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Domination in linear fractional-order distributed systems

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This paper investigates the notion of domination in linear fractional-order distributed systems in a finite-dimensional state. The objective is to compare or classify the input operators with respect to the output ones, and we present the characterization and property results of this concept. Then, we examine the relationship between controllability and the notion of domination. Finally, we provide a numerical example to illustrate our results.

Keywords: *fractional-order; distributed systems; controllability; domination.*

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1. Introduction

In the field of control theory, it is indisputable that certain controls outperform others. This perspective opens up a wide realm for the classification of input operators, ultimately giving rise to the concept known as domination.

The concept of domination was initially introduced and treated for lumped and distributed systems [1], then the asymptotic case. Subsequently, dominance was introduced and developed for continuous systems in both the parabolic and the hyperbolic cases [2]. It involves researching the potential for categorizing input operators and categorizing output operators based on duality. An extension of domination to the regional case is given in [3]. The regional aspect of this problem arises from the fact that a system may dominate another in a region but not on the basis of the system's entire geometrical support. The authors of [4] study a broadening to a class of distributed discrete systems and, as a result, they examine the case of discrete diffusion processes additionally to sensors and actuators.

Recent years have seen intense research into fractional differential systems [5–8], yielding a wealth of intriguing findings. The study of controllability and observability in fractional-order differential systems continues to receive significant attention in the rapidly changing field of control theory and system analysis. Fractional dynamical systems in finite-dimensional spaces that are both linear and non-linear are controllable [5]. The Mittag–Leffler matrix function and Schauder's fixed point theorem are both used to derive the necessary conditions for controllability. In [9], as a generalization of the deterministic situation, the authors studied the exact and complete controllability of linear stochastic fractional systems. A sufficient controllability condition via the Schauder fixed point theorem for nonlinear fractional dynamical systems has been obtained in [6]. The case of linear fractional-order finite-dimensional dynamical control systems with delays is considered in different works [7, 8].

The relationship between controllability and domination has been studied in several studies [2], but in the case of finite-dimensional linear fractional-order systems, domination is still not discussed, and that is the purpose of this paper.

Initially, let us consider a class of linear fractional-order control systems expressed by

$$(S) \quad \begin{cases} {}^{c}_{0}D^{\alpha}_{\tau} z(\tau) = Az(\tau) + B_1 u_1(\tau) + B_2 u_2(\tau), \quad 0 < \tau < \mathcal{T}, \quad 0 < \alpha < 1, \\ z(0) = z_0 \end{cases} \tag{1}$$

with $A \in \mathcal{M}_n(\mathbb{R})$, $B_1 \in \mathcal{M}_{[n,p]}(\mathbb{R})$, $B_2 \in \mathcal{M}_{[n,m]}(\mathbb{R})$, $u_1 \in L^2(0, \mathcal{T}; \mathbb{R}^p)$, $u_2 \in L^2(0, \mathcal{T}; \mathbb{R}^m)$ and ${}^{c}_{0}D^{\alpha}_{\tau}$ designates the Caputo fractional-order derivative, where

$$ {}_0^c D_\tau^\alpha z(\tau) = \begin{cases} \frac{1}{\Gamma(1-\alpha)} \int_0^\tau (\tau - s)^{-\alpha} \dot{z}(s)\, ds, & 0 < \alpha < 1, \\ \dot{z}(\tau), & \alpha = 1. \end{cases} $$

where Γ is Gamma function. The output equation of systems S is given by:

$$ y(\tau) = Cz(\tau) $$

with $C \in \mathcal{M}_{[q,n]}(\mathbb{R})$, we find

$$ z(\tau) = \mathscr{S}_0(\tau) z_0 + H_1^\alpha(\tau) u_1 + H_2^\alpha(\tau) u_2 $$

with

$$ \mathscr{S}_0(\tau) = \sum_{r=0}^{\infty} \frac{A^r \tau^{r\alpha}}{\Gamma(r\alpha + 1)}. $$

Then

$$ y(\tau) = C\mathscr{S}_0(\tau)\, z_0 + CH_1^\alpha(\tau)\, u_1 + CH_2^\alpha(\tau)\, u_2, $$

where H_i^α is the operator defined by

$$ H_1^\alpha(\tau) \colon L^2\left(0, \tau; \mathbb{R}^p\right) \longrightarrow \mathbb{R}^n, $$
$$ u_1 \longrightarrow \int_0^\tau \mathscr{S}(\tau - s)\, B_1\, u_1(s)\, ds $$

and

$$ H_2^\alpha(\tau) \colon L^2\left(0, \tau; \mathbb{R}^m\right) \longrightarrow \mathbb{R}^n, $$
$$ u_2 \longrightarrow \int_0^\tau \mathscr{S}(\tau - s)\, B_2\, u_2(s)\, ds $$

with

$$ \mathscr{S}(\tau) = \sum_{r=0}^{\infty} \frac{A^r \tau^{(r+1)\alpha - 1}}{\Gamma[(r+1)\,\alpha]}. $$

For $i = 1, 2$, we note $H_i^\alpha = H_i^\alpha(\mathcal{T})$.

The system (S) is stimulated by two input factors, the first one $B_2 u_2$ is considered as a disturbance caused by accidental or voluntary actions, and the second term $B_1 u_1$ is introduced in order to compensate the effect of disturbance at final time $\mathcal{T}$ by restoring the observation $(C\mathscr{S}_0(\mathcal{T}) z_0)$ to its normal state at final time $\mathcal{T}$ using an appropriate control applied through the control operator B_1. So, we can reformulate the problem under this form: $\forall u_2 \in L^2(0, \mathcal{T}; \mathbb{R}^m)$, does a control $u_1 \in L^2(0, \mathcal{T}; \mathbb{R}^p)$ such that:

$$ H_2^\alpha u_2 + H_1^\alpha u_1 = 0 $$

exist?

This brings us to the concept of domination which entails to study the comparison (or classification) of input operators, with respecting the output one.

This paper is structured in the following way: in section 2, we determine and we define the domination concept for fractional-order systems and we give some properties for the characterization of the domination and some examples to confirm the procured results. In section 3, we discuss the connection through the notion of controllability and domination. In the last section, we present a conclusion that summarize all the previous results.

2. Definitions and characterizations

Definition 1. *If* $\operatorname{Im}(CH_2^\alpha) \subset \operatorname{Im}(CH_1^\alpha)$, *we denote that* B_1 *dominates* B_2 *with respecting the operator* C *on* $[0, \mathcal{T}]$. *We note, in this situation* $B_2 \underset{C}{\leqslant} B_1$.

2.1. Characterizations

The coming result offers a characterization of domination with respecting the output operator C.

Proposition 1. These mathematical characteristics are equivalent

1. The input operator (B_1) dominates (B_2) with respecting the operator C.

2. For any $u_2 \in L^2(0,\mathcal{T};\mathbb{R}^m)$, there exists $u_1 \in L^2(0,\mathcal{T};\mathbb{R}^p)$ such that:
$$CH_1^\alpha u_1 + CH_2^\alpha u_2 = 0. \tag{2}$$
3. $\mathrm{Ker}[B_1^*\mathscr{S}^*(\mathcal{T}-\cdot)C^*] \subset \mathrm{Ker}[B_2^*\mathscr{S}^*(\mathcal{T}-\cdot)C^*]$.
4. $\exists\beta>0$ such that for any $\omega \in \mathbb{R}^q$, we have
$$\|B_2^*\mathscr{S}^*(\mathcal{T}-\cdot)C^*\omega\|_{L^2(0,\mathcal{T}:\mathbb{R}^m)} \leqslant \beta\,\|B_1^*\mathscr{S}^*(\mathcal{T}-\cdot)C^*\omega\|_{L^2(0,\mathcal{T}:\,\mathbb{R}^p)}\,. \tag{3}$$

Proof. The equivalence between the assertions is due to definition and from that, if $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ are Banach spaces, and $\mathcal{P} \in \mathcal{L}(\mathcal{X},\mathcal{Z})$, $Q \in \mathcal{L}(\mathcal{Y},\mathcal{Z})$, we have $\mathrm{Im}(\mathcal{P}) \subset \mathrm{Im}(\mathcal{Q})$ if and only if
$$\exists\beta>0,\ \forall z^* \in \mathcal{Z}' / \,\|\mathcal{P}^*z^*\|_{\mathcal{X}'} \leqslant \beta\,\|\mathcal{Q}^*z^*\|_{\mathcal{Y}'}\,,$$
with $\mathcal{X}'$, $\mathcal{Y}'$ and $\mathcal{Z}'$ denote the dual spaces of $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$, respectively. ■

We consider now, the domination Gramian of the system (1).

Definition 2. *Let $q \geqslant 1$, the domination Gramian of system (1) is the symmetric $q\times q$-matrix defined by*
$$\mathcal{D}^\alpha(\mathcal{T}) = \int_0^{\mathcal{T}} C\mathscr{S}(\mathcal{T}-s)\,B_1\,B_1^*\,\mathscr{S}(\mathcal{T}-s)^*\,C^*(\mathcal{T}-s)^{2(1-\alpha)}ds.$$

Remark 1. We have, for all $\Psi \in \mathbb{R}^q$,
$$\Psi^*\,\mathcal{D}^\alpha(\mathcal{T})\,\Psi = \int_0^{\mathcal{T}} \left\|B_1^*\,\mathscr{S}(\mathcal{T}-s)^*\,C^*(\mathcal{T}-s)^{1-\alpha}\,\Psi\right\|^2 ds.$$
Hence the domination Gramian $\mathcal{D}^\alpha(\mathcal{T})$ is a symmetric nonnegative matrix.

We give afterwards a second characterization of the notion domination.

Theorem 1. *Let $\bar{\mathcal{D}}^\alpha(\mathcal{T}) = \mathcal{D}^\alpha(\mathcal{T})|_{\mathrm{Im}(C)}$, B_1 dominates B_2 on $[0,\mathcal{T}]$, with respecting operator C if and only if the matrix $\bar{\mathcal{D}}^\alpha(\mathcal{T})$ is invertible in $\mathrm{Im}(C)$.*

Proof. Firstly, we suppose that $\bar{\mathcal{D}}^\alpha(\mathcal{T})$ is invertible in $\mathrm{Im}(C)$ and demonstrate that B_1 dominates B_2 with respecting operator C, on $[0,\mathcal{T}]$. For $u_1 \in L^2(0,\mathcal{T};\mathbb{R}^p)$ defined by:
$$u_1(s) = B_1^*\,\mathscr{S}(\mathcal{T}-s)^*\,C^*(\mathcal{T}-s)^{2(1-\alpha)}\,\bar{\mathcal{D}}^\alpha(\mathcal{T})^{-1}\left(-CH_2^\alpha u_2\right),$$
for $s \in [0,\mathcal{T}]$. We find
$$\begin{aligned} y(\mathcal{T}) &= C\mathscr{S}_0(\mathcal{T})z_0 + \int_0^{\mathcal{T}} C\mathscr{S}(\mathcal{T}-s)B_1B_1^*\mathscr{S}(\mathcal{T}-s)^*C^*(\mathcal{T}-s)^{2(1-\alpha)}ds\,\mathcal{D}^\alpha(\mathcal{T})^{-1}\left(-CH_2^\alpha u_2\right) + CH_2^\alpha u_2 \\ &= C\mathscr{S}_0(\mathcal{T})z_0. \end{aligned}$$
then B_1 dominates B_2 with respecting the operator C, on $[0,\mathcal{T}]$.

Let us now surmise that $\bar{\mathcal{D}}^\alpha(\mathcal{T})$ is not invertible in $\mathrm{Im}(C)$, then $\exists\Psi \in \mathrm{Im}(C)\backslash\{0\}$ such that $\bar{\mathcal{D}}^\alpha(\mathcal{T})\Psi = 0$.

Particularly, $\Psi^*\bar{\mathcal{D}}^\alpha(\mathcal{T})\Psi = 0$, signifies that,
$$\int_0^{\mathcal{T}} \Psi^*C\mathscr{S}(\mathcal{T}-s)\,B_1\,B_1^*\mathscr{S}(\mathcal{T}-s)^*C^*(\mathcal{T}-s)^{2(1-\alpha)}\,\Psi\,ds = 0.$$
From Remark 1, we obtain
$$\int_0^{\mathcal{T}} \left\|B_1^*\mathscr{S}(\mathcal{T}-s)^*C^*(\mathcal{T}-s)^{1-\alpha}\Psi\right\|^2 ds = 0.$$
Which implies that
$$\Psi^*(\mathcal{T}-s)^{1-\alpha}C\mathscr{S}(\mathcal{T}-s)B_1 = 0, \quad s \in [0,\mathcal{T}],$$
then
$$B_1^*\mathscr{S}(\mathcal{T}-s)^*C^*\Psi = 0,$$
consequently the inequality (3) is not verified.

So, $\bar{\mathcal{D}}^\alpha(\mathcal{T})$ is invertible. ■

Proposition 2. If $\operatorname{rank}\big(CB_1 \;\; CAB_1 \; \ldots \; CA^{n-1}B_1\big) = q$, then B_1 dominates any operator B_2 with respecting the operator C.

Proof. Applying Cayley–Hamilton theorem, we find

$$\operatorname{rank}\big(CB_1 \;\; CAB_1 \; \ldots \; CA^{n-1}B_1\big) = q \Longleftrightarrow \begin{pmatrix} (CB_1)^* \\ (CAB_1)^* \\ \ldots \\ (C(A)^{n-1}B_1)^* \end{pmatrix}_{(np,q)} y = 0;\ \forall y \in \mathbb{R}^q \Longrightarrow y = 0$$

$$\Longleftrightarrow \operatorname{Ker}\left[(CH_1^{\alpha})^*\right] = \{0\}.$$

So, $\operatorname{Ker}\left[(CH_1^{\alpha})^*\right] = \{0\}$, implies that $\operatorname{Ker}\left[(CH_1^{\alpha})^*\right] \subset \operatorname{Ker}\left[(CH_2^{\alpha})^*\right]$ as result, B_1 dominates B_2 with respecting C on $[0,\mathcal{T}]$. ■

Remark 2. 1. This condition can still be obtained $\operatorname{rank}\big(CB_1\, CAB_1 \; \ldots \; CA^{n-1}B_1\big) = q$ even if the system $\mathcal{S}_1$ is not controllable on $[0,\mathcal{T}]$. 2. B_1 can dominates any operator B_2 with respecting C on $[0,\mathcal{T}]$ without requiring $\operatorname{rank}\big(CB_1\, CAB_1 \; \ldots \; CA^{n-1}B_1\big) = q$.

For clarity, let us look at this example.

Example 1. 1. In the situation where $p = q = 1$ and $n = 2$

$$A = \begin{pmatrix} 1 & 3 \\ 0 & 0.5 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 1 \end{pmatrix}.$$

We have the controllability matrix as

$$\begin{pmatrix} B_1 & AB_1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

and its rank is $1 \neq 2$. As a result, our system is not controllable on $[0,\mathcal{T}]$. In contrast,

$$\begin{pmatrix} CB_1 & CAB_1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \end{pmatrix}$$

and its rank is $q = 1$, so B_1 dominates any operator B_2 with respecting C on $[0,\mathcal{T}]$.

2. When $q = n = 2$ and $p = 1$

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$
$$C = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \quad \omega = \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix},$$

we obtain

$$\mathscr{S}(\mathcal{T}-s) = \begin{pmatrix} \frac{(\mathcal{T}-s)^{\alpha-1}}{\Gamma(\alpha)} & \frac{(\mathcal{T}-s)^{2\alpha-1}}{\Gamma(2\alpha)} \\ 0 & \frac{(\mathcal{T}-s)^{\alpha-1}}{\Gamma(\alpha)} \end{pmatrix}.$$

We have

$$B_2^*\mathscr{S}(\mathcal{T}-s)^*C^*\omega = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{(\mathcal{T}-s)^{\alpha-1}}{\Gamma(\alpha)} & 0 \\ \frac{(\mathcal{T}-s)^{2\alpha-1}}{\Gamma(2\alpha)} & \frac{(\mathcal{T}-s)^{\alpha-1}}{\Gamma(\alpha)} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}$$
$$= \begin{pmatrix} 0 \\ \frac{(\mathcal{T}-s)^{\alpha-1}}{\Gamma(\alpha)}(\omega_1+\omega_2) \end{pmatrix}$$

as well as

$$B_1^*\mathscr{S}(\mathcal{T}-s)^*C^*\omega = 2\,\frac{(\mathcal{T}-s)^{\alpha-1}}{\Gamma(\alpha)}(\omega_1+\omega_2).$$

Then,

$$\|B_1^*\mathscr{S}(\mathcal{T}-s)^*C^*\omega\|^2_{L^2(0,\mathcal{T};R)} = 4\int_0^{\mathcal{T}} \frac{(\mathcal{T}-s)^{2(\alpha-1)}}{\Gamma(\alpha)^2}(\omega_1+\omega_2)^2\,ds$$

and

$$\|B_2^*\mathscr{S}(\mathcal{T}-s)^*C^*\omega\|^2_{L^2(0,\mathcal{T};\mathbb{R}^2)} = \int_0^{\mathcal{T}} \frac{(\mathcal{T}-s)^{2(\alpha-1)}}{\Gamma(\alpha)^2} \left\| \begin{pmatrix} 0 \\ \omega_1+\omega_2 \end{pmatrix} \right\|^2 ds$$

$$= \int_0^{\mathcal{T}} \frac{(\mathcal{T}-s)^{2(\alpha-1)}}{\Gamma(\alpha)^2} (\omega_1+\omega_2)^2 \, ds$$

therefore

$$\|B_2^* \mathscr{S}(\mathcal{T}-s)^* C^* \omega\|_{L^2(0,\mathcal{T};\mathbb{R}^2)} \leqslant \beta \, \|B_1^* \mathscr{S}(\mathcal{T}-s)^* C^* \omega\|_{L^2(0,\mathcal{T};R)} .$$

So, the condition (3) in Proposition 1 is verified with $\beta = 2$.

Resultantly, B_1 dominates B_2 with respecting C on $[0,\mathcal{T}]$, even if

$$\operatorname{rank}(CB_1 \;\; CAB_1) = \operatorname{rank}\begin{pmatrix} 2 & 0 \\ 2 & 0 \end{pmatrix} = 1 \neq 2.$$

3. Domination and controllability

Proposition 3. 1. If the system

$$(\mathcal{S}_1) \quad \begin{cases} {}_0^c D_\tau^\alpha z(\tau) = Az(\tau) + B_1 u(\tau), \quad 0 < \tau < \mathcal{T}, \quad 0 < \alpha < 1, \\ z(0) = z_0 \end{cases}$$

is controllable on $[0,\mathcal{T}]$, then B_1 dominates any operator B_2 with respecting C, on $[0,\mathcal{T}]$.

2. The reciproque is false.

Proof.

1. We assume that the linear control system $(\mathcal{S}_1)$ is controllable on $[0,\mathcal{T}] \Longleftrightarrow \operatorname{Im}(H_1^\alpha) = \mathbb{R}^n$.

So,

$$\operatorname{Im}(CH_1^\alpha) = \operatorname{Im}(C),$$

then

$$\operatorname{Im}(CH_2^\alpha) \subset \operatorname{Im}(CH_1^\alpha).$$

As result, B_1 dominates B_2 with respecting C, on $[0,\mathcal{T}]$.

2. **Counter example.** Considering the situation, when $n=2$, $p=1, q=2$

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

$$C = \begin{pmatrix} 1 & 0 \end{pmatrix}, \qquad \omega = \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}$$

we obtain

$$\mathscr{S}(\mathcal{T}-s) = \begin{pmatrix} \frac{(\mathcal{T}-s)^{\alpha-1}}{\Gamma(\alpha)} & \frac{(\mathcal{T}-s)^{2\alpha-1}}{\Gamma(2\alpha)} \\ 0 & \frac{(\mathcal{T}-s)^{\alpha-1}}{\Gamma(\alpha)} \end{pmatrix}.$$

We have

$$B_2^* \mathscr{S}(\mathcal{T}-s)^* C^* \omega = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{(\mathcal{T}-s)^{\alpha-1}}{\Gamma(\alpha)} & 0 \\ \frac{(\mathcal{T}-s)^{2\alpha-1}}{\Gamma(2\alpha)} & \frac{(\mathcal{T}-s)^{\alpha-1}}{\Gamma(\alpha)} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \omega$$

$$= \begin{pmatrix} \frac{(\mathcal{T}-s)^{\alpha-1}}{\Gamma(\alpha)} (\omega) \\ 0 \end{pmatrix}$$

and

$$B_1^* \mathscr{S}(\mathcal{T}-s)^* C^* \omega = \frac{(\mathcal{T}-s)^{\alpha-1}}{\Gamma(\alpha)} (\omega),$$

then

$$\|B_2^* \mathscr{S}^*(\mathcal{T}-s) C^* \omega\|_{L^2(0,\mathcal{T}\colon \mathbb{R}^2)} = \|B_1^* \mathscr{S}^*(\mathcal{T}-s) C^* \omega\|_{L^2(0,\mathcal{T}\colon \mathbb{R}^1)} . \tag{4}$$

Consequently, B_1 dominates B_2 on $[0,\mathcal{T}]$ with respecting C, even though

$$\operatorname{rank}\begin{pmatrix} B_1 & AB_1 \end{pmatrix} = \operatorname{rank}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = 1 \neq 2.$$

So, the controllability is not achievable for the system on $[0,\mathcal{T}]$. ∎

Proposition 4. B_1 dominates B_2 on $[0,\mathcal{T}]$, with respecting C if and only if

$$\operatorname{Im}\left(CB_2\ \ CAB_2\ \ldots\ CA^{n-1}B_2\right) \subset \operatorname{Im}\left(CB_1\ \ CAB_1\ \ldots\ CA^{n-1}B_1\right).$$

Proof. Using Proposition 1, B_1 dominates B_2 with respecting C on $[0,\mathcal{T}]$ if and only if,

$$\operatorname{Ker}\left[(CH_1^{\alpha})^*\right] \subset \operatorname{Ker}\left[(CH_2^{\alpha})^*\right].$$

Applying the Cayley–Hamilton theorem, we find for $i=\{1,2\}$

$$y \in \operatorname{Ker}\left[(CH_i^{\alpha})^*\right] \Longleftrightarrow \begin{pmatrix} (CB_i)^* \\ (CAB_i)^* \\ \ldots \\ (C(A)^{n-1}B_i)^* \end{pmatrix}_{(np,q)} y=0; \quad \forall y \in \mathbb{R}^q.$$

Then

$$\operatorname{Ker}\begin{pmatrix} (CB_i)^* \\ (CAB_i)^* \\ \ldots \\ (C(A)^{n-1}B_i)^* \end{pmatrix} = \operatorname{Ker}\left[(CH_i^{\alpha})^*\right].$$

Resultantly, B_1 dominates B_2 with respecting C on $[0,\mathcal{T}]$, if and only if

$$\operatorname{Im}\left(CB_2\ \ CAB_2\ \ldots\ CA^{n-1}B_2\right) \subset \operatorname{Im}\left(CB_1\ \ CAB_1\ \ldots\ CA^{n-1}B_1\right).$$ ■

Numerical example

Let $n=2$, $q=2$, $p=1$, and

$$A=\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad B_1=\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad C=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad B_2=\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

with the following u_2 term

$$u_2(\tau)=\begin{pmatrix} 0 \\ \Gamma\left(\frac{1}{4}\right)\tau \end{pmatrix}.$$

For $\alpha=\frac{1}{4}$, we find

$$\mathscr{S}(\mathcal{T}-\tau)=\begin{pmatrix} \frac{(\mathcal{T}-\tau)^{\frac{-3}{4}}}{\Gamma\left(\frac{1}{4}\right)} & 0 \\ \frac{(\mathcal{T}-\tau)^{\frac{-1}{2}}}{\Gamma\left(\frac{1}{2}\right)} & \frac{(\mathcal{T}-\tau)^{\frac{-3}{4}}}{\Gamma\left(\frac{1}{4}\right)} \end{pmatrix}.$$

Now, let us define u_1 by:

$$u_1(\tau)=B_1^*\mathscr{S}(\mathcal{T}-\tau)^*C^*(\mathcal{T}-\tau)^{2(1-\alpha)}\bar{\mathcal{D}}^{\alpha}(\mathcal{T})^{-1}\left(-CH_2^{\alpha}u_2\right),$$

it is clear that the control verify very well the equation (2) in Proposition 1.

So, we get

$$u_1(\tau)=96\,\Gamma\left(\tfrac{1}{2}\right)(\mathcal{T}-\tau)^{\frac{3}{4}}-120\,\Gamma\left(\tfrac{1}{2}\right)\mathcal{T}^{\frac{-1}{4}}(\mathcal{T}-\tau).$$

We suppose, for the sake of simplicity, that $z_0=0$, then $y_{(0,0)}=0$, and

$$y_{u_1,u_2}(\tau)=\begin{pmatrix} \frac{96\,\Gamma\left(\frac{1}{2}\right)(\mathcal{T}-\tau)^{\frac{3}{4}}\tau}{\Gamma\left(\frac{1}{4}\right)}-\frac{480\,\Gamma\left(\frac{1}{2}\right)\mathcal{T}^{\frac{-1}{4}}\tau^{\frac{5}{4}}}{5\Gamma\left(\frac{1}{4}\right)}. \\ \frac{384}{5}\mathcal{T}^{-1}\tau^{\frac{5}{4}}-\frac{240}{3}\mathcal{T}^{\frac{-1}{4}}\tau^{\frac{3}{2}}+\frac{16}{5}\tau^{\frac{5}{4}} \end{pmatrix}, \quad y_{0,u_2}(\tau)=\begin{pmatrix} 0 \\ \frac{16}{5}\tau^{\frac{5}{4}} \end{pmatrix}.$$

The values of the gamma function evaluted at $\frac{1}{4}$ and $\frac{1}{2}$ are respectivly:

$$\Gamma\left(\tfrac{1}{4}\right)\approx 3.62560990822, \quad \Gamma\left(\tfrac{1}{2}\right)=\sqrt{\pi}.$$

After fixing $\mathcal{T}=12$ we get the following numerical simulation which perform the previous developments.

Figure 1 shows that the control u_1 ensure the compensation of the effect of term B_2u_2, by returning the observation at the final time $\mathcal{T}$ to its regular situation which is $y_{0,0}=0$.

Figures 2 and 3 display respectively, the observation y_{0,u_2} and the evolution of the control term u_1.

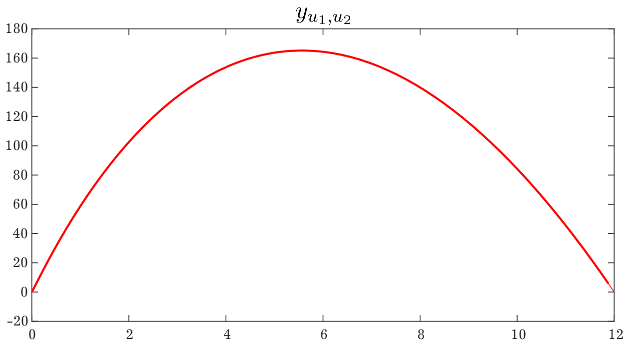

Fig. 1. Representation of y_{u_1,u_2}.

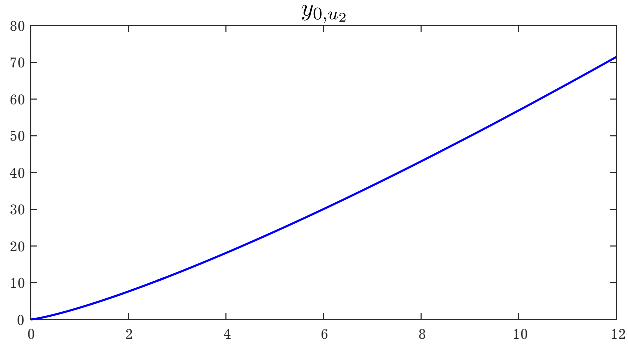

Fig. 2. Representation of y_{0,u_2}.

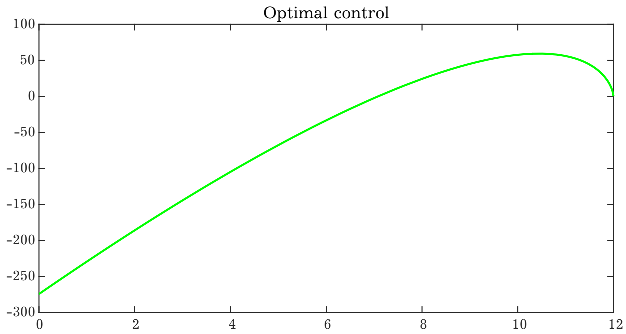

Fig. 3. Representation of the dominant control u_1.

4. Conclusion

Our paper discusses the classification of distributed parameter systems through the utilization of the domination concept. In this research, we extend this concept to include fractional linear-order dynamical systems where the Caputo fractional derivative is considered. Characterization results and main properties are presented, and not only the sufficient condition that ensures such domination was proved but also the sufficient and necessary conditions were discussed. Finally, we give some examples to illustrate our results.

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Поняття домінування в лінійних розподілених системах дробового порядку

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У цій статті досліджується поняття домінування у лінійних розподілених системах дробового порядку в скінченновимірному стані. Мета полягає в тому, щоб порівняти або класифікувати вхідні оператори відносно вихідних, і подати характеристики та властивості цієї концепції. Потім досліджено зв'язок між керованістю та поняттям домінування. Наведено числовий приклад, щоб проілюструвати отримані результати.

Ключові слова: *дробовий порядок; розподілені системи; керованість; домінування.*