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Global stability of fractional partial differential equations applied to the biological system modeling a viral infection with Hattaf time-fractional derivative

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In this article, we study the global stability of fractional partial differential equations applied to the biological system modeling a viral infection. The reaction in the proposed biological system is described by the new generalized Hattaf fractional (GHF) derivative. However, the diffusion is modeled by the Laplacian operator.

Keywords: *fractional partial differential equation; Hattaf time-fractional derivative; diffusion; global stability.*

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1. Introduction

A partial differential equation (PDE) describes a relationship between an unknown function and its partial derivatives. Partial differential equations (PDEs) are prevalent across various domains in physics and engineering. Furthermore, there has been a notable surge in their application in various fields such as biology [1], chemistry [2], ecology [3, 4], epidemiology [5, 6], computer science [7] (especially in the context of image manipulation and graphics), as well as in economics [8, 9], particularly in finance [10], in recent years.

Certain authors have shown interest in developing Lyapunov functionals to demonstrate the global stability of dynamical systems. For instance, Wang and Wang [11] introduced a mathematical model in order to simulate hepatitis B virus (HBV) infection taking into account spatial dependence. Based on the same model given in [11], Wang et al. [12] incorporated a delay to represent the duration required for infected cells to generate virions after viral entry. AlAgha and Elaiw [13] investigated the global stability of a system of PDEs with six dimensions that characterize the dynamics of human immunodeficiency virus type one (HIV-1) in the existence of humoral immune response and heterogeneous diffusion. Hattaf and Yousfi [14] formulated a mathematical model for HBV infection featuring dual modes of transmission, enabling the dynamics of HBV DNA-containing capsids and three distributed delays. In [15], they investigated an HBV model by introducing spatial diffusion, a general incidence rate, and time delays subject to homogeneous Neumann boundary. Other recent works for global stability of dynamical systems have been investigated in [16–18].

Fractional differential equations (FDEs) have been increasingly employed in various fields to describe the temporal movements of diverse systems in recent years. These types of equations represent the generalization of the ordinary differential equations (ODEs). Nevertheless, fractional partial differential equations (FPDEs) are a generalization of PDEs and can serve as an effective tool for modeling phenomena with memory or possessing hereditary properties in spatiotemporal dynamics.

There are few works on the study of the stability of biological systems using FPDEs. For example, Nawaz et al. [19] studied the stability of a fractional diffusive SEAIR model with nonlinear incidence

rate. In [20], Hattaf and Yousfi established the global stability of some fractional biological models with and without diffusion using the Caputo fractional derivative. Hamadneh et al. [21] studied the stability of a fractional discrete glycolysis reaction–diffusion model based on the works in [22, 23] and the method of discretization employed in [24].

Recently, Hattaf [25] proposed a new generalized definition of the fractional derivative with non-singular kernel for Caputo and Riemann–Liouville types. This definition was used to study the influence of the memory effect on the movements of certain dynamical systems in biology and epidemiology. Also, this definition generalized the most widely used fractional derivatives with non-singular kernels found in existing literature. For example, the fractional derivative of Atangana–Baleanu [26], the weighted fractional derivative of Atangana–Baleanu [27] and the fractional derivative of Caputo–Fabrizio [28].

In this article, we aim to study the global stability of FPDEs applied to a biological system modeling viral infection, including GHF derivative. To achieve this, the model formulation and preliminaries are provided in Section 2. The global stability of the model's equilibria is discussed in Section 3. The conclusion of our work is presented in Section 4.

2. Model formulation and preliminaries

Within this part of the article, we give the definitions and demonstrate some fundamental results concerning the partial GHF derivative with nonsingular kernel that will be needed in the following.

Definition 1 (Ref. [25]). *Let $\rho \in [0,1)$, $\gamma, \lambda > 0$ and $g \in H^1(a,d)$. The GHF derivative of order ρ in Caputo sense of the function g with respect to the weight function $\omega(t)$ is defined as stated*

$$\partial_{a,t,w}^{\rho,\lambda,\gamma} g(x,t) = \frac{\mathcal{N}(\rho)}{w(t)(1-\rho)} \int_a^t E_\lambda\big[-\mu_\rho (t-x)^\gamma\big] \frac{\partial}{\partial \xi}\big(w(\xi) g(x,\xi)\big)\, d\xi, \tag{1}$$

where $w \in C^1(a,b)$, w', $w > 0$ on $[a,d]$, $\mu_\rho = \frac{\rho}{1-\rho}$, $\mathcal{N}(\rho)$ is a normalization function such that $\mathcal{N}(0) = \mathcal{N}(1) = 1$ and $E_\lambda(t) = \sum_{i=0}^{+\infty} \frac{t^i}{\Gamma(\lambda i+1)}$, $\lambda > 0$ is the Mittag–Leffler function of parameter λ.

In the above definition, $H^1(a,d)$ is the Sobolev space of the order one defined as stated

$$H^1(a,d) = \big\{ w \in L^2(a,d) \colon w' \in L^2(a,d) \big\}. \tag{2}$$

Corollary 1 (Ref. [29]). *Let $W(x,t) \in \mathbb{R}^+$ be a continuously differentiable function and $W^\star > 0$. Then, for any time $t \geqslant t_0$, we have*

$$\partial_{0,w,t}^{\rho,\lambda,\gamma} \left[W(x,t) - W^\star - W^\star \ln \frac{W(x,t)}{W^\star} \right] \leqslant \left(1 - \frac{W^\star}{W(x,t)}\right) \partial_{0,w,t}^{\rho,\lambda,\gamma} W(x,t). \tag{3}$$

For simplicity, denote $\partial_{0,w,t}^{\rho,\lambda,\lambda}$ by $\partial_t^{\rho,\lambda}$. Now, we extend the model given in [15] by using the GHF derivative for the purpose of describing the dynamics of HBV infection under the effects of diffusion and memory. Hence, we propose the following model:

$$\begin{cases} \partial_t^{\rho,\lambda} U_1 = \sigma - d_1 U_1(x,t) - g(U_1(x,t), U_2(x,t), U_3(x,t)) U_3(x,t), \\ \partial_t^{\rho,\lambda} U_2 = g(U_1(x,t), U_2(x,t), U_3(x,t)) U_3(x,t) - d_2 U_2(x,t), \\ \partial_t^{\rho,\lambda} U_3 = D\Delta U_3 + \delta U_2(x,t) - d_3 U_3(x,t), \end{cases} \tag{4}$$

where the general incidence function $g(y,z,w)$ is supposed to be continuously differentiable in the interior of $\mathbb{R}_+^3$ and satisfies the subsequent fundamental hypotheses:

$$g(0,z,w) = 0, \quad \text{for all } z \geqslant 0 \text{ and } w \geqslant 0; \tag{H_1}$$

$$\frac{\partial g}{\partial y}(y,z,w) > 0, \quad \text{for all } y > 0,\ z \geqslant 0 \text{ and } w \geqslant 0; \tag{H_2}$$

$$\frac{\partial g}{\partial z}(y,z,w) \leqslant 0 \text{ and } \frac{\partial g}{\partial w}(y,z,w) \leqslant 0, \quad \text{for all } y \geqslant 0,\ z \geqslant 0 \text{ and } w \geqslant 0. \tag{H_3}$$

In biological context, $U_1(y,t)$ is the density of cells without infection at location x and time t, $U_2(x,t)$ denotes the density of cells that have been infected at location x and time t and $U_3(x,t)$ is the density

of free virus at location x and time t. The parameter σ represents the rate at which uninfected cells are recruited, δ is the rate at which infected cells generate free virus particles, d_1, d_2 and d_3 are respectively, the mortality rates of uninfected cells, infected cells and free virus. D is the diffusion coefficient. $\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ denotes the Laplacian operator.

In this article, we investigate system (4) with Neumann boundary conditions, outlined as stated:

$$\frac{\partial U_3}{\partial \nu} = 0 \ \text{ on } \ \partial\Omega \times (0, +\infty),$$

and initial conditions:

$$U_1(\vartheta, 0) = \Phi_1(\vartheta) \geqslant 0, \quad U_2(\vartheta, 0) = \Phi_2(\vartheta) \geqslant 0, \quad U_3(\vartheta, 0) = \Phi_3(\vartheta) \geqslant 0, \quad \vartheta \in \bar{\Omega}, \tag{5}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$ and $\frac{\partial U_3}{\partial \nu}$ represents the outward normal derivative on $\partial\Omega$.

3. Global stability of equilibria

Based on the results given by Hattaf et al. in [30], the basic reproduction number of the virus, when spatial dependence is not considered, is determined by:

$$\mathcal{R}_0 = \frac{\sigma}{d_2 d_3}\, g\left(\frac{\sigma}{d_1}, 0, 0\right).$$

It is easy to see that if $\mathcal{R}_0 \leqslant 1$, then equilibrium $E_f(\frac{\sigma}{d_1}, 0, 0)$ represents the only stable state, signifying the disappearance of the free virus. The subsequent theorem establishes that there exists one and only one chronic infection equilibrium if $\mathcal{R}_0 > 1$.

Theorem 1. *(i) When $\mathcal{R}_0 \leqslant 1$, then the model (4) has a single infection-free equilibrium of the form $E_f(\frac{\sigma}{d_1}, 0, 0)$. (ii) When $\mathcal{R}_0 > 1$, then the model (4) has a single chronic infection equilibrium of the form $E^\star(U_1^\star, U_2^\star, U_3^\star)$ where $U_1^\star \in \left(\frac{\sigma}{d_1}, 0\right), U_2^\star > 0$ and $U_3^\star > 0$.*

Proof. For any equilibrium, the subsequent equations satisfy:

$$\sigma - d_1 U_1 - U_3\, g(U_1, U_2, U_3) = 0, \tag{6}$$

$$U_1\, g(U_1, U_2, U_3) - d_1 U_3 = 0, \tag{7}$$

$$\delta U_3 - \lambda_3 U_3 = 0. \tag{8}$$

By (6)–(8), we get the following equation

$$g\left(U_1, \frac{\sigma - d_1 U_1}{d_2}, \frac{\delta(\sigma - d_1 U_1)}{d_2 \lambda_3}\right) = \frac{d_2 \lambda_3}{\delta}. \tag{9}$$

We have $U_2 = \frac{\sigma - d_1 U_1}{d_2} \geqslant 0$, which implies that $U_1 \geqslant \frac{\sigma}{d_1}$.

Then, we do not have any equilibrium point if $U_1 > \frac{\sigma}{d_1}$.

Now, let h be a function defined on interval $[0, \frac{\sigma}{d_1}]$ by:

$$h(U_1) = g\left(U_1, \frac{\sigma - d_1 U_1}{d_2}, \frac{\delta(\sigma - d_1 U_1)}{d_2 \lambda_3}\right) - \frac{d_2 \lambda_3}{\delta}.$$

We have $h(0) = -\frac{d_2 \lambda_3}{\delta} < 0$, $h(\frac{\sigma}{d_1}) = \frac{d_2 \lambda_3}{\delta}(\mathcal{R}_0 - 1) > 0$ and

$$h'(U_1) = \frac{\partial g}{\partial U_1} - \frac{d_1}{d_2}\frac{\partial g}{\partial U_2} - \frac{\delta d_1}{d_2 \delta_3}\frac{\partial g}{\partial U_3} > 0.$$

Hence, there exists a unique infection equilibrium $E^\star(U_1^\star, U_2^\star, U_3^\star)$, with $U_1^\star \in (0, \frac{\sigma}{d_1})$, $U_2^\star > 0$ and $U_3^\star > 0$. ∎

Now, we rigorously study the global stability of E_f.

Theorem 2. *The disease-free equilibrium E_f is globally stable whenever $\mathcal{R}_0 \leqslant 1$.*

Proof. Consider the Lyapunov functional for system (4)–(5) at E_f in the subsequent manner:

$$L_0 = \int_\Omega \left\{ U_1(x,t) - U_0 - \int_{U_0}^{U_1(x,t)} \frac{g(U_0, 0, 0)}{g(s, 0, 0)}\, ds + U_2(x,t) + \frac{d_2}{\delta} U_3(x,t) \right\} dx,$$

where $U_0 = \sigma/d_1$. For simplicity, we consider the subsequent notations: $z = z(x,t)$, for any $z \in \{U_1, U_2, U_3\}$. The time Hattaf derivative of L_0 along the solution of system (4) satisfies

$$\begin{aligned}
\partial_t^{\rho,\lambda} L_0 &= \int_\Omega \left\{ \partial_t^{\rho,\lambda} u - \partial_t^{\rho,\lambda} \left(\int_{U_0}^{U_1(x,t)} \frac{g(U_0,0,0)}{g(s,0,0)}\, ds \right) + \partial_t^{\rho,\lambda} U_2 + \frac{d_2}{\delta} \partial_t^{\rho,\lambda} U_3 \right\} dx, \\
&\leqslant \int_\Omega \left\{ \sigma - d_1 U_1 - \frac{g(U_0,0,0)}{g(U_1,0,0)} \partial_t^{\rho,\lambda} U_1 - d - 2U_2 + \frac{d_1}{\delta} D\Delta U_3 + d_1 U_2 - \frac{d_1}{\delta} d_3 U_3 \right\} dx, \\
&\leqslant \int_\Omega \left\{ d_1 U_0 (1 - \frac{g(U_0,0,0)}{g(U_1,0,0)}) - d_1 U_1 \left(1 - \frac{g(U_0,0,0)}{g(U_1,0,0)} \right) + \frac{g(U_0,0,0)}{g(U_1,0,0)} g(U_1,U_2,U_3) U_3 \right\} \\
&\quad + \left\{ \frac{d_2}{\delta} D\Delta U_3 - \frac{d_2}{\delta} d_3 U_3 \right\} dx, \\
&\leqslant \int_\Omega \left\{ (d_1 U_0 - d_1 U_1) \left(1 - \frac{g(U_0,0,0)}{g(U_1,0,0)} \right) + \frac{g(U_0,0,0)}{g(U_1,0,0)} g(U_1,U_2,U_3) U_3 - d_2 U_2 + \frac{d_2}{\delta} D\Delta U_3 \right\} \\
&\quad + \left\{ d_1 U_2 - \frac{d_2}{\delta} d_3 U_3 \right\} dx, \\
&\leqslant \int_\Omega \left\{ d_1 U_0 \left(1 - \frac{U_1}{U_0} \right) \left(1 - \frac{g(U_0,0,0)}{g(U_1,0,0)} \right) + \frac{d_2 d_3}{\delta} \left(\frac{g(U_1,U_2,U_3)}{g(U_1,0,0)} \mathcal{R}_0 - 1 \right) U_3 \right\} dx.
\end{aligned}$$

According to hypothesis (H_2), we have

$$\left(1 - \frac{U_1}{U_0} \right) \left(1 - \frac{g\,(U_0,0,0)}{g(U_1,0,0)} \right) \leqslant 0.$$

Then $\partial_{0,w}^{\rho,\lambda} L_0(x,y) \leqslant 0$. By applying Theorem 5 in [31], we conclude that the equilibrium E_f of (4) is globally stable if $\mathcal{R}_0 \leqslant 1$. ∎

Next, we need to define adequate conditions to ensure the global stability of $E^\star$. Hence, we introduce the following hypothesis:

$$\left(1 - \frac{U_1}{U_0} \right) \left(\frac{g(U_1,U_2^\star,U_3^\star)}{g(U_1,U_2,U_3)} - \frac{U_3}{U_3^\star} \right) < 0, \quad \text{for all } U_1,U_2,U_3 > 0. \tag{H_4}$$

Remark 1. The condition (H_4) is verified by numerous types of incidence functions as the saturation incidence if $g(U_1,U_2,U_3) = \frac{\lambda U_1}{1+\delta U_3}$, the mass action when $g(U_1,U_2,U_3) = \lambda U_1$, Beddington–DeAngelis response [32, 33] when $g(U_1,U_2,U_3) = \frac{\lambda U_1}{1+\delta_1 U_1+\delta_2 U_3}$, Crowley–Martin response [34] when $g(U_1,U_2,U_3) = \frac{\lambda U_1}{1+\delta_1 U_1+\delta_2 U_3+\delta_1\delta_2 U_1 U_3}$ and the more generalized Hattaf–Yousfi functional response [35] of the form $g(U_1,U_2,U_3) = \frac{\lambda U_1}{\delta_0+\delta_1 U_1+\delta_2 U_3+\delta_3 U_1 U_3}$, where $\lambda \geqslant 0$ is the rate at which the infection progresses and $\delta_1,\delta_2,\delta_3 \geqslant 0$ are constants. Over the past few years, this generalized incidence function is employed in [36–39].

The subsequent theorem confirms the global stability of $E^\star$.

Theorem 3. *Assume $\mathcal{R}_0 > 1$ and (H_4) holds. Then the chronic infection equilibrium $E^\star$ is globally stable.*

Proof. Consider the subsequent Lyapunov functional:

$$L_1 = \int_\Omega \left\{ U_1 - U_1^* - \int_{U_1^*}^{U_1} \frac{g(U_1^*,U_2^*,U_3^*)}{g(s,U_2^*,U_3^*)}\, d_1 s + U_2^* H\left(\frac{U_2}{U_2^*} \right) + \frac{d_2}{\delta}\, U_3^* H\left(\frac{U_3}{U_3^*} \right) \right\} dx, \tag{10}$$

where $H(y) = y - 1 - \ln y$, $y > 0$. Obviously, $H(y)$ achieves its minimum value 0 at $y = 1$. Then, $H(y) \geqslant 0$. On the other hand, we have

$$\begin{aligned}
\partial_t^{\rho,\lambda} L_1 &\leqslant \int_\Omega \left\{ \left(1 - \frac{g(U_1^\star,U_2^\star,U_3^\star)}{g(U_1,U_2^\star,U_3^\star)} \right) \partial_t^{\rho,\lambda} U_1 + \left(1 - \frac{U_3^\star}{U_2} \right) \partial_t^{\rho,\lambda} U_2 + \frac{d_2}{\delta} \left(1 - \frac{U_3^\star}{U_3} \right) \partial_t^{\rho,\lambda} U_3 \right\} dx, \\
&\leqslant \int_\Omega \left\{ \left(1 - \frac{g(U_1^\star,U_2^\star,U_3^\star)}{g(U_1,U_2^\star,U_3^\star)} \right) \left(d_1 U_1^\star + g(U_1^\star,U_2^\star,U_3^\star) U_3^\star - d_1 U_1 - g(U_1,U_2,U_3) U_3 \right) \right\}
\end{aligned}$$

$$
\begin{aligned}
&+\left\{g(U_1,U_2,U_3)U_3-\frac{U_2^\star}{U_2}g(U_1,U_2,U_3)U_3+d_2U_2^\star+\frac{d_2}{\delta}D\Delta U_3-\frac{d_2d_3}{\delta}U_3-\frac{d_2}{\delta}\frac{U_3^\star}{U_3}D\Delta U_3\right\}\\
&+\left\{-d_2\frac{U_3^\star}{U_3}U_2+\frac{d_2d_3}{\delta}U_3^\star\right\}dx,\\
\leqslant&\int_\Omega d_1U_1^\star\left(1-\frac{U_1}{U_1^\star}\right)\left(1-\frac{g(U_1^\star,U_2^\star,U_3^\star)}{g(U_1,U_2^\star,U_3^\star)}\right)+g(U_1^\star,U_2^\star,U_3^\star)U_3^\star-\frac{g(U_1^\star,U_2^\star,U_3^\star)}{g(U_1,U_2^\star,U_3^\star)}g\big(U_1^\star,U_2^\star,U_3^\star\big)U_3^\star\\
&+\left\{\frac{g(U_1^\star,U_2^\star,U_3^\star)}{g(U_1,U_2^\star,U_3^\star)}g(U_1,U_2,U_3)U_3-\frac{U_2^\star}{U_2}g(U_1,U_2,U_3)U_3+d_2U_3^\star\frac{d_2}{\delta}D\Delta U_3\right.\\
&\qquad\left.-\frac{d_2d_3}{\delta}U_3-\frac{d_2}{\delta}\frac{U_3^\star}{U_3}D\Delta U_3\right\}+\left\{-d_2\frac{U_3^\star}{U_3}U_2+\frac{d_2d_3}{\delta}U_3^\star\right\}dx,\\
\leqslant&\int_\Omega d_1U_1^\star\left(1-\frac{U_1}{U_1^\star}\right)\left(1-\frac{g(U_1^\star,U_2^\star,U_3^\star)}{g(U_1,U_2^\star,U_3^\star)}\right)\\
&+\left\{d_2U_2^\star\left(-1-\frac{U_3}{U_3^\star}+\frac{U_3}{U_3^\star}\frac{g(U_1,U_2,U_3)}{g(U_1,U_2^\star,U_3^\star)}-\frac{g(U_1,U_2,U_3)}{g(U_1^\star,U_2^\star,U_3^\star)}\frac{U_3}{U_3^\star}\frac{U_2^*}{U_2}-\frac{U_3^*}{U_3}\frac{U_2^\star}{U_2}\right)\right\}\\
&-d_2U_2^\star\left(4-\frac{g(U_1^\star,U_2^\star,U_3^\star)}{g(U_1,U_2^\star,U_3^\star)}-\frac{g(U_1,U_2^\star,U_3^\star)}{g(U_1,U_2,U_3)}-\frac{g(U_1,U_2,U_3)}{g(U_1^\star,U_2^\star,U_3^\star)}\frac{U_3}{U_3^\star}\frac{U_2^\star}{U_2}-\frac{U_3^\star}{U_3}\frac{U_2}{U_2^\star}\right)dx\\
&-\frac{d_2U_3^\star}{\delta}D\int_\Omega\frac{\Delta U_3}{U_3}dx.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\partial_t^{\rho,\lambda}L_1\leqslant&\int_\Omega d_1U_1^\star\left(1-\frac{U_1}{U_1^\star}\right)\left(1-\frac{g(U_1^\star,U_2^\star,U_3^\star)}{g(U_1,U_2^\star,U_3^\star)}\right)\\
&+d_1U_2^*\left(-1-\frac{U_3}{U_3^\star}+\frac{U_3}{U_3^\star}\frac{g(U_1,U_2,U_3)}{g(U_1,U_2^\star,U_3^\star)}+\frac{g(U_1,U_2^\star,U_3^\star)}{g(U_1,U_2,U_3)}\right)\\
&-d_1U_2^\star\left(H\left(\frac{g(U_1^\star,U_2^\star,U_3^\star)}{g(U_1,U_2^\star,U_3^\star)}\right)+\ln\left(\frac{g(U_1^\star,U_2^\star,U_3^\star)}{g(U_1,U_2^\star,U_3^\star)}\right)\right)\\
&-d_1U_2^\star\left(H\left(\frac{g(U_1,U_2^\star,U_3^\star)}{g(U_1,U_2,U_3)}\right)+\ln\left(\frac{g(U_1,U_2^\star,U_3^\star)}{g(U_1,U_2,U_3)}\right)+H\left(\frac{U_3^\star U_2}{U_3U_2^\star}\right)+\ln\left(\frac{U_3^\star U_2}{U_3U_2^\star}\right)\right)\\
&-d_1U_2^\star\left(H\left(\frac{U_3}{U_3^\star}\frac{U_2^\star}{U_2}\frac{g(U_1,U_2,U_3)}{g(U_1^\star,U_2^\star,U_3^\star)}\right)+\ln\left(\frac{U_3}{U_3^\star}\frac{U_2^\star}{U_2}\frac{g(U_1,U_2,U_3)}{g(U_1^\star,U_2^\star,U_3^\star)}\right)\right)dx-\frac{d_1U_3^\star}{\delta}D\int_\Omega\frac{\Delta U_3}{U_3}dx.
\end{aligned}
$$

Then

$$
\begin{aligned}
\partial_t^{\rho,\lambda}L_1\leqslant&\int_\Omega d_1U_1^\star\left(1-\frac{U_1}{U_1^\star}\right)\left(1-\frac{g(U_1^\star,U_2^\star,U_3^\star)}{g(U_1,U_2^\star,U_3^\star)}\right)\\
&+d_1U_2^\star\left(-1-\frac{U_3}{U_3^\star}+\frac{U_3}{U_3^\star}\frac{g(U_1,U_2,U_3)}{g(U_1,U_2^\star,U_3^\star)}+\frac{g(U_1,U_2^\star,U_3^\star)}{g(U_1,U_2,U_3)}\right)\\
&-d_1U_2^\star\left(H\left(\frac{g(U_1^\star,U_2^\star,U_3^\star)}{g(U_1,U_2^\star,U_3^\star)}\right)+H\left(\frac{g(U_1,U_2^\star,U_3^\star)}{g(U_1,U_2,U_3)}\right)+H\left(\frac{U_3}{U_3^\star}\frac{U_2^\star}{U_2}\frac{g(U_1,U_2,U_3)}{g(U_1^\star,U_2^\star,U_3^\star)}\right)\right.\\
&\qquad\left.+H\left(\frac{U_3^\star U_2}{U_3U_2^\star}\right)\right)dx-\frac{d_2U_3^\star}{\delta}D\int_\Omega\frac{|\Delta U_3|^2}{U_3^2}dx,\\
\leqslant&\int_\Omega d_1U_1^\star\left(1-\frac{U_1}{U_1^\star}\right)\left(1-\frac{g(U_1^\star,U_2^\star,U_3^\star)}{g(U_1,U_2^\star,U_3^\star)}\right)\\
&+d_2U_2^\star\left(-1-\frac{U_3}{U_3^\star}+\frac{U_3}{U_3^\star}\frac{g(U_1,U_2,U_3)}{g(U_1,U_2^\star,U_3^\star)}+\frac{g(U_1,U_2^\star,U_3^\star)}{g(U_1,U_2,U_3)}\right)\\
&-d_1U_2^\star\left(H\left(\frac{g(U_1^\star,U_2^\star,U_3^\star)}{g(U_1,U_2^*,U_3^\star)}\right)+H\left(\frac{g(U_1,U_2^\star,U_3^\star)}{g(U_1,U_2,U_3)}\right)\right.\\
&\qquad\left.+H\left(\frac{U_3}{U_3^\star}\frac{U_2^\star}{U_2}\frac{g(U_1,U_2,U_3)}{g(U_1^\star,U_2^\star,U_3^\star)}\right)+H\left(\frac{U_3^\star U_2}{U_3U_2^\star}\right)\right)dx-\frac{d_1U_3^\star}{\delta}D\left(\sum_i\int_\Omega\frac{1}{U_3}\frac{\partial}{\partial x_i}\left(\frac{\partial U_3}{\partial x_i}\right)dx\right),
\end{aligned}
$$

where

$$\sum_i \int_\Omega \frac{1}{U_3}\frac{\partial}{\partial x_i}\left(\frac{\partial U_3}{\partial x_i}\right)dx = \int_{\partial\Omega}\frac{1}{U_3}\sum_i \frac{\partial U_3}{\partial x_i}\,d\sigma + \int_\Omega \frac{|\Delta U_3|^2}{U_3^2}\,dx.$$

Consequently,

$$\begin{aligned}
\partial_t^{\rho,\lambda} L_1 \leqslant \int_\Omega & d_1 U_1^\star\left(1-\frac{U_1}{U_1^\star}\right)\left(1-\frac{g(U_1^\star,U_2^\star,U_3^\star)}{g(U_1,U_2^\star,U_3^\star)}\right)\\
&+ d_2 U_2^\star\left(-1-\frac{U_3}{U_3^\star}+\frac{U_3}{U_3^\star}\frac{g(U_1,U_2,U_3)}{g(U_1,U_2^\star,U_3^\star)}+\frac{g(U_1,U_2^\star,U_3^\star)}{g(U_1,U_2,U_3)}\right)\\
&- d_2 U_2^\star\left(H\left(\frac{g(U_1^\star,U_2^\star,U_3^\star)}{g(U_1,U_2^\star,U_3^\star)}\right)+H\left(\frac{g(U_1,U_2^\star,U_3^\star)}{g(U_1,U_2,U_3)}\right)+H\left(\frac{U_3}{U_3^\star}\frac{U_2^\star}{U_2}\frac{g(U_1,U_2,U_3)}{g(U_1^\star,U_2^\star,U_3^\star)}\right)\right.\\
&\left.+ H\left(\frac{U_3^\star U_2}{U_3 U_2^\star}\right)\right)-\frac{d_2 U_3^\star}{\delta}D\int_\Omega\frac{|\Delta U_3|^2}{U_3^2}\,dx.
\end{aligned}$$

Since $g(U_1,U_2,U_3)$ is strictly monotonically increasing with respect to U_1, we get

$$\left(1-\frac{U_1}{U_1^\star}\right)\left(1-\frac{g\,(U_1^\star,U_2^\star,U_3^\star)}{g\,(U_1,U_2^\star,U_3^\star)}\right)\leqslant 0.$$

According to the assumption (H_4), we get

$$-1-\frac{U_3}{U_3^\star}+\frac{g(U_1,U_2^\star,U_3^\star)}{g(U_1,U_2,U_3)}+\frac{U_3}{U_3^\star}\frac{g(U_1,U_2,U_3)}{g\,(U_1,U_2^\star,U_3^\star)}=\left(1-\frac{g(U_1,U_2,U_3)}{g(U_1,U_2^\star,U_3^\star)}\right)\left(\frac{g(U_1,U_2^\star,U_3^\star)}{g(U_1,U_2,U_3)}-\frac{U_3}{U_3^\star}\right)\leqslant 0.$$

Since $H(z)\geqslant 0$ for $z>0$, we have $\partial_t^{\rho,\lambda}L_1\leqslant 0$. By applying Theorem 5 in [31], we conclude that the equilibrium E^* of (4) is globally stable if $\mathcal{R}_0>1$. ∎

4. Conclusion

In this article, we have established a fractional virus dynamics model with general incidence rate and Hattaf time-fractional derivative. By employing appropriate Lyapunov functionals, we have studied the global stability of both the disease-free equilibrium E_f and the chronic infection equilibrium $E^\star$. We have shown that E_f is globally stable if the basic reproduction number satisfies $\mathcal{R}_0\leqslant 1$. In this case, the virus cannot maintain the infection and will eventually disappear. When $\mathcal{R}_0>1$, the virus persists and $E^\star$ is globally stable.

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Глобальна стійкість диференціальних рівнянь у дробових похідних, які застосовані до біологічної системи, що моделює вірусну інфекцію

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У статті вивчається глобальна стійкість диференціальних рівнянь у дробових похідних, які застосовані до біологічної системи, що моделює вірусну інфекцію. Реакція в запропонованій біологічній системі описана новою узагальненою дробовою похідною Хаттафа (GHF), проте дифузія моделюється оператором Лапласа.

Ключові слова: *диференціальне рівняння в дробових похідних; дробова похідна Хаттафа; дифузія; глобальна стійкість.*