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Solving non-linear functional equations by relaxed new iterative method

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For solving various equations of the form $v = f + N(v)$, the new iterative method and the new algorithm proposed by V. Daftardar–Gejji et al. [Daftardar–Gejji V., Jafari H. J. Math. Anal. Appl. **316** (2), 753–763 (2006); Kumar M., Jhinga A., Daftardar–Gejji V. Int. J. Appl. Comp. Math. **6** (2), 26 (2020)] are been employed successfully and accurately. Our aim in this paper is to present a relaxed new iterative method by introducing a controlled parameter ω in order to extend these methods. According to the values of the parameter ω, we discuss and provide the convergence analysis. The proposed algorithm is fast, effective and simple to implement as compared to the existing one. Numerous non-linear equations are solved to show the applicability and efficiency of the algorithm compared to the other methods.

Keywords: *nonlinear functional equations; new iterative method; Daftardar–Gejji and Jafari method; relaxed new iterative method; Adomian decomposition method; homotopy perturbation method; variational iterative method.*

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1. Introduction

A variety of problems in biology, chemistry, physics, and engineering can be modeled and formulated in terms of nonlinear functional equations as:

$$L(v) - R(v) - N(v) = g, \tag{1}$$

where g is the source term, $N(v)$ represents the nonlinear terms, L is the highest order derivative which is assumed to be easily invertible, R is a linear differential operator of order less than L, and v is the unknown function.

We can write (1) as follow, without loss of generality

$$v = f + L^{-1}(g + R(v) + N(v)) = f + F(v), \tag{2}$$

where F the nonlinear operator and f is a given function.

These nonlinear functional equations model many complex phenomena encountered in various branches of Science. Most of these equations do not have exact solutions, hence iterative/numerical methods have to be explored. Generally, equations (1) or (2) represent non-linear functional system of equations, linear/non-linear differential equations, ordinary differential equations ODEs, integral equations IEs, partial differential equations PDEs, differential algebraic equations DAEs, differential equations involving fractional order, systems of ODE/PDE/DAE, and so on.

To find the exact solution of the problem (1) or (2) is not easy and becomes complicated when the functional operator involves nonlinear partial differential equations and fractional derivatives. Hence, several analytical and approximate method have to be explored. In this exploration, various decomposition methods are proposed such as Adomian decomposition method (ADM) [1], homotopy perturbation method (HPM) [2], variational iterative method (VIM) [3], new iterative method (NIM), called also Daftardar–Gejji and Jafari method (DGJ) [4], and so on.

Initially proposed by Daftardar–Gejji and Jafari in [4], the NIM (or DGJ) method has been used by several authors to obtain an approximation of the solution of their problems. A recent overview of the method is given in [5]. Many research studies have been carried out to solve a nonlinear functional problem using the NIM method. The authors use and implement the method, to solve linear/nonlinear partial differential equations of integer and fractional order [6], to solve fractional differential equations [7], and to solve linear/nonlinear fractional diffusion-wave equations on finite domains with Dirichlet boundary conditions [8]. In [9], the authors explore the utility of the DGJ method to obtain approximate solution of the hyperbolic telegraph equation. In [10,11], the authors use the NIM method for solving fractional physical differential equations and fractional version of logistic equation. In [12], the author use the method to present a new finite-difference predictor–corrector method to solve nonlinear fractional differential equations along with its error and stability. In [13], the convergence of the NIM method has been studied and sufficiency conditions for its convergence towards the exact solution have been presented.

In [14], Radid et al., propose a SOR-like new iterative method to improve NIM and solve the spread of a nonfatal disease problem as well as the prey and predator problem.

Recently Manoj Kumar et al. [15] have proposed a new algorithm (NA) for solving a variety of non-linear functional equations problems as a simple reformulation of the NIM method which reduces a computation and is easily implemented.

Our aim in this paper is to present a relaxed new iterative method RNIM by introducing a relaxation parameter ω in order to extend and improve the NIM method as well as the New Algorithm (NA). According to the values of the parameter ω, we discuss and provide the convergence analysis.

Our contribution here can be summarized in the following steps:

— introducing a parameter ω to define a relaxed new iterative method;
— an application to new iterative method and new algorithm are presented;
— the convergence of the proposed method is discussed according to the values of the parameter ω;
— some numerical examples are given as illustration to confirm theoretical result;
— case where our method converges when the other methods fail.

Finally the general conclusion is presented.

2. Relaxed new iterative method

In this section, based on the new iterative method proposed in [4, 5], we discuss the basic idea of our proposed method. Then, we give a reformulation of the proposed method in order to extend and improve the new algorithm (NA) proposed by Kumar et al. in [15].

Let consider the general functional equation (2) written in the following form:

$$v(t) = f(t) + F(v(t)), \tag{3}$$

where f is a given function, F is a nonlinear continuous operator defined from a Banach space $\mathbf{H} \to \mathbf{H}$ equipped with the operator norm $\|F\| = \max_{\|v\|=1} \|F(v)\|$. We are looking for a solution u of (3) in the series form

$$v(t) = \sum_{i=0}^{\infty} v_i(t).$$

2.1. Basic idea of the proposed method

The idea of the proposed method is defined by the following process: let introduce a relaxation parameter ω such that $0 < \omega < 2$ and present the following recurrence formula

$$\begin{aligned} &v_0 = f, \\ &v_1 = \omega\, F(v_0), \\ &v_2 = \omega\,(F(v_0 + v_1) - F(v_0)) + (1-\omega)v_1, \\ &\dots \\ &v_i = \omega\,(F(v_0 + v_1 + v_2 + \ldots + v_{i-1}) - F(v_0 + v_1 + v_2 + \ldots + v_{i-2})) \\ &\qquad +(1-\omega)v_{i-1}, \quad i = 2, 3, \ldots. \end{aligned} \tag{4}$$

Addition of the above equations gives:

$$\sum_{k=0}^{i} v_k = f + \omega\left(F(v_0) + \sum_{k=1}^{i-1}\left\{F\Big(\sum_{j=0}^{k} v_j\Big) - F\Big(\sum_{j=0}^{k-1} v_j\Big)\right\}\right) + (1-\omega)\sum_{k=1}^{i-1} v_k$$
$$= \omega\left(f + F\Big(\sum_{k=0}^{i-1} v_k\Big)\right) + (1-\omega)\sum_{k=0}^{i-1} v_k, \quad i = 1, 2, \dots.$$

As $i \to \infty$, we get $\sum_{k=0}^{\infty} v_k = \omega\big(f + F(\sum_{k=0}^{\infty} v_k)\big) + (1-\omega)\sum_{k=0}^{\infty} v_k$.

Therefore, $v = \sum_{k=0}^{\infty} v_k$ is also solution of (3). Since,

$$v = f + F(v).$$

Remark 1. For $\omega = 1$, we found the classical new iterative method defined in [4,5] and the conditions of its convergence have been discussed in [13].

2.2. Relaxed new algorithm: SOR NA

We introduce a Relaxed New Algorithm noted SOR NA as follows: let us denote the approximate solution of the $(n+1)$-term of (3) by SR_n, i.e.

$$SR_n = \sum_{k=0}^{n} v_k, \quad n \in \mathbb{N}.$$

Taking into account the recurrence relation (4), we obtain the following algorithm to calculate the SR_n's:

$$\begin{cases} SR_0 = f, \\ SR_n = \omega(SR_0 + F(SR_{n-1})) + (1-\omega)SR_{n-1}, \quad n = 1, 2, 3, \dots. \end{cases} \tag{5}$$

where $\lim\limits_{n\to\infty} SR_n = v$ is the required solution.

Remark 2. The relaxed new algorithm can be seen as a reformulation of the relaxed new iterative method that considerably reduces the complexity as well as the computational cost. For $\omega = 1$, we found the new algorithm defined in [15] by

$$\begin{cases} S_0 = f, \\ S_n = S_0 + F(S_{n-1}), \quad n = 1, 2, 3, \dots, \end{cases} \tag{6}$$

where

$$S_n = \sum_{i=0}^{n} v_i, \quad n \in \mathbb{N},$$

and v_i's are determined by the following recurrence formula

$$\begin{aligned} &v_0 = f, \\ &v_1 = F(v_0), \\ &v_2 = F(v_0 + v_1) - F(v_0), \\ &\dots \\ &v_k = F(v_0 + v_1 + \dots + v_{k-1}) - F(v_0 + v_1 + \dots + v_{k-2}), \quad k = 2, 3, \dots. \end{aligned}$$

3. Convergence analysis

In order to study the convergence of our SOR NA, let consider

$$\overline{F}\colon \mathbf{H} \to \mathbf{H},$$
$$u \to v = \overline{F}(u),$$

such that

$$v = \overline{F}(u) = \omega\,(f + F(u)) + (1-\omega)u, \tag{7}$$

and recall the following definition.

Definition 1 (Ref. [16]). *An operator G is said to be Fréchet differentiable at $x \in \mathbf{H}$ if there exists a continuous linear operator $A\colon \mathbf{H} \to \mathbf{H}$ such that*

$$G(x+h) - G(x) = Ah + g(x,h),$$

where

$$\lim_{\|h\|\to 0} \frac{\|g(x,h)\|}{\|h\|} = 0.$$

A is called the Fréchet derivative of G at x and is denoted by $G'(x)$.

The following theorem describes the convergence of the SOR NA.

Theorem 1. *If F is Fréchet differentiable and continuous then $\overline{F}$ is also Fréchet differentiable and continuous and if $\|F'\| = r < 1$ and $0 < \omega < 2$ then $\|\overline{F}'\| = \rho < 1$ and the sequence of successive iterations $\{SR_n\}$ given in (5) converges uniformly to $\lim_{n\to\infty} SR_n = u^*$ a solution of (3) i.e. $u^* = f + F(u^*)$.*

Proof. Suppose that F is continuous and Fréchet differentiable with bounded Fréchet derivative F' i.e.

$$\begin{aligned}\lim_{h\to 0} \frac{\overline{F}(u+h) - \overline{F}(u)}{h} &= \lim_{h\to 0} \frac{\omega\,(f + F(u+h)) + (1-\omega)(u+h) - \omega\,(f+F(u)) - (1-\omega)u}{h}\\ &= \lim_{h\to 0} \frac{\omega\,(F(u+h) - F(u)) + (1-\omega)h}{h}\\ &= \lim_{h\to 0} \omega \frac{F(u) - F(u)}{h} + (1-\omega)I.\end{aligned}$$

As F is Fréchet differentiable with bounded Fréchet derivative F', then

$$\lim_{h\to 0} \frac{\overline{F}(u+h) - \overline{F}(u)}{h} = \omega F' + (1-\omega)I + o(h),$$

where I denotes the identity operator and o is an operator such that $\lim_{h\to 0} o(h) = 0$. So, $\overline{F}$ is also Fréchet differentiable with bound derivative $F' + (1-\omega)I$.

From (5) and using mean value inequality for Banach spaces [16], we have

$$\begin{aligned}\|SR_{j+1} - SR_j\| &= \big\|\omega(SR_0 + F(SR_j) + (1-\omega)SR_j - \omega(SR_0 + F(SR_{j-1})) - (1-\omega)SR_{j-1}\big\|\\ &= \big\|\omega F(SR_j) + (1-\omega)SR_j - \omega F(SR_{j-1}) - (1-\omega)SR_{j-1}\big\|\\ &\leqslant |\omega r + 1 - \omega|\,\big\|SR_j - SR_{j-1}\big\|\\ &\leqslant \rho\big\|SR_j - SR_{j-1}\big\|.\end{aligned}$$

Let $\rho = |\omega r + 1 - \omega|$,

- if $\omega = 1$, then $\rho = r < 1$;
- if $\omega \neq 1$, then $0 < 1 - r < 1 \Rightarrow 0 < \omega(1-r) < \omega \Rightarrow 1 - \omega < 1 - \omega(1-r) < 1$. As $-2 < -\omega < 0$ then we have: $-1 < 1 - \omega(1-r) < 1$ which implies $\rho = |\omega r + 1 - \omega| = |1 - \omega(1-r)| < 1$.

Therefore

$$\begin{aligned}\big\|SR_{j+1} - SR_j\big\| \leqslant \rho\big\|SR_j - SR_{j-1}\big\| &\leqslant \rho^2\big\|SR_{j-1} - SR_{j-2}\big\|\\ &\dots\\ &\leqslant \rho^j\big\|SR_1 - SR_0\big\|.\end{aligned}$$

Now, SR_n can be written as

$$SR_n = SR_0 + \sum_{j=0}^{n-1}(SR_{j+1} - SR_j).$$

By Weierstrass M-test, $\sum_{j=0}^{\infty} \rho^j \|SR_1 - SR_0\|$ converges, since $\{SR_n\}$ converge uniformly to a continuous function u^*, which is a required solution of (3). ∎

Proposition 1. Under Theorem 1 assumptions, if u_n and $\overline{u}_n$ are obtained by (5) and (6) respectively, and both ρ and r are less than one and if $1 < \omega < \frac{1+r}{1-r}$, then the rate of convergence of $\sum_{n=0}^{\infty} \overline{u}_n$ is higher than $\sum_{n=0}^{\infty} u_n$ (i.e. $\rho < r$).

Proof. Under Theorem 1, we have $\rho = |1-\omega(1-r)|$, $r < 1$ and $\omega > 1$.

Then, $\omega(1-r) > 1-r$ which implies $1-\omega(1-r) < r$.

Also, from $-\omega > -\frac{1+r}{1-r}$, we obtain $-r < 1-\omega(1-r)$.

Finally, $\rho = |1-\omega(1-r)| < r$. ∎

4. Illustrative examples

In this section, we will use the Relaxed New Algorithm (denoted SOR NA) established in previous section taking into account of the previous remark (2) to solve a variety of problems.

Example 1 (Ref. [15]). Consider the following nonlinear algebraic equation

$$3u^5 - 6u^4 + 12u^3 - 3u^2 - 82u + 76 = 0. \tag{8}$$

Equation (8) can be written as

$$u = \frac{76}{82} + \frac{1}{82}\left[3u^5 - 6u^4 + 12u^3 - 3u^2\right] = f + F(u),$$

where $f = \frac{76}{82}$ and $F(u) = \frac{1}{82}\left[3u^5 - 6u^4 + 12u^3 - 3u^2\right]$.

Note that $\|F'\| \leqslant 0.9878 < 1$.

In view of the SOR NA (5), we get

$$SR_0 = 0.926829,$$
$$SR_1 = 0.0561118\,\omega + 0.926829,$$
$$SR_2 = 2.03507\ 10^{-8}\omega^6 + 9.55357\ 10^{-7}\omega^5 + 0.000033452\,\omega^4 + 0.00089565\,\omega^3 - 0.044257\,\omega^2 + 0.112224\,\omega + 0.926829,$$
$$\ldots.$$

Table 1. Comparison of approximation solution for k iterations.

k	S_k	SR_k
0	0.9268293	0.9268293
1	0.9829411	0.9997747
2	0.995726	0.9999926
3	0.9989114	0.9999998
4	0.9997216	1.0
5	0.9999287	1.0
6	0.9999817	1.0
7	0.9999953	1.0
8	0.9999988	1.0
9	0.9999997	1.0
10	1.0	1.0

According to $\omega = 1.3$, we give a comparison between the values from $\{SR_n\}$ sequence (5) and those from $\{S_n\}$ sequence (6) in Table 1.

From Table 1, we remark that by taking $\omega = 1.3$ our proposed method converges in a few iterations to the exact solution $u = 1$ compared to the other methods. Hence,

$$\lim_{n\to\infty} SR_n = 1.$$

Example 2 (Ref. [15]). Consider the following nonlinear fractional differential equation:

$$\frac{d^\alpha u(t)}{dt^\alpha} - \frac{2}{5}u(t)^2 + \frac{1}{10}u(t) = 0, \quad 0 < \alpha \leqslant 1, \quad t \geqslant 0, \tag{9}$$

with the initial condition:

$$u(0) = \frac{1}{5}. \tag{10}$$

The exact solution of Eqs. (9) and (10) for $\alpha = 1$, is given by:

$$u(t) = (e^{\frac{t}{10}} + 4)^{-1}. \tag{11}$$

By integrating the Eqs. (9) and (10), we obtain

$$u(t) = u(0) + I_t^\alpha\left(\frac{2}{5}u(t)^2 - \frac{1}{10}u(t)\right) = \frac{1}{5} + I_t^\alpha\left(\frac{2}{5}u(t)^2 - \frac{1}{10}x(t)\right)$$
$$= f(t) + F(u(t)).$$

Take $f(t) = \frac{1}{5}$ and $F(u(t)) = I_t^\alpha\left(\frac{2}{5}u(t)^2 - \frac{1}{10}u(t)\right)$.

Here, $\|F'\| \leqslant 0.9 < 1$.

Taking into account the recurrence relation (5), we obtain

$$SR_0 = f = \frac{1}{5},$$

$$SR_1 = -\left(\frac{t^{\alpha}\omega}{250\,\alpha\,\Gamma(\alpha)}\right) + \frac{1}{5},$$

$$SR_2 = \frac{t^{\alpha}\omega^2}{250\,\Gamma(\alpha+1)} - \frac{3t^{2\alpha}\omega^2}{12500\,\Gamma(\alpha+1)^2} + \frac{t^{3\alpha}\omega^3}{156250\,\Gamma(\alpha+1)^3} - \frac{t^{\alpha}\omega}{125\,\Gamma(\alpha+1)} + \frac{1}{5},$$

$$\begin{aligned} SR_3 = {} & \frac{3t^{\alpha}\omega^2}{250\,\Gamma(\alpha+1)} - \frac{t^{\alpha}\omega^3}{250\,\Gamma(\alpha+1)} - \frac{3t^{\alpha}\omega}{250\,\Gamma(\alpha+1)} - \frac{9t^{2\alpha}\omega^2}{12500\,\Gamma(\alpha+1)^2} + \frac{3t^{2\alpha}\omega^3}{6250\,\Gamma(\alpha+1)^2} \\ & + \frac{11t^{3\alpha}\omega^3}{625000\,\Gamma(\alpha+1)^3} - \frac{(t^{3\alpha})\omega^4}{31250\,\Gamma(\alpha+1)^3} + \frac{t^{(3\alpha)}\omega^5}{156250\,\Gamma(\alpha+1)^3} + \frac{3t^{4\alpha}\omega^4}{1562500\,\Gamma(\alpha+1)^4} \\ & - \frac{3t^{4\alpha}\omega^5}{3906250\,\Gamma(\alpha+1)^4} - \frac{7t^{5\alpha}\omega^5}{390625000\,\Gamma(\alpha+1)^5} + \frac{t^{5\alpha}\omega^6}{48828125\,\Gamma(\alpha+1)^5} - \frac{3t^{6\alpha}\omega^6}{2441406250\,\Gamma(\alpha+1)^6} \\ & + \frac{t^{7\alpha}\omega^7}{61035156250\,\Gamma(\alpha+1)^7} + \frac{1}{5}, \end{aligned}$$

$\dots$

According to $\omega = 0.8$, we give a comparison between the values from $\{S_n\}$ sequence (6) and those from $\{SR_n\}$ sequence (5) in Figure 1.

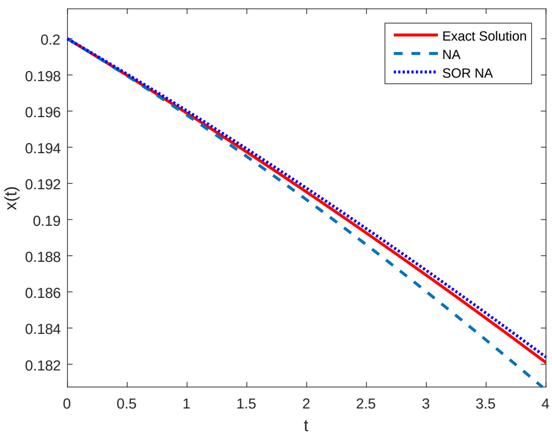

Fig. 1. For $\alpha = 1$: exact solution (11), the 5-term (6) and (5) approximations of Eq. (9).

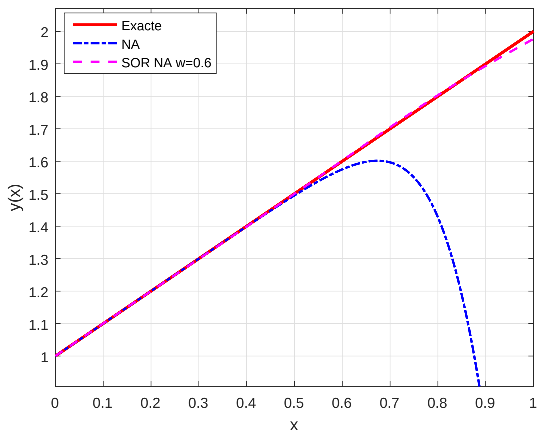

Fig. 2. Five-term (5) and (6) approximations of Eq. (12) compared to the exact solution.

Hence, the approximate solution of Eqs. (9) and (10) given by new algorithm method (6) and by our proposed method (5) are plotted in Figure 1 for $\alpha = 1$, and compared to the exact solution (11).

Example 3. Consider the initial value problem

$$\begin{cases} y' + \left(1 + x^2\right) y^2 = x^4 + 2x^3 + 2x^2 + 2x + 2, \\ y(0) = 1. \end{cases} \tag{12}$$

Equation (12) can be written as

$$y = \frac{x^5}{5} + \frac{x^4}{2} + \frac{2x^3}{3} + x^2 + 2x + 1 - \int_0^x \left(1+\tau^2\right) y^2(\tau)\, d\tau = f + F(y),$$

where $f = \frac{x^5}{5} + \frac{x^4}{2} + \frac{2x^3}{3} + x^2 + 2x + 1$ and $F(y) = -\int_0^x \left(1+\tau^2\right) y^2(\tau)\, d\tau$.

In view of the SOR NA (5), we get

$$SR_0 = \frac{x^5}{5} + \frac{x^4}{2} + \frac{2x^3}{3} + x^2 + 2x + 1,$$

$$SR_1 = -\frac{w}{325}x^{13} - \frac{w}{60}x^{12} - \frac{167w}{3300}x^{11} - \frac{19w}{150}x^{10} - \frac{497w}{1620}x^9 - \frac{3w}{5}x^8 - \frac{311w}{315}x^7 - \frac{68w}{45}x^6 + \left(\frac{1}{5} - \frac{32w}{15}\right)x^5 + \left(\frac{1}{2} - \frac{7w}{3}\right)x^4 + \left(\frac{2}{3} - \frac{7w}{3}\right)x^3 + (1-2w)x^2 + (2-w)x + 1,$$

$$\ldots.$$

According to $\omega = 0.6$, we give a comparison between the values from $\{SR_n\}$ sequence (5) and those from $\{S_n\}$ sequence (6) in Figure 2.

The exact solution of (12) is given by $y(x) = 1 + x$.

Remark 3. In this example, we show in Figure 2 that our proposed method converges but the NIM method and its modification given by (6) fails.

Example 4. In this example, the fourth-order Runge–Kutta (RK4) approximation solutions will be considered as a benchmark for the considered problems.

Consider the system of nonlinear differential equations:

$$\begin{cases} x' = 2y^2, & x(0) = 1, \\ y' = e^{-t}x, & y(0) = 1, \\ z' = y + z, & z(0) = 0. \end{cases} \tag{13}$$

Integrating (13), we get

$$x = 1 + 2\int_0^t y^2 dt = f_1 + F_1(x, y, z),$$

$$y = 1 + \int_0^t e^{-t}x\, dt = f_2 + F_2(x, y, z),$$

$$z = \int_0^t (y + z)\, dt = F_3(x, y, z).$$

In view of the SOR NA (5), we get

$$SR_{1,0} = f_1 = 1,$$

$$SR_{2,0} = f_2 = 1,$$

$$SR_{3,0} = f_3 = 0,$$

$$SR_{1,1} = \omega(2t + 1) - w + 1,$$

$$SR_{2,1} = 1 - \omega(e^{-t} - 2) - w,$$

$$SR_{3,1} = t\,\omega,$$

$$SR_{1,2} = \omega\left(2t - t^2(2e^{-t} - 2) + \frac{2t^3(e^{-t} - 1)^2}{3} + 1\right) - (\omega - 1)(\omega(2t + 1) - \omega + 1),$$

$$SR_{2,2} = \omega\left(te^{-t}(t^2 + 1) + 1\right) + (\omega - 1)\left(\omega + \omega(e^{-t} - 2) - 1\right),$$

$$SR_{3,2} = \omega\left(t - t^2\left(\frac{e^{-t}}{2} - \frac{1}{2}\right) + \frac{t^3}{2}\right) - t\omega(\omega - 1),$$

$$\ldots.$$

According to values of ω, we give a comparison between the SOR NA given by (5), the New Algorithm denote NA given by (6) and the fourth- order Runge–Kutta in Figure 3.

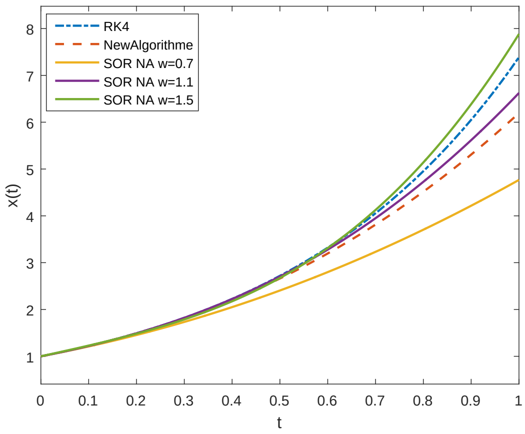

Fig. 3. Four-term (5) and (6) approximations of Eq. (12) compared to the exact solution.

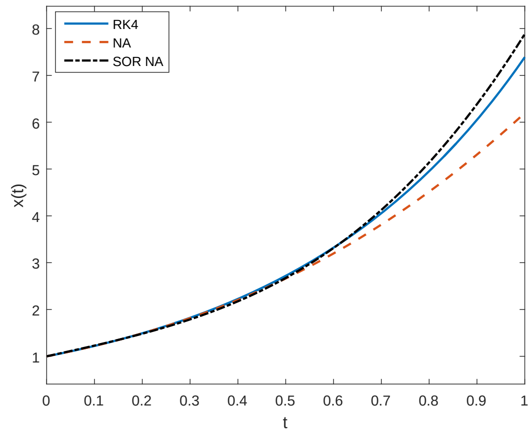

Fig. 4. Five-term (5) and (6) approximations of Eq. (12) compared to the exact solution.

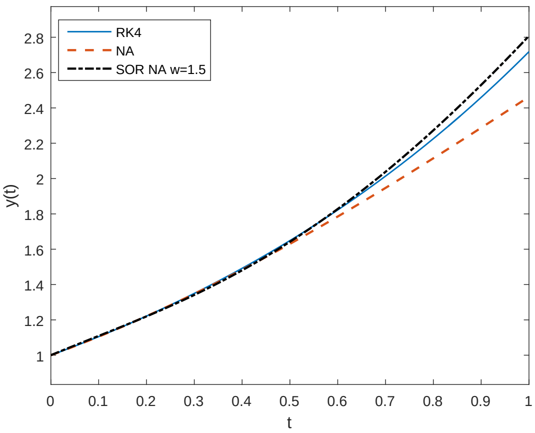

Fig. 5. Five-term (5) and (6) approximations of Eq. (12) compared to the exact solution.

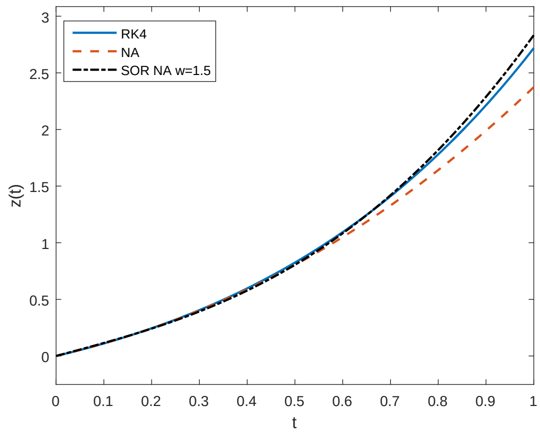

Fig. 6. Five-term (5) and (6) approximations of Eq. (12) compared to the exact solution.

5. Conclusion

In this paper we have introduced an relaxed new iterative method RNIM and relaxed new algorithm SOR NA as a new decomposition to solve the non-linear functional equation of the form $v = f + F(v)$. According to the values of ω, the proposed method gives new variants of new iterative method and improves and accelerates the new algorithm for NIM method [15]. We present the convergence analysis for our proposed algorithm under certain assumptions by taking account of a valid interval for a parameter ω. To demonstrate its usefulness and confirm our theoretical and numerical results, a number of non-trivial examples are presented and solved. We show that our proposed algorithm is suitable for obtaining exact solutions and gives a clear speedup as compared to the other algorithms.

In the section 4, example 3, we show that our proposed method converges to the exact solution when the others methods fail.

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Розв'язування нелінійних функціональних рівнянь новим ітераційним методом

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Для розв'язування різноманітних рівнянь виду $v = f + N(v)$, запропоновано В. Дафтардар–Геджі та ін. новий ітераційний метод і новий алгоритм [Daftardar–Gejji V., Jafari H. J. Math. Anal. Appl. **316** (2), 753–763 (2006); Kumar M., Jhinga A., Daftardar–Gejji V. Int. J. Appl. Comp. Math. **6** (2), 26 (2020)], які використовуються успішно і точно. Наша мета в цій статті полягає в тому, щоб подати послаблений новий ітеративний метод шляхом введення контрольованого параметра ω для розширення цих методів. За значеннями параметра ω обговорюємо та здійснюємо аналіз збіжності. Запропонований алгоритм є швидким, ефективним і простим у реалізації порівняно з існуючим. Численні нелінійні рівняння розв'язуються, щоб показати застосовність та ефективність алгоритму порівняно з іншими методами.

Ключові слова: *нелінійні функціональні рівняння; новий ітераційний метод; метод Дафтардар–Геджі та Джафарі; послаблений новий ітераційний метод; метод декомпозиції Адоміана; метод гомотопічних збурення; варіаційний ітераційний метод.*