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Yevhen Kharchenko[1], Dmytro Butrynskyi[2]

[1] Department of Strength of Materials and Structural Mechanics, Lviv Polytechnic National University, 12, S. Bandery str., Lviv, Ukraine, E-mail: yevhen.v. kharchenko@lpnu.ua, ORCID 0000-0001-5957-9238

[2] Department of Robotics and Integrated Mechanical Engineering Technologies, Lviv Polytechnic National University, 12, S. Bandery str., Lviv, Ukraine, E-mail: dmytro.i.butrynskyi@lpnu.ua, ORCID 0009-0007-4328-2895

INTERACTION OF THE FLOW OF FLUSHING FLUID WITH THE DRILL PIPES AND THE WALL OF THE WELL

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Abstract. The steady-state flow of flushing fluid in the annular space of a drill pipe string in the case when the cross section of the channel is an eccentric ring is considered. The drilling mud is considered as a viscoplastic non-Newtonian fluid. The method for calculating the hydraulic parameters of the flushing fluid flow is based on the use of iterative methods of solving nonlinear equations and numerical analysis methods. The method can be used to calculate the hydraulic parameters of the flow in cases of laminar, structural and turbulent modes of fluid motion.

Keywords: drilling fluid flow, non-Newtonian fluid, eccentric annulus.

Introduction

Drilling of deep wells is traditionally carried out with the use of a flushing fluid (drilling mud). The drilling mud is pumped under pressure into the drill string, inside which it moves to the bit (bottom hole), captures the drilled rock (cuttings) and then moves away from the bit in the annulus between the drill string and the well wall.

In addition to the function of removing the drilled rock, the drilling mud transfers hydraulic energy to the bore bit motor; maintains the stability of the borehole soil; cools and lubricates the bit; contributes to the stability of the drill pipe string and performs other functions.

In terms of rheological properties, modern drilling fluids are viscoplastic non-Newtonian fluids (mostly with a nonlinear dependence of viscosity on the shear strain rate).

The annulus is the area between the cylindrical surfaces of the drill string and the wellbore wall. During the drilling and operation of a well, the center of the drill string may shift relative to the center of the well, and mud may accumulate on the well walls. As a result, the cross-section of the annulus takes the form of an eccentric ring. This leads to significant changes in such hydraulic characteristics of the drilling mud flow as the distribution of velocities in the cross section, the average velocity of fluid movement, hydraulic resistance, the value and distribution of forces of interaction between the flow of flushing fluid and the drill pipe string.

This paper proposes an approach to calculating the distribution of velocities and forces of interaction between the flow of mud with the drill string and the borehole wall for laminar, structural, and turbulent flows of non-Newtonian mud in the eccentric annulus. The approach is based on the use of iterative methods for solving nonlinear equations and numerical analysis methods.

Problem Statement

The steady-state flow of drilling fluid in the annulus of a drill pipe string whose cross section is an eccentric annulus, is considered. The inner radius of the drill pipe is denoted as r_1, the outer radius as r_2 and the borehole diameter as r_3. The area of the mentioned annulus is bounded inside by the cylindrical surface of the pipes, and outside by the cylindrical surface of the borehole wall. The drilling mud is considered as a viscoplastic non-Newtonian fluid (mostly with a nonlinear dependence of viscosity on the shear strain rate).

For the considered flow of drilling mud, we accept the following assumptions:

- the fluid is incompressible, isotropic, viscoplastic;
- the flow is steady-state, plane-parallel, the flow parameters do not depend on time and the longitudinal coordinate z;
- the flow is due to the pressure gradient;
- the annular channel is vertical, gravity is applied along the channel.

The solution to the problem is the drilling fluid flow parameters (cross-sectional velocity distribution, average fluid velocity, hydraulic resistance), as well as the distribution of forces of interaction between the flushing fluid flow and the drill pipe string and the borehole wall.

Review of Modern Information Sources on the Subject of the Paper

The problem of fluid flow through a channel whose cross-section has the form of an eccentric ring is a classic one. It has wide practical application in many fields of engineering. Due to the need to model industrial technological processes, the study of fluid flows in eccentric cross-sections began in the middle of the last century.

For the flow of a Newtonian fluid in an eccentric channel, the analytical solution of the problem was most likely first obtained in [1], the numerical solution using the finite difference method was first obtained in [2]. In the case of a non-Newtonian fluid flow in an eccentric annular channel, the analytical solution of the problem is unknown. This can be explained by significant mathematical difficulties in constructing a solution of the boundary value problem for a system of nonlinear partial differential equations. Over the past decades, various experimental studies of hydrodynamic processes in pipes have been conducted, approximate and asymptotic approaches to solving such problems have been proposed and applied and numerical studies of fluid flows have been carried out.

The results of experimental studies of the behavior of non-Newtonian fluid flow in an eccentric channel are presented in [3–7]. Particularly noteworthy are the experiments performed with high accuracy of study the laminar motion of a non-Newtonian fluid are described in [8]. In this paper, the velocity profiles for concentric and eccentric (with a relative eccentricity of 0.8) annular channels under the laminar flow regime are investigated and the results obtained by the authors are compared with previously known and literature data.

The approach to the approximate consideration of the geometry of the eccentric region with the subsequent imposition of additional hypotheses on the modes of fluid motion and the distribution of non-Newtonian fluid velocities during the construction of an approximate solution proved to be effective. In particular, a solution of the problem with the replacement of the eccentric region by expanding it into an equivalent slot with vertical sides and variable height was constructed in [9].

In [10] and other works by these authors, a refined slot geometry was used to model the flow of a non-Newtonian fluid through an eccentric annulus; using the hypothesis of a small velocity gradient in the circular direction, as well as the symmetry of the velocity gradient relative to the midline between the channel boundaries, an analytical expression of the flow velocity at points in the region was obtained; the decrease in the flow velocity in a narrow region of an eccentric annular channel was investigated as a function of eccentricity.

In [11] an approximate representation of an eccentric ring by a set of radial sectors of concentric rings with different outer radii was used.

Over the past decades there has been great development of numerical finite-difference methods of the fluid dynamics (CFD) as well as the development of appropriate software; the study of non-Newtonian fluid flows in eccentric and concentric annuli using these methods was carried out, in particular, in [12–23].

In [12] the numerical modeling of the flow of non-Newtonian fluids for annular channels of arbitrary shape is considered; the numerical solution is constructed using curvilinear coordinate systems. To test the developed numerical algorithm, an annular concentric region is considered, for which the numerical results obtained for several types of fluids are compared with the analytical solution obtained by the same authors.

Paper [13] is devoted to numerical modeling and study of fluid flow in irregular annular channels. Non-orthogonal curvilinear coordinate systems are used; to avoid difficulties in the convergence of the numerical algorithm in the area where the shear stress is less than the yield strength, the smoothed rheological dependence proposed in [15] was used.

In the previously mentioned work [8], a numerical solution to the problem was obtained; a variant of the finite volume method was used with the use of a non-orthogonal grid and a second-order central-difference scheme. A comparison of the obtained results of numerical modeling with the experimental data obtained by the authors is presented and the high accuracy of the developed numerical models is shown.

Paper [15] is devoted to the numerical modeling of uniaxial steady-state flow of drilling mud through an eccentric annular cross-section. The classical rheological model of Bingham [16] was used for the fluid. The solution was obtained using a finite element method in the form of an augmented Lagrangian method. The discretization of the domain by a grid of triangular elements is used. Difficulties associated with the discontinuity of the Bingham rheological model are overcome by using the Lagrange multiplier and an augmented Lagrangian / Uzawa method. The solution of the problem is obtained for the steady-state flow mode with a given average velocity. The good convergence of the applied numerical method and the influence of the finite element mesh refinement on the solution accuracy are shown. The influence of eccentricity on the flow shape is analyzed. Based on the obtained results, an engineering method for estimating pressure losses is proposed.

In [17] a power-law rheological model of the fluid medium was used to study the laminar mode of motion of a viscous fluid. The geometry of the cross-section of an eccentric annulus is mapped onto a unit circle using a coordinate transformation. The system of nonlinear equations is solved using a finite-difference approximation of the second order of accuracy. Numerical solutions are presented for the ratio of circle radii in the range of 0.2...0.8 and the relative eccentricity of 0...0.8. It is shown that both the rheological characteristics of the fluid and the eccentricity of annulus have a significant effect on the flow behavior. At certain values of eccentricity, the flow is blocked in the narrow segment of the gap and the peak velocity increases in the gap of large width segment; the significant velocity gradients in the circular direction are found.

Despite the significant development and practical application of fluid dynamics numerical methods for solution of considered problems, in particular given in [18–23], development of effective algorithms for calculating the parameters of non-Newtonian fluids flows without the use of finite-difference discretization of the flow domain is still a scientific and practical interest. The development of an effective method for analyzing the interaction forces of the flushing fluid flow with the drill string and the well wall for a wide range of geometric parameters of the mechanical system and characteristics of fluid flow modes remains of great relevance for today.

Objectives and Problems of Research

The purpose of this study is to develop a method for calculating the hydraulic parameters of the flow of non-Newtonian flushing fluid in the eccentric annulus of a well, to determine the velocity distribution across the flow cross section, as well as the forces of interaction between the fluid flow and the drill pipe string and the well wall for a wide range of geometric characteristics of the channel, rheological parameters, and flow modes.

To achieve this goal, the following tasks are considered: analysis of the current state of the research problem, selection of a rheological model of the drilling mud, substantiation of the accepted hypotheses and assumptions, formulation of an effective mathematical model of the drilling mud movement in the annulus, development of an iterative solving algorithm of the research problem, analysis of the results.

Main Material Presentation

The tangential stresses that occur on the surfaces of drill pipes and well walls during flushing depend on the rheological properties of the fluid, the flow mode, and the geometric parameters of the cross-sections of the flows. In oilfield practice, viscous liquids, such as water, and viscoplastic liquids, which include clay and other drilling fluids, are common. Viscous fluids are obeying the Newton's law of viscosity

$$\tau = \mu \frac{dv}{dr}, \tag{1}$$

where τ is the tangential stress; μ is the dynamic viscosity coefficient; dv/dr is the velocity gradient.

The rheological characteristic of viscoplastic liquids is given in the form of the Bingham dependence

$$\tau = \eta \frac{dv}{dr} + \tau_0 \operatorname{sign} \tau, \tag{2}$$

where η is the structural toughness coefficient; τ_0 is the dynamic shear stress.

A viscous fluid is characterized by laminar and turbulent modes of motion, while a viscoplastic fluid is characterized by structural and turbulent modes.

On the inner surface of a pipe of circular cross-section, the tangential stress is written in the form [25].

$$\tau_1 = \frac{\lambda\rho}{8} w^2 \operatorname{sign} \omega, \tag{3}$$

where ρ and w are the density and average cross-sectional velocity of the fluid; λ is the drag coefficient used in the Darcy – Weisbach formula.

For the laminar regime, the coefficient λ is determined by the formula

$$\lambda = \frac{64}{\mathrm{Re}}, \tag{4}$$

where Re is the Reynolds parameter,

$$\mathrm{Re} = \frac{2|w| r_1 \rho}{\mu}, \tag{5}$$

where r_1 is the inner radius of the pipe.

The critical value of the Reynolds parameter, which corresponds to the transition from the laminar to the turbulent mode of motion, is $\mathrm{Re}_k = 2320$.

If the rheological characteristics of the fluid and the radius of the pipe are known, then, according to (5), the critical value of the average flow rate will be

$$w_k = \frac{\mathrm{Re}_k \, \mu}{2 r_1 \rho}. \tag{6}$$

A number of empirical dependencies have been obtained for the turbulent mode of viscous fluid motion [26, 27]. For the problems of well hydrodynamics, the Blasius formula provides sufficient accuracy

$$\lambda = \frac{0{,}3164}{\sqrt[4]{\mathrm{Re}}}. \tag{7}$$

Considering the movement of a viscoplastic fluid in the structural mode, the drag coefficient is determined by the formula [27]

$$\lambda = \frac{64}{\mathrm{Re}^*}, \tag{8}$$

where Re^* is the generalized Reynolds parameter,

$$\mathrm{Re}^* = \frac{6\,\mathrm{Re}}{6 + \mathrm{Sen}}, \tag{9}$$

here Re and Sen - Reynolds and St. Venant criteria that matter

$$\mathrm{Re} = \frac{2|w| r_1 \rho}{\eta}; \quad \mathrm{Sen} = \frac{2\tau_0 r_1}{\eta |w|}. \tag{10}$$

The analysis of empirical and semi-empirical formulas for determining λ in the case of turbulent motion of a viscoplastic fluid [26, 27] shows that sufficient accuracy can be obtained using the Nikuradze formula

$$\lambda = 0{,}0032 + \frac{0{,}221}{\mathrm{Re}^{0{,}237}}, \tag{11}$$

if

$$2500 \le \mathrm{Re}^* \le 5000; \quad 20000 \le \mathrm{He} \le 140000,$$

where $\mathrm{He} = \mathrm{Re} \cdot \mathrm{Sen}$ is the Hedstrom parameter.

Dependence (11) was obtained based on the logarithmic law of velocity distribution across the cross-section of the flow. The critical velocity corresponding to the transition from the structural mode of motion to the turbulent mode is determined by the formula [28]

$$w_k = \alpha \sqrt{\frac{\tau_0}{\rho}}, \tag{12}$$

where α = 25 m^{-1} is a constant.

Taking into account (3)–(5), we obtain the tangential stresses for the laminar motion of a viscous fluid in the form

$$\tau_1 = \frac{4\mu}{r_1} w. \tag{13}$$

Similarly, using formulas (3), (8)–(10), we determine the tangential stresses in the case of structural motion of a viscoplastic fluid

$$\tau_1 = \frac{4\eta}{r_1} w + \frac{4}{3} \tau_0 \operatorname{sign} w. \tag{14}$$

As can be seen from expressions (13), (14), the laminar and structural modes are characterized by linear dependences of the tangential stresses on the inner wall of the pipe from the average flow rate. The turbulent regime, according to (3), (7), (11), is characterized by rather complex nonlinear dependences of these quantities. However, a change in the Reynolds parameter in a small range has little effect on the drag coefficient λ. This makes it possible to use an approximate value of this coefficient $\lambda = \lambda_k$, which is calculated by formula (7) or (11), taking into account $\mathrm{Re} = \mathrm{Re}_k$. Then the expression of tangential stresses (3) for turbulent motion takes the form

$$\tau_1 = \frac{\lambda_k}{8} \rho w^2 \operatorname{sign} w. \tag{15}$$

Using dependences (13)–(15) and formulas (6), (12), it is possible to ensure sufficient accuracy in solving a wide range of problems of hydromechanics of oil and gas well drilling. It should be noted that in the presence of intense longitudinal vibrations of the column, the average relative fluid velocity should be considered as w in the above formulas.

Determining the tangential stresses on the outer surface of the drill pipe and on the borehole wall is a more complex task. It involves a detailed study of the velocity distribution in the flow cross section. Let’s consider a generalized algorithm for calculating the parameters of hydrodynamic processes in the annulus, which is based on the use of iterative methods for solving nonlinear algebraic equations and numerical analysis methods.

In the case of an eccentric pipe arrangement, the distance from its outer surface to the borehole wall is determined according to [29],

$$\delta = c\cos\varphi - r_2 + \sqrt{r_3^2 - c^2\sin^2\varphi}, \tag{16}$$

where c is the eccentricity of the well and casing axes; r_2, r_3 is the radius of the outer surface of the pipe and well; φ is the angle of inclination of the beam coming from the center of the casing cross section.

In general, the range of values of the angle φ, which corresponds to the full coverage of the annulus, is within $-\varphi_0 \le \varphi \le \varphi_0$. If the drill string does not touch the borehole wall or contacts it along the line, then $\varphi_0 = \pi$. Due to the formation of a clay crust on the surfaces to be washed, the radius r_2 increases and the radius r_3 decreases. When the casing is pressed against the borehole wall, the annulus is deformed, and the region of the angle φ, corresponding to the live section of the annulus, decreases. In this case, the boundary values of φ are found in the region ($-\pi, \pi$), solving the transcendental equation obtained from (16):

$$33\, c\cos\varphi_0 - r_2 + \sqrt{r_3^2 - c^2\sin^2\varphi_0} = 0,. \tag{17}$$

Considering an equilibrium of a differentially small flow element of unit length bounded by radial planes forming the angle $d\varphi$ and cylindrical surfaces with radii r and $r + dr$, we obtain the equations

$$\frac{d\tau}{dr} = -\frac{1}{r}\tau + h, \tag{18}$$

where τ is the tangential stress occurring on a plane perpendicular to the radius; r is the radial coordinate; h is the specific load,

$$h = \frac{dp}{dx} - g\rho, \tag{19}$$

x is the longitudinal spatial coordinate; p is the average pressure in the cross section.

The equilibrium equation (18) is written using the quasi-stationarity hypothesis, and the tangential stresses in the radial planes are assumed to be zero.

The solution of the differential equation (18) is given by the dependence

$$\tau = \frac{r}{2}h + \frac{r_2}{r}\left(\tau_2 - \frac{r_2}{2}h\right), \tag{20}$$

where $\tau_2 = \tau(r_2)$ is the tangential stress on the outer surface of the pipe.

Due to the lack of boundary conditions that would allow us to determine the unknown τ_2 in relation (20), we will consider the differential equations obtained directly from (2):

$$\frac{dv}{dr} = \frac{\tau}{\eta} - \frac{\tau_0}{\eta}\operatorname{sign}\tau, \ \text{ if } |\tau| > \tau_0; \ \frac{dv}{dr} = 0, \text{ if } |\tau| \le \tau_0. \tag{21}$$

The boundary conditions for the integration of equations (21) are given in the form

$$v(r_2) = v_2, \quad v(r_2 + \delta) = 0. \tag{22}$$

Taking into account (20), equation (21) is written as

$$\frac{dv}{dr} = \frac{r}{2\eta}h + \frac{r_2}{r\eta}\left(\tau_2 - \frac{r_2}{2}h\right) - \frac{\tau_0}{\eta}\operatorname{sign}\tau, \ \text{ if } |\tau| < \tau_0; \ \frac{dv}{dr} = 0, \text{ if } |\tau| \le \tau_0. \tag{23}$$

The integral of the differential equations (23) for any value of h must satisfy the condition

$$F(\tau_2) = v(r_2 + \delta) = 0. \tag{24}$$

To find a solution to equations (24), we apply Newton's iterative formula

$$\tau_2^{k+1} = \tau_2^k - \left[\left(\frac{\partial F^k}{\partial \tau_2}\right)_{\tau_2 = \tau_2^k}\right]^{-1} F^k\left(\tau_2^k\right), \tag{25}$$

where $k+1$ is the iteration number.

Derivative

$$\frac{\partial F}{\partial \tau_2} = \frac{\partial v(r_2 + \delta)}{\partial \tau_2}$$

in expression (25) is calculated by integrating in the range from r_2 to $r_2 + \delta$ the equations obtained by differentiating (23) by the argument τ_2:

$$\frac{d}{dr}\left(\frac{\partial v}{\partial \tau_2}\right) = \frac{r_2}{r\eta}, \ \text{ if } |\tau| > \tau_0; \ \frac{d}{dr}\left(\frac{\partial v}{\partial \tau_2}\right) = 0, \ \text{if} |\tau| \le \tau_0. \tag{26}$$

Since the speed of $v(r_2) = v_2$ is considered as an independent variable, the initial condition that the solution of equations (26) must satisfy is written as

$$\frac{dv(r_2)}{d\tau_2} = 0. \tag{27}$$

To determine the stress τ_2 for a fixed value of h, we first set the value of this stress arbitrarily, then integrate equation (26) in the range from r_2 to $r_2 + \delta$, taking into account (20) and (27), refine τ_2 using formula (25), analyze the accuracy of the calculation of the desired value, and, if necessary, repeat the calculation in the same sequence. We stop the iteration process if the following condition is met

$$\text{abs}\left(\tau_2^{k+1} - \tau_2^k\right) \le \varepsilon_\tau, \tag{28}$$

where ε_τ is the absolute error *f*.

The tangential stresses on the borehole wall are determined by formula (20):

$$\tau_3 = \tau(r_2 + \delta) = \frac{r_2 + \delta}{2} h + \frac{r_2}{r_2 + \delta}\left(\tau_2 - \frac{r_2}{2} h\right). \tag{29}$$

If $v_2 = 0$ and for all values of r corresponding to a fixed angular coordinate φ, the condition $\tau \le \tau_0$ is satisfied, then there is no fluid motion in this radial plane. We assume that at the moment of time when the motion begins, the approximate relationship

$$\tau_2 = -\tau_3 = -\frac{\delta h}{2}. \tag{30}$$

In most cases, the specific load h is unknown, since the average pressure in the cross-section of the flow is also unknown. To determine it from the value of the average velocity w, we write the following equation:

$$\Phi(h) = -w + \frac{2}{s}\int_0^{\varphi_0}\int_{r_2}^{r_2+\delta} vr dr d\varphi = 0, \tag{31}$$

Where s is the cross-sectional area of the flow,

$$s = 2\int_{0}^{\varphi_0}\int_{r_2}^{r_2+\delta} r dr d\varphi. \tag{32}$$

The iterative formula for solving equation (31) is given as

$$h^{m+1} = h^m - \left[\left(\frac{d\Phi^m}{dh}\right)_{h=h_m}\right]^{-1}\Phi^m\left(h^m\right), \tag{33}$$

where $m+1$ is the iteration number.

The full derivative of $d\Phi/dh$, included in expression (33), is written as

$$\frac{d\Phi}{dh} = \frac{2}{s}\int_{0}^{\varphi_0}\int_{r_2}^{r_2+\delta}\left(\frac{\partial v}{\partial h} + \frac{\partial v}{\partial \tau_2}\frac{d\tau_2}{dh}\right) r dr d\varphi. \tag{34}$$

To determine the partial derivative of $\partial v/\partial \tau_2$, we use equation (26), and the partial derivative of $\partial v/\partial h$, we use equation

$$\begin{aligned} &\frac{d}{dr}\left(\frac{\partial v}{\partial h}\right) = \frac{1}{2\eta}\left(r - \frac{r_2^2}{r}\right), \ \text{if } |\tau| > \tau_0; \\ &\frac{d}{dr}\left(\frac{\partial v}{\partial h}\right) = 0, \ \text{if } |\tau| \le \tau_0, \end{aligned} \tag{35}$$

obtained by differentiating (23) by the parameter h.

The initial condition, which must be satisfied by the solution of equations (35), is written similarly to (27) in the form

$$\frac{\partial v(r_2)}{\partial h} = 0. \tag{36}$$

Since the zero value of the functional (24) is ensured for any specific load h, its full derivative satisfies the relation

$$\frac{dF}{dh} = \frac{\partial F}{\partial h} + \frac{\partial F}{\partial \tau_2}\frac{d\tau_2}{dh} = 0. \tag{37}$$

From equation (37) we obtain

$$\frac{d\tau_2}{dh} = -\left(\frac{\partial F}{\partial \tau_2}\right)^{-1}\frac{\partial F}{\partial h}. \tag{38}$$

The partial derivatives included in formula (38) are defined as the values of the unknowns of equations (26), (35) at the ends of the integration intervals.

Taking into account (38), we transform expression (34) to the form

$$\frac{d\Phi}{dh} = \frac{2}{s}\int_{0}^{\varphi_0}\int_{r_2}^{r_2+\delta}\left(\frac{\partial v}{\partial h} - \frac{\partial v}{\partial \tau_2}\left(\frac{\partial F}{\partial \tau_2}\right)^{-1}\frac{\partial F}{\partial h}\right) r dr d\varphi. \tag{39}$$

Using the iterative formula (33), taking into account (39), we calculate h with the accuracy determined by the condition

$$\text{abs}\left(h^{m+1} - h^m\right) \le \varepsilon_h, \tag{40}$$

where ε_h is the absolute calculation error.

In this iterative process, solving equation (24) is a local procedure that must be performed at each step.

Having calculated τ_2, τ_3 and h taking into account the imposed conditions (28), (40), we find the distributed forces with which the fluid flow acts on the outer surface of the column $q_{\tau 2}$ and the well wall $q_{\tau 3}$

$$q_{\tau 2} = 2r_2 \int_0^{\varphi_0} \tau_2 d\varphi; \quad q_{\tau 3} = 2\int_0^{\varphi_0} \tau_3 (r_2 + \delta) d\varphi. \tag{41}$$

For the laminar flow of a viscous fluid, the velocity gradient is written similarly to (23):

$$\frac{dv}{dr} = \frac{r}{2\mu} h + \frac{r_2}{r\mu}\left(\tau_2 - \frac{r_2}{2} h\right). \tag{42}$$

Integrating (42) taking into account the first boundary condition (22), we obtain

$$v = \frac{h}{4\mu}\left(r^2 - r_2^2\right) + \frac{r_2}{\mu}\left(\tau_2 - \frac{r_2}{2} h\right)\ln\frac{r}{r_2} + v_2. \tag{43}$$

Using the second boundary condition (22) and expression (43), we find the tangential stresses on the outer surface of the drill pipe

$$\tau_2 = \frac{r_2}{2} h + \left(4r_2 \ln\frac{r_2}{r_2 + \delta}\right)^{-1}\left[4v_2\mu + h\left(\delta^2 + 2r_2\delta\right)\right]. \tag{44}$$

Taking into account (20), (44), we determine the tangential stresses on the well wall

$$r_3 = \frac{r_2 + \delta}{2} h + \left[4(r_2 + \delta)\ln\frac{r_2}{r_2 + \delta}\right]^{-1}\left[4v_2\mu + h\delta(\delta + 2r_2)\right]. \tag{45}$$

In order to determine the tangential stresses using formulas (44), (45), it is necessary to have the value of the specific load h, which is found from equation (31). The full derivative $d\Phi/dh$, which appears in (33), is written in the form

$$\frac{d\Phi}{dh} = \frac{2}{s}\int_0^{\varphi_0}\int_{r_2}^{r_2+\delta} \frac{dv}{dh} r dr d\varphi, \tag{46}$$

and as follows from (43), (44)

$$\frac{dv}{dh} = \frac{r^2 - r_2^2}{4\mu} + \frac{\delta^2 + 2r_2\delta}{4\mu}\frac{\ln r - \ln r_2}{\ln r_2 - \ln(r_2 + \delta)}. \tag{47}$$

When calculating the laminar flow mode of a viscous fluid, there is no need to numerically solve differential equations (26), (35). At the next iteration of finding the value of h, equation (47) is subject to integration. We use the iterative formula in the form of (33), taking into account dependence (46). After determining the tangential stresses (44), (45), we find the distributed forces of interaction between the fluid and the column and the walls of the well using formulas (41).

Let us consider the turbulent motion of fluids in a space of annular cross-section.

As is known [29], the influence of the constant component τ_0 in the study of turbulent motion of a Bingham fluid can be neglected. In this case, the rheological characteristics of the viscous (1) and viscoplastic media become identical, which allows us to apply a single method for calculating hydrodynamic processes. The problem of velocity distribution in the cross-section of the flow is solved taking into account the eccentric location of the column and well axes.

Using the power law [26], the dependence of the fluid velocity on the radial coordinate is given by

$$\begin{aligned} &v = \psi_2, \ \text{if} \ \ r_2 \le r \le r_2 + \delta; \\ &v = \psi_3, \ \text{if} \, r_2 + \delta_2 \le r \le r_2 + \delta, \end{aligned} \tag{48}$$

where δ_2 is the distance from the outer surface of the column to the point of conjugation of the velocity distribution curves; ψ_2, ψ_3 is the functions determined by the dependencies

$$\psi_2 = v_2 + k\left(\frac{|\tau_2|}{\rho}\right)^{\frac{n+1}{2}}\left(\frac{r-r_2}{\nu}\right)^n \operatorname{sign}\tau_2;$$
$$\psi_3 = -k\left(\frac{|\tau_3|}{\rho}\right)^{\frac{n+1}{2}}\left(\frac{\delta+r_2-r}{\nu}\right)^n \operatorname{sign}\tau_3, \tag{49}$$

where ν is the kinematic viscosity coefficient; k and n are values that depend on the Reynolds parameter and acquire values in the region of Blasius' law $k = 8{,}74;\ n = 1/7$.

At the point $r = r_2 + \delta_2$, the condition of conjugation of curves should be used

$$\psi_2(r_2 + \delta_2) = \psi_3(r_2 + \delta_2), \tag{50}$$

which expresses the equality of the velocities given by the dependencies $\psi_2(r)$ and $\psi_3(r)$.

In addition, at the junction of these curves (if $r = r_2 + \delta_2$), the following equations are satisfied

$$\tau = 0,\ \text{if}\ \tau_1\tau_2 < 0. \tag{51}$$

$$\frac{d\psi_2}{dr} = \frac{d\psi_3}{dr},\ \text{if}\ \tau_1\tau_2 \geq 0. \tag{52}$$

From the equilibrium condition of a differentially small volume of fluid of unit length bounded by radial planes forming the angle $d\varphi$ and cylindrical surfaces with radii r_2 and $r_2 + \delta$, we obtain

$$\tau_2 r_2 - \tau_3(r_2 + \delta) + h\delta\left(\frac{\delta}{2} + r_2\right) = 0. \tag{53}$$

Taking into account (51), we write the equilibrium equation for a differentially small volume of fluid bounded by radii r_2 and $r_2 + \delta_2$:

$$\tau_2 r_2 + h\delta_2\left(\frac{\delta_2}{2} + r_2\right) = 0. \tag{54}$$

Depending on the conditions of conjugation of curves $\psi_2(r)$ and $\psi_3(r)$, the unknown flow parameters τ_2, τ_3, δ_2 for a given specific load h are determined by solving equations (50), (52), (53) or (50), (53), (54).

We represent these systems of equations as a single matrix equality

$$F(q) = 0, \tag{55}$$

where

$$F = \operatorname{col}(F_1, F_2, F_3), \quad q = \operatorname{col}(\tau_2, \tau_3, \delta_2), \tag{56}$$

and

$$F_1 = v_2 + k\left(\frac{|\tau_2|}{\rho}\right)^{\frac{n+1}{2}}\left(\frac{\delta_2}{\nu}\right)^n \operatorname{sign}\tau_2 + k\left(\frac{|\tau_3|}{\rho}\right)^{\frac{n+1}{2}}\left(\frac{\delta-\delta_2}{\nu}\right)^n \operatorname{sign}\tau_2,$$
$$F_2 = \tau_2 r_2 - \tau_3(r_2 + \delta) + h\delta\left(\frac{\delta}{2} + r_2\right);$$
$$F_3 = |\tau_2|^{\frac{n+1}{2}}\delta_2^{n-1}\operatorname{sign}\tau_2 - |\tau_3|^{\frac{n+1}{2}}(\delta - \delta_2)^{n-1}\operatorname{sign}\tau_3,\ \text{if}\ \tau_1\tau_2 \geq 0; \tag{57}$$
$$F_3 = \tau_2 r_2 + h\delta_2\left(\frac{\delta_2}{2} + r_2\right),\ \text{if}\ \ \tau_1\tau_3 < 0.$$

Applying Newton's formula, we obtain an iterative formula for solving the matrix equation (55) in the form

$$q^{k+1} = q^k - \left[\left(\frac{\partial F}{\partial q} \right)_{q=q^k} \right]^{-1} F\left(q^k\right). \tag{58}$$

The components of the matrix $\partial F / \partial q$, taking into account (56), (57), are written as

$$\begin{gathered}
\frac{\partial F_1}{\partial \tau_2} = kn_2 \frac{\left|\tau_2\right|^{n_1}}{\rho^{n_2}} \left(\frac{\delta_2}{\nu} \right) \operatorname{sign} \tau_2, \quad \frac{\partial F_1}{\partial \tau_3} = kn_2 \frac{\left|\tau_3\right|^{n_1}}{\rho^{n_2}} \left(\frac{\delta - \delta_2}{\nu} \right) \operatorname{sign} \tau_3, \\
\frac{\partial F_1}{\partial \delta_2} = kn \left(\frac{\left|\tau_2\right|}{\rho} \right)^{n_2} \frac{\delta_2^{n-1}}{\nu^n} \operatorname{sign} \tau_2 - kn \left(\frac{\left|\tau_3\right|}{\rho} \right)^{\frac{n+1}{2}} \frac{\left(\delta - \delta_2\right)^{n-1}}{\nu^n} \operatorname{sign} \tau_3; \\
\frac{\partial F_2}{\partial \tau_2} = r_2, \quad \frac{\partial F_2}{\partial \tau_3} = -r_2 - \delta, \quad \frac{\partial F_2}{\partial \delta_2} = 0; \\
\frac{\partial F_3}{\partial \tau_3} = n_2 \left|\tau_2\right|^{n_1} \delta_2^{n-1} \operatorname{sign} \tau_2, \quad \frac{\partial F_3}{\partial \tau_3} = -\left|\tau_3\right|^{n_1} \left(\delta - \delta_2\right)^{n-1} \operatorname{sign} \tau_3, \\
\frac{\partial F_3}{\partial \delta_2} = (n-1)\left|\tau_2\right|^{n_2} \delta_2^{n-2} \operatorname{sign} \tau_2 + (n-1)\left|\tau_3\right|^{n_2} \left(\delta - \delta_2\right)^{n-2} \operatorname{sign} \tau_3, \text{ if } \tau_2 \tau_3 \geq 0; \\
\frac{\partial F_3}{\partial \tau_2} = r_2, \quad \frac{\partial F_3}{\partial \tau_3} = 0, \quad \frac{\partial F_3}{\partial \delta_2} = h\left(r + \delta_2\right), \text{ if } \tau_2 \tau_3 < 0,
\end{gathered} \tag{59}$$

where

$$n_1 = (n-1)/2; \; n_2 = (n+1)/2.$$

Setting an arbitrary initial value of the components of the matrix-column q, using the iterative formula (58), taking into account (57), (59), we determine the tangential stresses τ_2 and τ_3 and the distance δ_2.

The condition for ensuring the required calculation accuracy is as follows

$$\operatorname{abs}\left(q^{k+1} - q^k\right) \leq \varepsilon, \tag{60}$$

where $\varepsilon = \operatorname{col}\left(\varepsilon_{\tau 2}, \varepsilon_{\tau 3}, \varepsilon_{\delta 2}\right)$ is a column matrix of permissible errors.

If the specific load of h is not known, it should be determined by the value of the average flow rate from condition (31), using the iterative formula (33).

Taking into account (48), we transform expression (31) to the form

$$\Phi(h) = -w + \frac{2}{s} \int_0^{\varphi_0} \left[\int_{r_2}^{r_2+\delta_2} \psi_2 r dr + \int_{r_2+\delta_2}^{r_2+\delta} \psi_3 r dr \right] d\varphi, \tag{61}$$

given that the values of ψ_2 and ψ_3 are functions of the independent variables h and r.

Using the implicit dependence $\Phi(h)$ in the form (61) and the relation (49), using Leibniz's rule of differentiation of certain integrals by a parameter, we obtain

$$\begin{gathered}
\frac{d\Phi}{dh} = \frac{2}{s} \int_0^{\varphi_0} \left[\int_{r_2}^{r_2+\delta_2} \frac{\partial \psi_2}{\partial \tau_2} \frac{d\tau_2}{dh} r dr + \int_{r_2+\delta_2}^{r_2+\delta} \frac{\partial \psi_3}{\partial \tau_3} \frac{d\tau_3}{dh} r dr + \right. \\
\left. + \left(r_2 + \delta_2\right)\left(\psi_2\left(r_2 + \delta_2, h\right) - \psi_3\left(r_2 + \delta_2, h\right)\right) \frac{d\delta_2}{dh} \right] d\varphi,
\end{gathered} \tag{62}$$

where

$$\frac{\partial\psi_2}{\partial\tau_2} = kn_2\frac{|\tau_2|^{n_1}}{\rho^{n_2}}\left(\frac{r-r_2}{\nu}\right)^n \operatorname{sign}\tau_2;$$
$$\frac{\partial\psi_3}{\partial\tau_3} = -kn_2\frac{|\tau_3|^{n_1}}{\rho^{n_2}}\left(\frac{\delta+r_2-r}{\nu}\right)^n \operatorname{sign}\tau_3;$$
$$\psi_2(r_2+\delta_2,h) = v_2 + k\left(\frac{|\tau_2|}{\rho}\right)^{n_2}\left(\frac{\delta_2}{\nu}\right)^n \operatorname{sign}\tau_2;$$
$$\psi_3(r_2+\delta_2,h) = -k\left(\frac{|\tau_3|}{\rho}\right)^{n_2}\left(\frac{\delta-\delta_2}{\nu}\right)^n \operatorname{sign}\tau_3. \tag{63}$$

Since equations (55) are satisfied by any value of the specific load, then, having differentiated them by h, we define the derivative of dq/dh in the form

$$\frac{dq}{dh} = -\left(\frac{\partial F}{\partial q}\right)^{-1}\frac{\partial F}{\partial h}. \tag{64}$$

We consider the component of the matrix-column dF/dq to be known, since its elements are calculated by formulas (59) at the stage of determining the parameters τ_2, τ_3, δ_2.

The elements of the matrix-column dq/dh, determined by formula (63), are the full derivatives included in expression (62).

The hydromechanical characteristics of the flow for the turbulent regime are found in the following sequence. Setting an arbitrary specific load h, from matrix equation (55), taking into account (58), (60), we determine the column matrix q. We refine the value of the specific load using expressions (33), (61)–(64). Check condition (40) and, if necessary, repeat the calculation.

We use the condition for the transition from the laminar or structural regime to the turbulent regime in the form [26].

$$v_m = w_k, \tag{65}$$

where v_m is the average integral velocity of the fluid on the beam coming from the center of the pipe cross-section,

$$v_m = \frac{1}{\delta}\int_{r_2}^{r_2+\delta} v\,dr; \tag{66}$$

w_k is the critical value of the average velocity found experimentally [30],

$$w_k = \frac{\nu}{2(r_3-r_2)}\left(1750 - 1000\frac{c}{r_3-r_2}\right). \tag{67}$$

As can be seen from expressions (65)–(67), laminar and turbulent, or structural and turbulent motion can occur simultaneously in the annular space. The value of the angle φ, which delimits the regions with different modes of motion, is determined from Eq. (65) and denoted as $\pm\varphi_1$. In this case, taking into account (31), (61), we write the equation for determining the specific load h in the form

$$\Phi(h) = -w + \Phi_1(h) + \Phi_2(h), \tag{68}$$

where

$$\Phi_1 = \frac{2}{s}\int_0^{\varphi_1}\left[\int_{r_2}^{r_2+\delta_2}\psi_2 r dr + \int_{r_2+\delta_2}^{r_2+\delta}\psi_3 r dr\right]d\varphi;$$

$$\Phi_2 = \frac{2}{s}\int_{\varphi_1}^{\varphi_0}\int_{r_2}^{r_2+\delta} v r dr d\varphi.$$

The iterative formula for solving equation (68) is (33), with the full derivative of $d\Phi/dh$ defined as

$$\frac{d\Phi}{dh} = \frac{d\Phi_1}{dh} + \frac{d\Phi_2}{dh}. \tag{69}$$

The derivative $d\Phi_1/dh$, included in Eq. (69), is found in the form (62), replacing the upper limit of integration by the parameter φ with φ_1, and the derivative $d\Phi_2/dh$ is found in the form (39), replacing the lower limit of this parameter with φ_1. We calculate the fluid velocities and tangential stresses for each of the regions in accordance with the algorithms for analyzing the laminar, structural, and turbulent regimes described above. The distributed forces of interaction between the fluid and the drill pipe string and the well walls are found using dependencies (41).

Conclusions

A method of calculation of the hydraulic parameters of the steady-state flow of non-Newtonian drilling fluid in the eccentric annulus of a drilling well has been developed. The method allows calculation of hydraulic flow parameters for laminar, structural, and turbulent modes of fluid motion, as well as interaction between the fluid flow and the drill string. The approach allows solving of the considered problem with sufficient accuracy without the application of computationally expensive finite-difference methods.

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