derivatives $\partial_{x_{1j}} u$, $j \in \{1,\ldots,n_1\}$, $\partial_{x_{1j}}\partial_{x_{1s}} u$, $\{j,s\} \subset \{1,\ldots,n_1\}$, and at any point $(t,x) \in \Pi_{(0,T]}$ the equation*
$$\big(S_B^L - A(t,x,\partial_{x_1})\big)u(t,x) = 0, \quad (t,x) \in \Pi_{(0,T]} \tag{18}$$
holds.

Note if the coefficients of the expression A do not depend on the spatial variables then the L-solutions are the ordinary classical solutions of the equation.

To formulate the theorem, we introduce the following notations and definitions:
$$d(x,\xi) := \sum_{i=1}^{3} |x_i - \xi_i|^{1/(2(i-1)+1)}, \quad d(t,x;\tau,\xi) := |t-\tau|^{1/2} + d(x,\xi),$$
$$\Delta_x^{\xi} := f(\cdot,x) - f(\cdot,\xi), \quad \Delta_{t,x}^{\tau,\xi} := f(t,x) - f(\tau,\xi),$$
where $\{t,\tau\} \subset \mathbb{R}$, $\{x,\xi\} \subset \mathbb{R}^n$, f is a some function.

Definition 3. *We will call the function $f(t,x)$, $(t,x) \in \Pi_{[0,T]}$ by B-Hölder function with the exponent $\alpha \in (0,1]$ in $\Pi_{[0,T]}$ if there exists a constant $H > 0$ such that for any $\{(t,x),(\tau,\xi)\} \subset \Pi_{[0,T]}$*
$$\Delta_{t,x}^{\tau,\xi} f(t,x) \leqslant H\big(d(t, X(t-\tau); \tau, \xi)\big)^{\alpha}.$$

Theorem 2. *Let the coefficients of the equation (11) satisfy the conditions $\mathbf{A}_1$, $\mathbf{A}_2$ as well as the following condition:*

$\mathbf{A}_3$. *The coefficients of the expression $A(t,x,\partial_{x_1})$ are bounded and B-Hölder with exponent $\alpha \in (0,1)$ in $\Pi_{[0,T]}$.*

Then for the equation (11) there exists a L-FSCP (there is a FSCP for the equation (18)) Z and
$$|\partial_{x_1}^{k_1} Z(t,x;\tau,\xi)| \leqslant C(t-\tau)^{-M-|k_1|/2} E_c(t,x;\tau,\xi), \quad |k_1| \leqslant 2;$$
$$|S_B^L Z(t,x;\tau,\xi)| \leqslant C(t-\tau)^{-M-1} E_c(t,x;\tau,\xi),$$
$$|\Delta_x^{x'} \partial_{x_1}^{k_1} Z(t,x;\tau,\xi)| \leqslant C(d(x,x')^{\alpha})(t-\tau)^{-M-(|k_1|+\alpha)/2}\big(E_c(t,x;\tau,\xi) + E_c(t,x';\tau,\xi)\big), \quad |k_1| \leqslant 2;$$
$$|\Delta_x^{x'} S_B^L Z(t,x;\tau,\xi)| \leqslant C\big(d(x,x')^{\alpha}\big)(t-\tau)^{-M-1-\alpha/2}\big(E_c(t,x;\tau,\xi) + E_c(t,x';\tau,\xi)\big);$$
$$\left|\int_{\mathbb{R}^n} \partial_{x_1}^{k_1} Z(t,x;\tau,\xi)\, d\xi\right| \leqslant C(t-\tau)^{-(|k_1|-\alpha)/2}, \quad 0 < |k_1| \leqslant 2;$$
$$\left|\int_{\mathbb{R}^n} S_B^L Z(t,x;\tau,\xi)\, d\xi\right| \leqslant C(t-\tau)^{-1+\alpha/2},$$
$$0 \leqslant \tau < t \leqslant T, \quad \{x,x',\xi\} \subset \mathbb{R}^n,$$
where C and c are positive constants.

Let the coefficients of the equation (11) satisfy the conditions $\mathbf{A}_1$–$\mathbf{A}_3$ as well as the next condition:

$\mathbf{A}_4$. The coefficients of the expression $A(t,x,\partial_{x_1})$ have bounded derivatives of the same form about which they stand and they are B-Hölder with exponent $\alpha \in (0,1)$ in $\Pi_{[0,T]}$.

Then there is the adjoint equation for the equation (11)
$$S_B^* v(\tau,\xi) - \sum_{i,j=1}^{n_1} \partial_{\xi_{1i}}\partial_{\xi_{1j}}(\overline{a}_{ij}(\tau,\xi)v(\tau,\xi)) + \sum_{i=1}^{n_1} \partial_{\xi_{1i}}(\overline{a}_i(\tau,\xi)v(\tau,\xi))$$
$$- \overline{a}_0(\tau,\xi)v(\tau,\xi) = 0, \quad (\tau,\xi) \in \Pi_{[0,T)}, \tag{19}$$

where

$$S_B^* := -\partial_\tau + \sum_{i=1}^{n_2}\Bigg(\sum_{j=1}^{n_1} b_{ji}^1 \xi_{1j}\Bigg)\partial_{\xi_{2i}} + \sum_{i=1}^{n_3}\Bigg(\sum_{j=1}^{n_2} b_{ji}^2 \xi_{2j}\Bigg)\partial_{\xi_{3i}},$$

and the condition $\mathbf{A}_3$ is fulfilled for the coefficients of this equation. Here, the dash above the coefficient means complex conjugation.

Theorem 3. *If the conditions $\mathbf{A}_1$–$\mathbf{A}_4$ are satisfied for the coefficients of the equation (11), then the adjoint equation (19) has L-FSCP Z^* which is related to the Z by the equality*

$$Z^*(\tau,\xi;t,x) = \overline{Z}(t,x;\tau,\xi), \quad 0 \leqslant \tau < t \leqslant T, \quad \{x,\xi\} \subset \mathbb{R}^n,$$

and Z has the correct convolution formula

$$Z(t,x;\tau,\xi) = \int_{\mathbb{R}^n} Z(t,x;\lambda,y)\, Z(\lambda,y;\tau,\xi)\, dy, \quad 0 \leqslant \tau < \lambda < t \leqslant T, \quad \{x,\xi\} \subset \mathbb{R}^n.$$

The statements of the Theorems 2 and 3 are formulated in [18]. We can obtain them by using the substitution of variables (15) and the results from the monograph [17] for the equations (16) from the class $\mathbf{E}_{22}$. Indeed, under the conditions $\mathbf{A}_3$ and $\mathbf{A}_4$ for the coefficients of the equation (11), the conditions $\hat{A}_3$ and $\hat{A}_4$ are valid, respectively, for the coefficients of the equation (16). The last conditions $\hat{A}_3$ and $\hat{A}_4$ differ from the conditions $\mathbf{A}_3$ and $\mathbf{A}_4$ only that $X(h)$ is replaced by

$$\hat{X}(h) := \big(UX'(h)\big)', \quad \hat{X}_{ij}(h) := \sum_{s=0}^{i-1} \frac{1}{s!} h^s \hat{x}_{(i-s)j}, \quad j \in \{1,\dots,n_i\}, \quad i \in \{1,2,3\}. \tag{20}$$

6. Classic FSCP

Below, we will use the following conditions for the coefficients of the equation (11):

$\mathbf{A}_5$. The coefficients of the expression $A(t,x,\partial_{x_1})$ (that is, the functions a_{ij}, a_i, a_0) are bounded, continuous on t on the segment $[0,T]$ and they satisfy the Hölder condition with respect to spatial variables in the following sense:

$$\exists H_1 > 0, \quad \exists \alpha_1 \in (0,1] \quad \forall (t,x) \in \Pi_{[0,T]}, \quad \forall z_1 \in \mathbb{R}^{n_1}\colon \big|\Delta_{x_1}^{z_1} a(t,x)\big| \leqslant H_1 |x_1 - z_1|^{\alpha_1},$$

$$\exists H_2 > 0, \quad \exists \alpha_2 \in (1/3,2/3] \quad \forall (t,x) \in \Pi_{[0,T]}, \quad \forall z_2 \in \mathbb{R}^{n_2}, \quad \forall h \in [0,T]\colon$$
$$|\Delta_{x_2}^{z_2} a(t,x)| \leqslant H_2 (h^{3\alpha_2/2} + |X_2(h) - z_2|^{\alpha_2}),$$

$$\exists H_3 > 0, \quad \exists \alpha_3 \in (3/5,4/5] \quad \forall (t,x) \in \Pi_{[0,T]}, \quad \forall z_3 \in \mathbb{R}^{n_3}, \quad \forall h \in [0,T]\colon$$
$$|\Delta_{x_3}^{z_3} a(t,x)| \leqslant H_3 \big(h^{5\alpha_3/2} + |X_3(h) - z_3|^{\alpha_3}\big).$$

$\mathbf{A}_6$. The coefficients of the expression $A(t,x,\partial_{x_1})$ (that is, the functions a_{ij}, a_i, a_0) satisfy the Hölder condition with respect to spatial variables in the following sense:

$$\exists H_4 > 0 \quad \forall (t,x) \in \Pi_{[0,T]}, \quad \forall z_i \in \mathbb{R}^{n_i}, \quad i \in \{1,2\}, \quad \forall h \in [0,T]\colon$$
$$\big|\Delta_{x_1}^{z_1}\Delta_{x_2}^{z_2} a(t,x)\big| \leqslant H_4 |x_1 - z_1|^{\alpha_1}\big(h^{3\alpha_2/2} + |X_2(h) - z_2|^{\alpha_2}\big),$$

$$\exists H_5 > 0 \quad \forall (t,x) \in \Pi_{[0,T]}, \quad \forall z_i \in \mathbb{R}^{n_i}, \quad i \in \{1,3\}, \quad \forall h \in [0,T]\colon$$
$$\big|\Delta_{x_1}^{z_1}\Delta_{x_3}^{z_3} a(t,x)\big| \leqslant H_5 |x_1 - z_1|^{\alpha_1}\big(h^{5\alpha_3/2} + |X_3(h) - z_3|^{\alpha_3}\big),$$

where the constants α_1, α_2 and α_3 are the same as in the condition $\mathbf{A}_5$.

It is obvious that at $h = 0$ the classical Hölder conditions for groups of spatial variables follow from the condition $\mathbf{A}_5$.

Theorem 4. *Let the coefficients of the equation (11) satisfy the conditions $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_5$, and $\mathbf{A}_6$. Then there is a classical FSCP Z for the equation and*

$$\begin{gathered}\big|\partial_x^k Z(t,x;\tau,\xi)\big| \leqslant C(t-\tau)^{-M-M_k} E_c(t,x;\tau,\xi), \quad |k_1|/2 + |k_2| + |k_3| \leqslant 1,\\ k = (k_1,k_2,k_3) \in \mathbb{Z}_+^n, \quad 0 \leqslant \tau < t \leqslant T, \quad \{x,\xi\} \subset \mathbb{R}^n;\\ |S_B Z(t,x;\tau,\xi)| \leqslant C(t-\tau)^{-M-1} E_c(t,x;\tau,\xi), \quad 0 \leqslant \tau < t \leqslant T, \quad \{x,\xi\} \subset \mathbb{R}^n.\end{gathered} \tag{21}$$

Here $M_k := \sum_{i=1}^3 (2(i-1)+1)|k_i|/2$ for $k \in \mathbb{Z}_+^n$.

We apply the non-degenerate substitution of variables (15) to the equation (11) and the conditions of the theorem. On the basis of the Statement 2, we obtain the equation (16) from the class $\mathbf{E}_{22}$, and from the conditions $\mathbf{A}_2$, $\mathbf{A}_5$, and $\mathbf{A}_6$ we get for this equation, respectively, the conditions $\hat{A}_2$, $\hat{A}_5$, and $\hat{A}_6$, which differ from the previous ones only by fact that the expression $X(h)$ is replaced by $\hat{X}(h)$ which are defined in (20). Using the results from the papers [19, 20] for equations from the class $\mathbf{E}_{22}$ (namely, Theorem 3 from [19, p. 23]) we obtain a proof of the statement of the Theorem 4.

Theorem 5. *Let the coefficients of the equation (11) satisfy the conditions of the Theorem 4 as well as the following condition:*

$\mathbf{A}_7$. *In $\Pi_{[0,T]}$, there are bounded derivatives $\partial_{x_{1i}}\partial_{x_{1j}}a_{ij}$ and $\partial_{x_{1i}}a_i$ which satisfy the Hölder condition for spatial variables in the sense of $\mathbf{A}_5$ and $\mathbf{A}_6$.*

Then there is classical FSCP Z^* for the adjoint equation (19); the function Z^* is related to the Z by the equality

$$Z^*(\tau,\xi;t,x) = Z(t,x;\tau,\xi), \quad 0 \leqslant \tau < t \leqslant T, \quad \{x,\xi\} \subset \mathbb{R}^n, \tag{22}$$

and for Z the convolution formula

$$Z(t,x;\tau,\xi) = \int_{\mathbb{R}^n} Z(t,x;\lambda,y)\, Z(\lambda,y;\tau,\xi)\, dy, \quad 0 \leqslant \tau < \lambda < t \leqslant T, \quad \{x,\xi\} \subset \mathbb{R}^n \tag{23}$$

is correct.

Note that the equality (22) means the normality property of the FSCP. The convolution formula (23) is also called the Chepmen–Kolmogorov equation. It expresses the important fact that the corresponding stochastic process is Markov process, namely a process without an aftereffect.

The proof of Theorem 5 is based on the Green–Ostrogradsky formula

$$\begin{aligned}\int_{t_1}^{t_2} d\theta \int_{B_R} (\overline{v}Lu - u\overline{L^*v})(\theta,y)\, dy &= \int_{B_R} (\overline{v}u)(\theta,y)|_{\theta=t_1}^{t_2} dy \\ &- \int_{t_1}^{t_2} d\theta \int_{\Gamma_R} \left(\sum_{j=1}^{n_2} \left(\sum_{s=1}^{n_1} b_{sj}^1 y_{1s} \right) \mu_{2j} + \sum_{j=1}^{n_3} \left(\sum_{s=1}^{n_2} b_{sj}^2 y_{2s} \right) \mu_{3j} \right) (\overline{v}u)(\theta,y)\, dS_y \\ &+ \int_{t_1}^{t_2} d\theta \int_{\Gamma_R} \sum_{j=1}^{n_1} B^j[v,u](\theta,y) \mu_{1j} dS_y, \end{aligned} \tag{24}$$

where $0 \leqslant t_1 < t_2 \leqslant T$, B_R is a sphere in $\mathbb{R}^n$ of radius R with its center at the origin, Γ_R is its boundary, $(\mu_{11},\ldots,\mu_{1n_1},\mu_{21},\ldots,\mu_{2n_2},\mu_{31},\ldots,\mu_{3n_3})$ is a unit vector of the outer normal to Γ_R, $L := S_B - A(\theta,y,\partial_{y_1})$, $L^* := S_B^* - A^*(\theta,y,\partial_{y_1})$,

$$B^j[v,u] := -\sum_{l=1}^{n_1} \left(a_{jl}\partial_{y_{1l}} u\overline{v} - u\partial_{y_{1l}}(a_{jl}\overline{v}) \right) + a_j u\overline{v}, \quad j \in \{1,\ldots,n_1\},$$

u and v are sufficiently smooth functions. The formula (24) is also correct for functions u and v which have continuous derivatives with respect to x_1 up to the second order and have the Lie derivatives $S_B^L u$ and $S_B^{*L} v$. This fact is received if we consider approximated for u and v sequences of sufficiently smooth functions, then we write for them the formula (24) and pass to the limit.

With such functions u and v, if we pass in the formula (24) to the limit as $R \to \infty$ then in the real-valued case we obtain the formula

$$\int_{t_1}^{t_2} d\theta \int_{\mathbb{R}^n} (vLu - uL^*v)(\theta,y)\, dy = \int_{\mathbb{R}^n} (vu)(\theta,y)|_{\theta=t_1}^{t_2} dy. \tag{25}$$

Using the estimates from Theorem 4 and the same estimates for Z^*, in the formula (25) we can put $u(\theta,y) = Z(\theta,y;\tau,\xi)$, $v(\theta,y) = Z^*(\theta,y;t,x)$, $t_1 = \tau + \varepsilon$, and $t_2 = t - \varepsilon$, where ε is a small positive number. In the obtained equality, passing to the limit as $\varepsilon \to 0$ we receive the formula (22).

The equality (23) is obtained in the same way, only it is necessary to take $t_1 = \lambda$. We will get equality

$$\int_{\mathbb{R}^n} Z^*(\lambda,y;t,x)\, Z(\lambda,y;\tau,\xi)\, dy = \int_{\mathbb{R}^n} Z^*(t-\varepsilon,y;t,x)\, Z(t-\varepsilon,y;\tau,\xi)\, dy,$$

in which it is necessary to pass to the limit as $\varepsilon \to 0$ and to use the formula (22).

Theorem 6 (Uniqueness of normal classical FSCP). *There is only one normal classical FSCP for which the estimates (21) hold.*

Let Z_1 and Z_2 be two normal classical FSCP of the equations (11) for which the estimates (21) are true. Let us put in the formula (25) $u(\theta, y) = Z_1(\theta, y; \tau, \xi)$, $v(\theta, y) = Z_2(t, x; \theta, y)$. Then we get equality

$$\int_{\mathbb{R}^n} Z_1(t_2, y; \tau, \xi)\, Z_2(t, x; t_2, y)\, dy = \int_{\mathbb{R}^n} Z_1(t_1, y; \tau, \xi)\, Z_2(t, x; t_1, y)\, dy \tag{26}$$

for arbitrary t_1 and t_2 from the interval (τ, t). From the arbitrariness of t_1 and t_2 it follows that the right and left parts in (26) do not depend on t_1 and t_2, and it is possible to pass to the limit in (26) by tending $t_1 \to \tau$, $t_2 \to t$. Doing it we get that

$$Z_1(t, x; \tau, \xi) = Z_2(t, x; \tau, \xi), \quad 0 \leqslant \tau < t \leqslant T, \quad \{x, \xi\} \subset \mathbb{R}^n.$$

Theorem 7 (Representations of the coefficients of the equation by the FSCP). *If the coefficients of the equation (11) satisfy the conditions of the Theorem 5 then the following formulas are correct for the coefficients and the classical FSCP Z of this equation:*

$$a_{ij}(t, x) = \lim_{\tau \to t} \left(\frac{1}{2(t-\tau)} \int_{\mathbb{R}^n} (y_{1i} - x_{1i})(y_{1j} - x_{1j}) Z(t, x; \tau, y)\, dy \right), \quad \{i, j\} \subset \{1, \dots, n_1\}, \tag{27}$$

$$a_i(t, x) = \lim_{\tau \to t} \left(\frac{1}{t-\tau} \int_{\mathbb{R}^n} (y_{1i} - x_{1i}) Z(t, x; \tau, y)\, dy \right), \quad i \in \{1, \dots, n_1\}, \tag{28}$$

$$a_0(t, x) = \lim_{\tau \to t} \left(\frac{1}{t-\tau} \left(\int_\tau^t d\theta \int_{\mathbb{R}^n} Z(t, x; \theta, y)\, dy - 1 \right) \right), \tag{29}$$

$$(t, x) \in \Pi_{(0,T]}.$$

We illustrate the method of proving the formulas on the example of the coefficient a_{11}. Let us put in the formula (25): $u(\theta, y) = (y_{11} - x_{11})^2$, $v(\theta, y) = Z(t, x; \theta, y)$. We get the equality

$$-\int_{t_1}^{t_2} d\theta \int_{\mathbb{R}^n} Z(t, x; \theta, y) \big(2a_{11}(\theta, y) + 2a_1(\theta, y)(y_{11} - x_{11}) + a_0(\theta, y)(y_{11} - x_{11})^2 \big)\, dy$$
$$= \int_{\mathbb{R}^n} (y_{11} - x_{11})^2 Z(t, x; \theta, y)\big|_{\theta = t_1}^{t_2} dy.$$

Here $t_1 = \tau$, $t_2 = t - \varepsilon$ and we pass to the limit as $\varepsilon \to 0$ and divide the result by $t - \tau$,

$$\frac{1}{t-\tau} \int_\tau^t d\theta \int_{\mathbb{R}^n} Z(t, x; \theta, y)\, a_{11}(\theta, y)\, dy = \frac{1}{2(t-\tau)} \int_{\mathbb{R}^n} (y_{11} - x_{11})^2 Z(t, x; \tau, y)\, dy$$
$$- \frac{1}{t-\tau} \int_\tau^t d\theta \int_{\mathbb{R}^n} \left(a_1(\theta, y)\,(y_{11} - x_{11}) + a_0(\theta, y) \frac{(y_{11} - x_{11})^2}{2} \right) Z(t, x; \theta, y)\, dy. \tag{30}$$

The proof of the formula (27) for $i = 1$, $j = 1$ follows from the last equality (30). Indeed, the limit as $\tau \to t$ of the left part of (30) is equal to $a_{11}(t, x)$ in the base of the properties of the FSCP Z and of the theorem about the average value for integrals. The second term of the right part of (30) tends to zero under the assumptions on the function a_1 and a_0.

The proof of the formula (27) for other values of i and j as well as the formulas (28) and (29) is carried out similarly.

Theorem 8 (Positivity of FSCP). *For the classical FSCP Z under the conditions of Theorem 5 for the coefficients of the equation (11) the inequality*

$$Z(t, x; \tau, \xi) > 0, \quad 0 \leqslant \tau < t \leqslant T, \quad \{x, \xi\} \subset \mathbb{R}^n$$

is valid.

The proof of the theorem is carried out similarly to the proof of properties 3.12 in the monograph [17, p. 213]. Namely, for a sequence of functions

$$v_\nu(t, x) := \int_{\mathbb{R}^n} Z(t, x; \tau, \xi) g_\nu(\xi) d\xi, \quad (t, x) \in \Pi_{(\tau, T]},$$

with some sequence of delta functions g_ν, $\nu \geqslant 1$, it is used the strong maximum principle and the following statement of the maximum principle for unbounded domains.

Lemma 1. *Let the coefficients of the equation (11) satisfy the conditions* $\mathbf{A}_1$, $\mathbf{A}_2$ *and the next condition:*

$\mathbf{A}_8$. *The coefficients* a_{ij}, a_i, $\{i,j\} \subset \{1,\dots,n_1\}$, *and* a_0 *are continuous functions in* $\Pi_{[0,T]}$ *and for all* $(t,x) \in \Pi_{[0,T]}$ *the estimates*

$$|a_{ij}(t,x)| \leqslant C_0(|x|^2+1), \quad |a_i(t,x)| \leqslant C_0(|x|+1), \quad |a_{ij}(t,x)| \leqslant C_0$$

are valid with some constant $C_0 > 0$; *and* $u\colon (0,T] \times \Omega \to \mathbb{R}$ *is a function which is continuous together with its Lie derivative and with its derivatives with respect to* x_1 *from the equation (11), where* $\Omega = \mathbb{R}^n \setminus B_{R_0}$, B_{R_0} *is a sphere in* $\mathbb{R}^n$ *of radius* $R_0 > 0$ *with the center at the origin, or* $\Omega = \mathbb{R}^n$. *If*

1) $(Lu)(t,x) \geqslant 0$, $(t,x) \in (0,T] \times \Omega$;

2) $\liminf_{(t,x)\to(t^0,x^0)} u(t,x) \geqslant 0$ *for any point* $(t^0,x^0) \in \partial((0,T] \times \Omega) \setminus \{t = T\}$;

3) uniformly with respect to $t \in (0,T)$ *there exists* $\liminf_{|x|\to\infty} u(t,x) \geqslant 0$.

Then $u(t,x) \geqslant 0$, $(t,x) \in (0,T] \times \Omega$.

Note that the properties of classical FSCP for the equation (16) from the class $\mathbf{E}_{22}$ similar to the statements of Theorems 5–8 were obtained in [21].

The results obtained in the paper can be used to receive the well-posedness of the Cauchy problem for the equation (11) in the classic sense.

7. Conclusions

Asian options on financial market have variables depending on the primary assets price trajectory. Among methods of constructing of Asian options mathematic models is to include such variables in the state space. An equivalent path-independent Markov problem is obtained. In this case, the price of Asian option satisfies partial differential equation of ultra-parabolic type. This equation has degenerations of parabolicity with respect to the part of spatial variables. Appropriate non-degenerate substitution of the spatial variables transforms such equation to ultra-parabolic Kolmogorov's equation with block structure. The class of such equations is denoted by E_{22}^B. It generates the well-known class of degenerated parabolic Kolmogorov's equations E_{22}. The class of equations E_{22} is well studied. For equations from the class E_{22}^B so called L-solutions were constructed and studied previously only.

In the paper the conditions for the coefficients are formulated under which the existence of classic fundamental solution of the Cauchy problem (further CFSCP) for parabolic Kolmogorov's equations with block structure from the class E_{22}^B is proved and their estimations are obtained. The following properties are proved also: the normality, the convolution formula (or the Chepmen–Kolmogorov equation), uniqueness of normal CFSCP, representations of the coefficients of the equation by the CFSCP, positivity of CFSCP.

The CFSCP for ultra-parabolic equations of Kolmogorov type in the Asian options models have essential values. They are the probability density of the transition between the states of the stochastic process in the appropriate Markov problem. The obtaining of conditions of the existence and the properties of the CFSCP has own value for the research of relevant Asian options models, for instance Markov process. Also, the obtained results are determinative to establish well-posedness Cauchy problem for the equations from the class E_{22}^B.

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Про фундаментальний розв'язок задачі Коші для ультрапараболічних рівнянь в моделях азійських опціонів

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Наше дослідження присвячене ультрапараболічним рівнянням із трьома групами просторових змінних, які виникли у задачах азіатських опціонів. Клас цих рівнянь, які задовольняють деякі умовам, було позначено $\mathbf{E}_{22}^{B}$. Цей клас є узагальненням відомого класу вироджених параболічних рівнянь типу Колмогорова $\mathbf{E}_{22}$. Раніше було побудовано так звані фундаментальні розв'язки L-типу для рівнянь із класу $\mathbf{E}_{22}^{B}$ та встановлено деякі їхні властивості. Головною особливістю дослідження було встановлення взаємно-однозначної відповідності між класами $\mathbf{E}_{22}^{B}$ та $\mathbf{E}_{22}$. У нашій роботі для рівнянь із класу $\mathbf{E}_{22}^{B}$ будуємо та вивчаємо класичні фундаментальні розв'язки задачі Коші. На коефіцієнти рівнянь накладаються спеціальні умови Гельдера щодо просторових змінних.

Ключові слова: *азійські опціони; ультрапараболічне рівняння типу Колмогорова; фундаментальний розв'язок задачі Коші.*