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A generalized diffusive IS-LM business cycle model with delays in gross product and capital stock

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In this paper, we suggest a diffusive and delayed IS-LM business cycle model with interest rate, general investment and money supply under homogeneous Neumann boundary conditions. The time delays are respectively incorporated into capital stock and gross product. We first demonstrate the model's sound mathematical and economic posing. By examining the corresponding characteristic equation, the local stability of the economic equilibrium and the existence of Hopf bifurcation are proved.

Keywords: *economics; diffusion; stability; time delays; Hopf bifurcation.*

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1. Introduction

In recent years, partial differential equations (PDEs) play an essential role to understand the dynamics of business cycles. In 2023, Elkarmouchi et al. [1] developed an IS-LM model that explains the spatiotemporal dynamics of business cycle under the services and goods market as well as the money market by taking the money supply as a constant. Such model includes numerous economic models that are known to exist in the literature like [2–4].

In general, we can say that the are two schools of economics that determine the nature of money supply. The first one is Chicago school that considers the money supply as an exogenous variable for the reason that the monetary authority can regulate and observe it, which is taken into account in some recent works [4–6]. The second one is Keynesian school views it as an endogenous variable because the monetary expansion as a result of different economic or non-economic factors, this idea is presented in some recent studies like [7–9].

In this paper by incorporating endogenous money supply, we suggest a diffusive IS-LM model with a two-delay capital accumulation equation. The first delay refers to the time delay between the decision to invest and when it is implemented, whereas the second one represents the period of time it takes for an investment to be produced. The system of nonlinear PDEs or reaction-diffusion equations below provides the model.

$$\begin{cases} \dfrac{\partial Y}{\partial t} = d_1 \Delta Y(t,x) + \alpha \left[I(Y(t,x), K(t,x), R(t,x)) + G(Y(t,x)) - T(Y(t,x)) - S(Y^D) \right], \\ \dfrac{\partial K}{\partial t} = d_2 \Delta K(t,x) + I(Y(t-\tau_1,x), K(t-\tau_2,x), R(t,x)) - \delta K(t,x), \\ \dfrac{\partial R}{\partial t} = d_3 \Delta R(t,x) + \beta [L(Y(t,x), R(t,x)) - M(t,x)], \\ \dfrac{\partial M}{\partial t} = d_4 \Delta M(t,x) + \psi M(t,x) [G(Y(t,x)) - T(Y(t,x))], \end{cases} \tag{1}$$

where the economic variables, $Y(t,x)$, $K(t,x)$, $R(t,x)$ and $M(t,x)$ indicate the gross product, the capital stock, the interest rate and the money supply at location x and time t, respectively. The diffusion coefficients of Y, K, R and M are d_1, d_2, d_3 and d_4, respectively, and Δ means the Laplacian operator. The demand for money or liquidity preference function is marked by $L(Y,R)$ while the investment is noted by $I(Y,K,R)$. The parameter α presents the adjustment coefficient in the goods market while β indicates the coefficient of adjustment in the money market. $G(Y)$, $T(Y)$ and $Y^D = Y - T(Y)$ refer to the Government expenditure, tax income and discretionary income, respectively. Finally, δ is depreciation rate of the capital stock and ψ is the adjustment coefficient in public funds. Additionally, we take into account the model (1) with initial conditions:

$$Y(t,x) = \Phi_1(t,x),\ K(t,x) = \Phi_2(t,x),\ R(t,x) = \Phi_3(t,x),\ M(t,x) = \Phi_4(t,x),\ (t,x) \in [-\tau,0]\times\overline{\Omega}, \tag{2}$$

and Neumann boundary conditions:

$$\frac{\partial Y}{\partial \nu} = \frac{\partial K}{\partial \nu} = \frac{\partial R}{\partial \nu} = \frac{\partial M}{\partial \nu} = 0, \quad \text{on} \quad (0,+\infty)\times\partial\Omega, \tag{3}$$

in which $\tau = \max\{\tau_1,\tau_2\}$, $\frac{\partial}{\partial\nu}$ means the outward normal derivative on the smooth boundary $\partial\Omega$ and Ω indicates the market capacity.

The remainder of the document is structured as below. In Section 2, we give some preliminary findings including the existence and the uniqueness of solutions of our model (1) also the existence of economic equilibrium. The characteristic equation of this system is examined in Section 3 through the construction a basis of phase space based on the eigenvectors of the Laplace operator. Local stability is proved, as well as the existence of Hopf bifurcation. The last section offers a brief conclusion.

2. Preliminary results

As in [4,6,8], before realizing the remainder, the following specific functions will be utilized:

• Savings S:

$$S(Y^D) = s\,Y^D,$$

where $0 < s < 1$;

• Government expenditure G:

$$G(Y) = G_0 + \frac{\gamma_0}{g_1 Y + g_2},$$

where G_0 represents positive autonomous public expenditure and γ_0, g_1, g_2 are positive parameters regulating G is susceptibility to changes in Y;

• Tax income T:

$$T(Y) = \delta_1 Y,$$

where δ_1 $(0 < \delta_1 < 1)$ is tax rate;

• Liquidity preference L:

$$L(Y,R) = \mathcal{L}(Y) - \gamma R,$$

in which γ represents the variation of demand of liquidity in relation to interest rate.

Let $\mathbb{X} = C\left(\overline{\Omega},\mathbb{R}^4\right)$ be the Banach space of continuous functions from $\overline{\Omega}$ into $\mathbb{R}^4$ and $\mathcal{C} = C\left([-\tau,0],\mathbb{X}\right)$ be the Banach space of continuous functions of $[-\tau,0]$ into $\mathbb{X}$ with standard uniform topology. For the sake of simplicity, we denote an element $\varphi \in \mathcal{C}$ and define it as a function from $[-\tau,0]\times\overline{\Omega}$ into $\mathbb{R}^4$ defined by $\varphi(s,x) = \varphi(s)(x)$. For any continuous function $\omega(\cdot)\colon [-\tau,b) \to \mathbb{X}$ for $b > 0$, we set $\omega_t \in \mathcal{C}$ by $\omega_t(s) = \omega(t+s)$ for $s \in [-\tau,0]$.

Theorem 1. *For each given initial* $\Phi = (\Phi_1,\Phi_2,\Phi_3,\Phi_4)^T \in \mathcal{C}$, *there exists a unique solution of problem (1)–(3) defined on* $[0,+\infty)$.

Proof. For each $\varphi = (\varphi_1,\varphi_2,\varphi_3,\varphi_4)^T \in \mathcal{C}$ and $x \in \overline{\Omega}$, we define $\tilde{G} = (\tilde{G}_1,\tilde{G}_2,\tilde{G}_3,\tilde{G}_4)\colon \mathcal{C} \to \mathbb{X}$ by

$$\begin{aligned}
&\tilde{G}_1(\varphi)(x) = \alpha\left[I(\varphi_1(0,x),\varphi_2(0,x),\varphi_3(0,x)) + G_0 + \frac{\gamma_0}{g_1\varphi_1(0,x)+g_2} + (s(\delta_1-1)-\delta_1)\varphi_1(0,x)\right],\\
&\tilde{G}_2(\varphi)(x) = I(\varphi_1(-\tau_1,x),\varphi_2(-\tau_2,x),\varphi_3(0,x)) - \delta\varphi_2(0,x),\\
&\tilde{G}_3(\varphi)(x) = \beta[\mathcal{L}(\varphi_1(0,x)) - \gamma\varphi_3(0,x) - \varphi_4(0,x)],
\end{aligned}$$

$$\tilde{G}_4(\varphi)(x) = \psi\varphi_4(0,x)\left[G_0 + \frac{\gamma_0}{g_1\varphi_1(0,x)+g_2} - \delta_1\varphi_1(0,x)\right].$$

The following abstract functional differential equation can therefore be used to rewrite the problem (1)–(3),

$$\begin{cases} n'(t) = Fn(t) + \tilde{G}(n_t), \ t > 0 \\ n(0) = \Phi \in \mathcal{C}, \end{cases} \tag{4}$$

where $n = (Y, K, R, M)^T$ and $Fn = (d_1\Delta Y, d_2\Delta K, d_3\Delta R, d_4\Delta M)^T$. It is apparent that $\tilde{G}$ is locally Lipschitz in $\mathcal{C}$, and as in [10], we draw the conclusion that the problem (4) has a unique local solution on $[0, T_{\max})$, where $T_{\max}$ denotes the maximal existence time for solution of system (4). ■

We need the following hypothesis in order to investigate whether the model's economic equilibrium (1) exists:

$$(H_1) \qquad I\left(\frac{G_0g_1 - \delta_1 g_2 + \sqrt{(\delta_1 g_2 + G_0 g_1)^2 + 4\delta_1 g_1\gamma_0}}{2\delta_1 g_1}, \frac{sY^D}{\delta}, 0\right) - S\big(Y^D\big) > 0.$$

Theorem 2. *If (H_1) holds, then (1) has a unique economic equilibrium defined by $E^*\big(Y^*, K^*, R^*, M^*\big)$, in which $Y^* = \frac{G_0g_1 - \delta_1 g_2 + \sqrt{(\delta_1 g_2 + G_0 g_1)^2 + 4\delta_1 g_1\gamma_0}}{2\delta_1 g_1}$, $K^* = \frac{s(1-\delta_1)Y^*}{\delta}$, R^* is the positive solution of the subsequent equation*

$$I\left(\frac{G_0g_1 - \delta_1 g_2 + \sqrt{(\delta_1 g_2 + G_0 g_1)^2 + 4\delta_1 g_1\gamma_0}}{2\delta_1 g_1}, \frac{s(1-\delta_1)Y}{\delta}, R\right) - s(1-\delta_1)Y = 0,$$

and $M^* = \mathcal{L}(Y^*) - \bar{\gamma}R^*$.

Proof. For the aforementioned system with certain functional forms, the unique equilibrium point is represented by the solution of the subsequent equations:

$$I(Y, K, R) = S(Y^D), \tag{5}$$

$$I(Y, K, R) = \delta K, \tag{6}$$

$$\mathcal{L}(Y) - \bar{\gamma}R - M = 0, \tag{7}$$

$$G(Y) = T(Y). \tag{8}$$

From (8),

$$Y = \frac{G_0g_1 - \delta_1 g_2 + \sqrt{(\delta_1 g_2 + G_0 g_1)^2 + 4\delta_1 g_1\gamma_0}}{2\delta_1 g_1}, \tag{9}$$

and from (5)–(6) we get

$$K = \frac{s(1-\delta_1)Y}{\delta}. \tag{10}$$

By replacing (9) and (10) into the first equation (5), one can get

$$I\left(\frac{G_0g_1 - \delta_1 g_2 + \sqrt{(\delta_1 g_2 + G_0 g_1)^2 + 4\delta_1 g_1\gamma_0}}{2\delta_1 g_1}, \frac{s(1-\delta_1)Y}{\delta}, R\right) - s(1-\delta_1)Y = 0.$$

Let Q be the function defined on the interval $[0, +\infty)$ by

$$Q(R) = I\left(\frac{G_0g_1 - \delta_1 g_2 + \sqrt{(\delta_1 g_2 + G_0 g_1)^2 + 4\delta_1 g_1\gamma_0}}{2\delta_1 g_1}, \frac{s(1-\delta_1)Y}{\delta}, R\right) - s(1-\delta_1)Y.$$

From (H_1), we obtain $Q(0) > 0$, $\lim\limits_{R\to+\infty} Q(R) = -\infty$ and $Q'(R) = \frac{\partial I}{\partial R} < 0$.

As a consequence, there exists a unique $R^* \in (0, +\infty)$ such that R^* is the positive solution of the equation $Q(R) = 0$. We get by using (7) $M^* = \mathcal{L}(Y^*) - \bar{\gamma}R^*$. This completes the proof. ■

3. Local stability and Hopf bifurcation analysis

Study the condition of local stability and Hopf bifurcation is what this part is all about. Let $y = Y - Y^*$, $k = K - K^*$, $r = R - R^*$ and $m = M - M^*$. One can obtain the linear part of (1) as follows

$$\begin{cases} \dfrac{\partial y}{\partial t} = d_1 \Delta y(t,x) + \alpha \left[\left(a - \dfrac{\gamma_0 g_1}{(g_1 Y^* + g_2)^2} + (s(\delta_1 - 1) - \delta_1) \right) y(t,x) + bk(t,x) + cr(t,x) \right], \\ \dfrac{\partial k}{\partial t} = d_2 \Delta k(t,x) + ay(t-\tau_1, x) + bk(t-\tau_2, x) - \delta k(t,x) + cr(t,x), \\ \dfrac{\partial r}{\partial t} = d_3 \Delta r(t,x) + \beta[l_1 y(t,x) - \gamma r(t,x) - m(t,x)], \\ \dfrac{\partial m}{\partial t} = d_4 \Delta m(t,x) + \psi M^* \left[\dfrac{\gamma_0 g_1}{(g_1 Y^* + g_2)^2} - \delta_1 \right] y(t,x), \\ \dfrac{\partial y}{\partial \nu} = \dfrac{\partial k}{\partial \nu} = \dfrac{\partial r}{\partial \nu} = \dfrac{\partial m}{\partial \nu} = 0, \quad x \in \partial\Omega, \quad t > 0, \end{cases} \tag{11}$$

where $a = \frac{\partial I}{\partial Y}(Y^*, K^*, R^*)$, $b = \frac{\partial I}{\partial K}(Y^*, K^*, R^*)$, $c = \frac{\partial I}{\partial R}(Y^*, K^*, R^*)$ and $l_1 = \mathcal{L}'(Y^*)$.

Letting $\zeta = C([-\tau, 0], X)$ represent the Banach space of continuous functions of $[-\tau, 0]$ into X, where X is defined by

$$X = \left\{ y, k, r, m \in W^{2,2}(\Omega) \colon \frac{\partial y(t,x)}{\partial \nu} = \frac{\partial k(t,x)}{\partial \nu} = \frac{\partial r(t,x)}{\partial \nu} = \frac{\partial m(t,x)}{\partial \nu} = 0,\ x \in \partial\Omega \right\},$$

with the inner product $\langle \cdot, \cdot \rangle$. Therefore, system (11) can be recast as an abstract differential equation in the phase space ζ as below,

$$W'(t) = D\Delta W + L(W_t), \tag{12}$$

where $W = (y, k, r, m)^T$, $D = \operatorname{diag}(d_1, d_2, d_3, d_4)$ and $L \colon \zeta \to X$ defined by

$$L(\phi) = \mathcal{A}_0 \phi(0) + \mathcal{A}_1 \phi(-\tau_1) + \mathcal{A}_2 \phi(-\tau_2), \tag{13}$$

with

$$\mathcal{A}_0 = \begin{pmatrix} \alpha \left(a - \frac{\gamma_0 g_1}{(g_1 Y^* + g_2)^2} + s(\delta_1 - 1) - \delta_1 \right) & \alpha b & \alpha c & 0 \\ 0 & \delta & c & 0 \\ \beta l_1 & 0 & -\beta\gamma & -\beta \\ \psi M^* \left[-\frac{\gamma_0 g_1}{(g_1 Y^* + g_2)^2} - \delta_1 \right] & 0 & 0 & 0 \end{pmatrix}, \quad \mathcal{A}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$\mathcal{A}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The characteristic of (12) is as below

$$\lambda y - D\Delta y - L\big(e^{\lambda} y\big) = 0, \quad y \in \operatorname{dom}(\Delta) \backslash \{0\}. \tag{14}$$

Let the eigenvalue of the operator Δ under the Neumann boundary conditions on X be $-k^2$ ($k \in \mathbb{N}$), then the corresponding eigenvectors have the subsequent shape:

$$\beta_k^1 = \begin{pmatrix} \gamma_k \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \beta_k^2 = \begin{pmatrix} 0 \\ \gamma_k \\ 0 \\ 0 \end{pmatrix}, \quad \beta_k^3 = \begin{pmatrix} 0 \\ 0 \\ \gamma_k \\ 0 \end{pmatrix}, \quad \beta_k^3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \gamma_k \end{pmatrix}, \quad \gamma_k = \cos(kx), \quad k \in \mathbb{N}$$

and a basis for the phase space X is constructed by $\{\beta_k^1, \beta_k^2, \beta_k^3, \beta_k^4\}_{k=0}^{+\infty}$. The phase space X can therefore be expanded in the form of Fourier, as shown below

$$y = \sum_{k=0}^{\infty} Y_k^T \begin{pmatrix} \beta_k^1 \\ \beta_k^2 \\ \beta_k^3 \\ \beta_k^4 \end{pmatrix}, \quad Y_k = \begin{pmatrix} \langle y, \beta_k^1 \rangle \\ \langle y, \beta_k^2 \rangle \\ \langle y, \beta_k^3 \rangle \\ \langle y, \beta_k^4 \rangle \end{pmatrix}. \tag{15}$$

Then by calculation, we obtain

$$L\left(\phi^T\begin{pmatrix}\beta_k^1\\ \beta_k^2\\ \beta_k^3\\ \beta_k^4\end{pmatrix}\right)=L(\phi)^T\begin{pmatrix}\beta_k^1\\ \beta_k^2\\ \beta_k^3\\ \beta_k^4\end{pmatrix},\quad k\in\mathbb{N}. \tag{16}$$

Substituting (16) and (15) into (14),

$$\sum_{k=0}^{\infty}Y_k^T\left[\left(\lambda I_4+\mathcal{D}k^2\right)-A\right]\cdot\begin{pmatrix}\beta_k^1\\ \beta_k^2\\ \beta_k^3\\ \beta_k^4\end{pmatrix}=0, \tag{17}$$

$$A=\begin{pmatrix}\alpha\big(a-\frac{\gamma_0g_1}{(g_1Y^*+g_2)^2}+s(\delta_1-1)-\delta_1\big) & \alpha b & \alpha c & 0\\ a\,e^{-\lambda\tau_1} & -\delta+b\,e^{-\lambda\tau_2} & c & 0\\ \beta l_1 & 0 & -\beta\gamma & -\beta\\ \psi M^*\big[-\frac{\gamma_0g_1}{(g_1Y^*+g_2)^2}-\delta_1\big] & 0 & 0 & 0\end{pmatrix}.$$

The following is the characteristic equation of (17)

$$\lambda^4+\lambda^3g_{3,k}+\lambda^2g_{2,k}+\lambda g_{1,k}+g_{0,k}+\left[h_3\lambda^3+h_{2,k}\lambda^2+h_{1,k}\lambda+h_{0,k}\right]e^{-\lambda\tau_2}+\left[r_2\lambda^2+r_{1,k}\lambda+r_{0,k}\right]e^{-\lambda\tau_1}=0, \tag{18}$$

where

$$\begin{aligned}
g_{3,k}&=d_1k^2-\alpha\left(a-\frac{\gamma_0g_1}{(g_1Y^*+g_2)^2}+s(\delta_1-1)-\delta_1\right)+d_2k^2+\delta+d_3k^2+\beta\gamma+d_4k^2,\\
g_{2,k}&=-\beta l_1\alpha c+(d_2k^2+\delta)(d_3k^2+\beta\gamma+d_4k^2)+d_4k^2(d_3k^2+\beta\gamma)\\
&\quad+\left[d_2k^2+\delta+d_3k^2+\beta\gamma+d_4k^2\right]\left[d_1k^2-\alpha\left(a-\frac{\gamma_0g_1}{(g_1Y^*+g_2)^2}+s(\delta_1-1)-\delta_1\right)\right],\\
g_{1,k}&=-\beta l_1\alpha c(b+d_2k^2+\delta+d_4k^2)-\beta\psi M^*\alpha c\left[\frac{\gamma_0g_1}{(g_1Y^*+g_2)^2}+\delta_1\right]+d_4k^2\\
&\quad\left[d_1k^2-\alpha\left(a-\frac{\gamma_0g_1}{(g_1Y^*+g_2)^2}+(s(\delta_1-1)-\delta_1)\right)\right](d_3k^2+\beta\gamma)+(d_2k^2+\delta)\\
&\quad(d_4k^2+d_3k^3+\beta\gamma)\left[d_1k^2-\alpha\left(a-\frac{\gamma_0g_1}{(g_1Y^*+g_2)^2}+s(\delta_1-1)-\delta_1\right)\right],\\
g_{0,k}&=-d_4k^2\beta l_1\alpha bc-\beta l_1\alpha cd_4k^2(d_2k^2+\delta)-b\beta\psi M^*\alpha c\left[\frac{\gamma_0g_1}{(g_1Y^*+g_2)^2}+\delta_1\right]\\
&\quad+(d_3k^2+\beta\gamma)(d_2k^2+\delta)\left[d_1k^2-\alpha\left(a-\frac{\gamma_0g_1}{(g_1Y^*+g_2)^2}+s(\delta_1-1)-\delta_1\right)\right]\\
&\quad d_4k^2-\beta\psi M^*\alpha c\left[\frac{\gamma_0g_1}{(g_1Y^*+g_2)^2}+\delta_1\right](d_2k^2+\delta),\\
h_3&=-b,\\
h_{2,k}&=-\left[d_1k^2-\alpha\left(a-\frac{\gamma_0g_1}{(g_1Y^*+g_2)^2}+s(\delta_1-1)-\delta_1\right)+d_3k^2+\beta\gamma+d_4k^2\right]b,\\
h_{1,k}&=\left[\beta l_1\alpha c-(d_3k^2+\beta\gamma)\left[d_1k^2-\alpha\left(a-\frac{\gamma_0g_1}{(g_1Y^*+g_2)^2}+s(\delta_1-1)-\delta_1\right)\right]\right]b\\
&\quad-\left[d_4k^2\left(d_1k^2-\alpha\left(a-\frac{\gamma_0g_1}{(g_1Y^*+g_2)^2}+s(\delta_1-1)-\delta_1\right)+d_3k^2+\beta\gamma\right)\right]b,\\
h_{0,k}&=-bd_4k^2(d_3k^2+\beta\gamma)\left[d_1k^2-\alpha\left(a-\frac{\gamma_0g_1}{(g_1Y^*+g_2)^2}+s(\delta_1-1)-\delta_1\right)\right]
\end{aligned}$$

$$+\left[\beta l_1\alpha c d_4 k^2+\beta\psi M^*\alpha c\left(\frac{\gamma_0 g_1}{(g_1Y^*+g_2)^2}+\delta_1\right)\right]b,$$
$$r_2=-ab,$$
$$r_{1,k}=-ab\big(d_3k^2+\beta\gamma+d_4k^2\big),$$
$$r_{0,k}=-abd_4k^2\big(d_3k^2+\beta\gamma\big).$$

$\lambda=0$ is not a solution to Eq. (18) for $k\in\mathbb{N}$.

When $\tau_1=\tau_2=0$, the equation (18) is

$$\lambda^4+\lambda^3(g_{3,k}+h_3)+\lambda^2(g_{2,k}+r_2+h_{2,k})+\lambda(g_{1,k}+r_1+h_{1,k})+g_{0,k}+r_0+h_{0,k}=0. \tag{19}$$

If $a<\frac{\gamma_0 g_1}{(g_1Y^*+g_2)^2}-s(\delta_1-1)+\delta_1$, then the coefficients of equation (19) satisfy:

$$g_{3,k}+h_3>0,\quad g_{0,k}+r_0+h_{0,k}>0,\quad (g_{3,k}+h_3)(g_{2,k}+r_2+h_{2,k})>(g_{1,k}+r_1+h_{1,k}),$$
$$(g_{3,k}+h_3)(g_{2,k}+r_2+h_{2,k})(g_{1,k}+r_1+h_{1,k})>(g_{3,k}+h_3)^2(g_{0,k}+r_0+h_{0,k})+(g_{1,k}+r_1+h_{1,k})^2.$$

According to the Routh–Hurwitz criterion, we find that the economic equilibrium E^* of system (1) is locally stable when $\tau_1=\tau_2=0$.

3.1. Case 1: $\tau_2=0$, $\tau_1\neq 0$

Now, when $\tau_2=0$ and $\tau_1\neq 0$ Eq. (18) corresponds to the subsequent equation:

$$\lambda^4+\lambda^3(h_3+g_{3,k})+\lambda^2(h_{2,k}+g_{2,k})+\lambda(h_{1,k}+g_{1,k})+g_{0,k}+h_{0,k}+\big[r_2\lambda^2+r_{1,k}\lambda+r_{0,k}\big]e^{-\lambda\tau_1}=0. \tag{20}$$

Suppose that Eq. (20) has a pair of conjugate purely imaginary roots $B\pm i\omega(\omega>0)$. Substituting $\lambda=i\omega(\omega>0)$ into (20) and separating the imaginary and real parts, it is evident that

$$\begin{cases}\omega^4-\omega^2(g_{2,k}+h_{2,k})+g_{0,k}+h_{0,k}=(r_2\omega^2-r_{0,k})\cos(\omega\tau_1)-\omega r_{1,k}\sin(\omega\tau_1),\\ \omega^3(g_{3,k}+h_{3,k})-\omega(g_{1,k}+h_{1,k})=(r_2\omega^2-r_{0,k})\sin(\omega\tau_1)+\omega r_{1,k}\cos(\omega\tau_1).\end{cases} \tag{21}$$

Therefore, the existence of purely imaginary roots of Eq. (20) is identical to the existence of solutions of Eq. (21). Define

$$E(\omega)=(r_2\omega^2-r_{0,k})^2+r_{1,k}^2\omega^2. \tag{22}$$

If $E\neq 0$, then by Eq. (21), we get

$$\sin(\omega\tau_1)=\frac{(r_2\omega^2-r_{0,k})M(\omega)-r_{1,k}\omega[\omega^4-\omega^2(g_{2,k}+h_{2,k})+g_{0,k}+h_{0,k}]}{E(\omega)},$$
$$\cos(\omega\tau_1)=\frac{\omega r_{1,k}M(\omega)+(r_2\omega^2-r_{0,k})[\omega^4-\omega^2(g_{2,k}+h_{2,k})+g_{0,k}+h_{0,k}]}{E(\omega)}, \tag{23}$$

where

$$M(\omega)=\omega^3(g_{3,k}+h_{3,k})-\omega(g_{1,k}+h_{1,k}).$$

Two equations of (23) are squared and added, and it results that

$$E^2(\omega)=\big[(r_2\omega^2-r_{0,k})M(\omega)-r_{1,k}\omega[\omega^4-\omega^2(g_{2,k}+h_{2,k})+g_{0,k}+h_{0,k}]\big]^2$$
$$+\big[\omega r_{1,k}M(\omega)+(r_2\omega^2-r_{0,k})[\omega^4-\omega^2(g_{2,k}+h_{2,k})+g_{0,k}+h_{0,k}]\big]^2.$$

Denote

$$W(\omega)=E^2(\omega)-\big[(r_2\omega^2-r_{0,k})M(\omega)-r_{1,k}\omega\left[\omega^4-\omega^2(g_{2,k}+h_{2,k})+g_{0,k}+h_{0,k}\right]\big]^2$$
$$-\big[\omega r_{1,k}M(\omega)+(r_2\omega^2-r_{0,k})\left[\omega^4-\omega^2(g_{2,k}+h_{2,k})+g_{0,k}+h_{0,k}\right]\big]^2. \tag{24}$$

By calculations, we get from (22) and (24) that

$$W(\omega)=f_{12}\omega^{12}+f_{10}\omega^{10}+f_8\omega^8+f_6\omega^6+f_4\omega^4+f_2\omega^2+f, \tag{25}$$

where

$$f_{12}=-r_2^2,$$
$$f_{10}=2r_2^2(g_{2,k}+h_{2,k})+r_2^2(h_{3,k}+g_{3,k})^2-r_{1,k}^2,$$

$$\begin{aligned}
f_8 &= r_2^4 + 2r_2^2(g_{3,k}+h_{3,k})(h_{1,k}+g_{1,k}) + [2r_2r_{0,k}-r_{1,k}](g_{3,k}+h_{3,k})^2 - r_{0,k}^2 - (g_{2,k}+h_{2,k})^2r_2^2\\
&\quad + [2r_{1,k}^2 - 2r_{0,k}r_2](g_{2,k}+h_{2,k}) - 2(h_{0,k}+g_{0,k})r_2^2,\\
f_6 &= [2r_{1,k}-4r_2r_{0,k}](g_{3,k}+h_{3,k})(h_{1,k}+g_{1,k}) + 2(g_{2,k}+h_{2,k})(h_{0,k}+g_{0,k})r_2^2\\
&\quad + [2r_{0,k}r_2 - 2r_{1,k}^2](h_{0,k}+g_{0,k}) + [2r_{0,k}r_2 - r_{1,k}^2](g_{2,k}+h_{2,k})^2 + (g_{3,k}+h_{3,k})^2\\
&\quad r_{0,k}^2 + 2r_{0,k}^2(g_{2,k}+h_{2,k}) - r_2^2(g_{1,k}+h_{1,k})^2,\\
f_4 &= r_{1,k}^4 - 4r_2^2r_{0,k}^2 + 2r_2r_{0,k}(h_{1,k}+g_{1,k})^2 - 2r_{0,k}(h_{1,k}+g_{1,k})(g_{3,k}+h_{3,k})\\
&\quad [2r_{1,k}^2 - 4r_{0,k}r_2](h_{0,k}+g_{0,k})(g_{2,k}+h_{2,k}) - r_{1,k}(h_{1,k}+g_{1,k})^2 - (g_{0,k}+h_{0,k})^2\\
&\quad r_2^2 - r_{0,k}^2(g_{2,k}+h_{2,k})^2 - (h_{0,k}+g_{0,k})r_{0,k}\\
f_2 &= r_{0,k}(h_{1,k}+g_{1,k})^2 + [r_{1,k}^2 + 2r_{0,k}r_2](h_{0,k}+g_{0,k})^2 + 2(h_{0,k}+g_{0,k})(g_{2,k}+h_{2,k}),\\
f &= r_{0,k}^4 - (h_{0,k}+g_{0,k})^2r_{0,k}^2.
\end{aligned}$$

Let $z=\omega^2$, Eq. (24) can be expressed as

$$h(z) := f_{12}z^6 + f_{10}z^5 + f_8z^4 + f_6z^3 + f_4z^2 + f_2z + f = 0. \tag{26}$$

Clearly, if $a < \frac{\gamma_0 g_1}{(g_1Y^*+g_2)^2} - s(\delta_1-1)+\delta_1$ and $h(z)=0$ has no positive roots, then the economic equilibrium E^* of system (1) is locally asymptotically stable for all $\tau_1 \geqslant 0$. Otherwise, for a certain $k_0 \in \mathbb{N}$, if Eq. (26) has positive roots, without losing the ability to generalize, we suppose that Eq. (26) with $k=k_0$ has six positive roots, namely $z_n (n=1,\ldots,6)$. Consequently, Eq. (24) has six positive roots $\omega_n = \sqrt{z_n}$ $(n=1,\ldots,6)$.

For $n=1,\ldots,6$, one can extract the matching $\tau_{1,n}^j > 0$ from (21) such that (20) has a pair of purely imaginary roots $\pm\mathrm{i}\,\omega_n$ offered by

$$\tau_{1,n}^j = \frac{1}{\omega_n}\arccos\frac{\omega_n r_{1,k}M(\omega_n) + \left(r_2\omega_n^2 - r_{0,k}\right)\left[\omega_n^4 - \omega_n^2(g_{2,k}+h_{2,k}) + g_{0,k}+h_{0,k}\right]}{E(\omega_n)} + \frac{2\pi j}{\omega_n},\ j\in\mathbb{N}. \tag{27}$$

Let the root of Eq. (20) be $\lambda(\tau_1)=\xi(\tau_1)+\mathrm{i}\,\omega(\tau_1)$ satisfying $\xi(\tau_{1,n}^j)=0$, $\omega(\tau_{1,n}^j)=\omega_n$. We now investigate the existence of Hopf bifurcation [11]. To this end, differentiating two sides of (20) with respect to τ_1. Thus, it follows

$$\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau_1}\right)^{-1} = \frac{4\lambda^3 + 3\lambda^2(g_{3,k}+h_3) + 2\lambda(g_{2,k}+h_{2,k}) + (g_{1,k}+h_{1,k})}{\lambda(r_2\lambda^2 + r_{1,k}\lambda + r_{0,k})e^{-\lambda\tau_1}} + \frac{2r_2\lambda + r_{1,k}}{\lambda(r_2\lambda^2 + r_{1,k}\lambda + r_{0,k})} - \frac{\tau_1}{\lambda}.$$

Denote

$$\tau_{1,0}^* = \tau_{1,n_0}^{(0)} = \min_{n\in\{1,\ldots,6\}}\left\{\tau_{1,n}^{(0)}\right\}, \quad \omega_0^* = \omega_{n_0}.$$

By a simple computation, one get that

$$\operatorname{sign}\left\{\frac{\mathrm{d}(\operatorname{Re}\lambda)}{\mathrm{d}\tau}\right\}_{\tau=\tau_{1,0}^*} = \operatorname{sign}\left\{\operatorname{Re}\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right)^{-1}\right\}_{\tau=\tau_{1,0}^*} = \operatorname{sign}\left\{\frac{h'(z_n^*)}{E(\omega_0^*)}\right\},$$

where $z_n^* = \omega_0^{*2}$. The results from the discussion above are as follows.

Theorem 3. *For $\tau_2=0$, suppose that $a < \frac{\gamma_0 g_1}{(g_1Y^*+g_2)^2} - s(\delta_1-1)+\delta_1$.*

(i) The economic equilibrium E^* of system (1) is locally asymptotically stable for all $\tau_1 \geqslant 0$, if $h(z)=0$ has no positive roots.

(ii) If $\operatorname{sign}\{h'(z_n^*)/E(\omega_0^*)\} > 0$, then system (1) undergoes a Hopf bifurcation at E^* when $\tau=\tau_{1,0}^*$. Furthermore, the economic equilibrium E^* of system (1) is unstable for $\tau > \tau_{1,0}^*$ and locally asymptotically stable for $\tau \in [0,\tau_{1,0}^*)$.

3.2. Case 2: $\tau_2 \neq 0$, $\tau_1 \neq 0$

In this subsection, we study Eq. (18) with $\tau_2 > 0$ and τ_1 in the stable regions. We take τ_2 as a parameter of bifurcation. By Ruan and Wei [12], we have the subsequent lemma.

Lemma 1. *If all roots of equation (20) have negative real parts for $\tau_1 > 0$, then there exists a $\tau_2^*(\tau_1) > 0$, such that when $0 \leqslant \tau_2 < \tau_2^*(\tau_1)$ all roots of equation (18) have negative real parts.*

Proof. The left side of the equation (18) is analytical in λ and τ_2. According to [12], the sum of the multiplicities of zeros on the left side of Eq. (18) in the open right half-plane can change when τ_2 varies only if a zero is on or crosses the imaginary axis. ∎

Theorem 4. *Let τ_1 in the stable regions and $a < \frac{\gamma_0 g_1}{(g_1 Y^* + g_2)^2} - s(\delta_1 - 1) + \delta_1$. Then we have*

- *(i) There exists a $\tau_2^*(\tau_1)$ such that the economic equilibrium E^* is locally asymptotically stable for $\tau_2 \in [0, \tau_2^*(\tau_1))$ when $h(z) = 0$ has no positive roots.*
- *(ii) For any $\tau_1 \in [0, \tau_{1,0})$, there exists a $\tau_2^*(\tau_1)$ such that the economic equilibrium E^* is locally asymptotically stable for $\tau_2 \in [0, \tau_2^*(\tau_1))$ when $\operatorname{sign}\{h'(z_n^*)/E(\omega_0^*)\} > 0$.*

Proof. Theorem 3 (i) and Lemma 1 lead directly to the proof of (i). Now, we prove (ii). Assume that $\operatorname{sign}\{h'(z_n^*)/E(\omega_0^*)\} > 0$ and we conclude using Theorem 3 that E^* is locally asymptotically stable for $\tau_1 \in [0, \tau_{1,0})$. Therefore, all roots of Eq. (20) have negative real parts. It follows from Lemma 1, that there exists a $\tau_2^*(\tau_1) > 0$, such that when $0 \leqslant \tau_2 < \tau_2^*(\tau_1)$ all roots of equation (18) have negative real parts. Thus, when $\tau_2 \in [0, \tau_2^*(\tau_1))$ we find that E^* is locally asymptotically stable. ∎

3.3. Special case

We examine the following IS-LM model in this subsection:

$$\begin{cases} \dfrac{\partial Y}{\partial t} = d_1 \Delta Y(t,x) + \alpha \left[I(Y(t,x), K(t,x), R(t,x)) + G(Y(t,x)) - T(Y(t,x)) - S(Y^D) \right], \\ \dfrac{\partial K}{\partial t} = d_2 \Delta K(t,x) + I(Y(t-\tau,x), K(t-\tau,x), R(t,x)) - \delta K(t,x), \\ \dfrac{\partial R}{\partial t} = d_3 \Delta R(t,x) + \beta [L(Y(t,x), R(t,x)) - M(t,x)], \\ \dfrac{\partial M}{\partial t} = d_4 \Delta M(t,x) + \psi M(t,x) [G(Y(t,x)) - T(Y(t,x))], \end{cases} \tag{28}$$

This system is a special case of system (1) with $\tau_1 = \tau_2 = \tau$. The following conclusions are drawn from Theorems 1 and 2.

Corollary 1.

- *(i) There exists a unique solution of problem (28) defined on $[0, +\infty)$ for any given initial $\Phi \in \mathcal{C}$.*
- *(ii) If (H_1) holds, then (28) has a unique economic equilibrium defined by $E^*(Y^*, K^*, R^*, M^*)$, where $Y^* = \frac{G_0 g_1 - \delta_1 g_2 + \sqrt{(\delta_1 g_2 + G_0 g_1)^2 + 4\delta_1 g_1 \gamma_0}}{2\delta_1 g_1}$, $K^* = \frac{s(1-\delta_1)Y^*}{\delta}$, R^* is the positive solution of the subsequent equation*

$$I\left(\frac{G_0 g_1 - \delta_1 g_2 + \sqrt{(\delta_1 g_2 + G_0 g_1)^2 + 4\delta_1 g_1 \gamma_0}}{2\delta_1 g_1}, \frac{s(1-\delta_1)Y}{\delta}, R \right) - s(1-\delta_1)Y = 0$$

and $M^* = \mathcal{L}(Y^*) - \bar{\gamma} R^*$.

The system stability analysis (28) is the subject of the following discussion. In this case, Eq. (18) becomes

$$\lambda^4 + \lambda^3 g_{3,k} + \lambda^2 g_{2,k} + \lambda g_{1,k} + g_{0,k} + e^{-\lambda\tau} \left[h_3 \lambda^3 + (r_2 + h_{2,k})\lambda^2 + (h_{1,k} + r_{1,k})\lambda + h_{0,k} + r_{0,k} \right] = 0, \tag{29}$$

When $\tau = 0$, all roots of Eq. (29) have negative real parts if $a < \frac{\gamma_0 g_1}{(g_1 Y^* + g_2)^2} - s(\delta_1 - 1) + \delta_1$. Then the economic equilibrium is locally asymptotically stable.

For $\tau > 0$, let $i\omega$ $(\omega > 0)$ is a root of (29), then we obtain

$$\begin{cases} -\omega^4 + \omega^2 g_{2,k} - g_{0,k} = T_1 \cos(\omega\tau) + T_2 \sin(\omega\tau), \\ \omega^3 g_{3,k} - \omega g_{1,k} = T_2 \cos(\omega\tau) - T_1 \sin(\omega\tau), \end{cases} \tag{30}$$

where

$$T_1 = r_{0,k} + h_{0,k} - \omega^2 (h_{2,k} + r_2), \quad T_2 = (r_{1,k} + h_{1,k})\omega - \omega^3 h_3.$$

Therefore, the existence of purely imaginary roots of Eq. (20) is equivalent to the existence of solutions of Eq. (30). Define

$$L(\omega) = T_2^2 + T_1^2. \tag{31}$$

If $L \neq 0$, then by Eq. (21), we get

$$\begin{aligned}\cos(\omega\tau) &= \frac{T_2(\omega^3 g_{3,k} - \omega g_{1,k}) + T_1(-\omega^4 + \omega^2 g_{2,k} - g_{0,k})}{L(\omega)},\\ \sin(\omega\tau) &= \frac{T_2(-\omega^4 + \omega^2 g_{2,k} - g_{0,k}) - T_1(\omega^3 g_{3,k} - \omega g_{1,k})}{L(\omega)}.\end{aligned} \tag{32}$$

Two equations of (32) are squared and added, and it results that

$$\begin{aligned}L^2(\omega) = &\left[T_2(-\omega^4 + \omega^2 g_{2,k} - g_{0,k}) - T_1(\omega^3 g_{3,k} - \omega g_{1,k})\right]^2\\ &+ \left[T_2(\omega^3 g_{3,k} - \omega g_{1,k}) + T_1(-\omega^4 + \omega^2 g_{2,k} - g_{0,k})\right]^2.\end{aligned} \tag{33}$$

Denote

$$\begin{aligned}V(\omega) = L^2(\omega) &- \left[T_2(-\omega^4 + \omega^2 g_{2,k} - g_{0,k}) - T_1(\omega^3 g_{3,k} - \omega g_{1,k})\right]^2\\ &- \left[T_2(\omega^3 g_{3,k} - \omega g_{1,k}) + T_1(-\omega^4 + \omega^2 g_{2,k} - g_{0,k})\right]^2.\end{aligned} \tag{34}$$

By calculations, we get from (31) and (34) that

$$V(\omega) = v_{14}\omega^{14} + v_{12}\omega^{12} + v_{10}\omega^{10} + v_8\omega^8 + v_6\omega^6 + v_4\omega^4 + v_2\omega^2 + v, \tag{35}$$

where

$$\begin{aligned}
v_{14} &= -h_3^2,\\
v_{12} &= h_3^4 + 2h_3(r_{1,k} + h_{1,k}) + 2h_3^2 g_{2,k} - (h_{2,k} + r_2)^2 - h_3^2 g_{3,k}^2,\\
v_{10} &= 2h_3^2(h_{2,k} + r_2)^2 - 2h_3^3(h_{1,k} + r_{1,k}) - 4h_3^3(h_{1,k} + r_{1,k}) - (h_{1,k} + r_{1,k})^2 - h_3 g_{2,k}(h_{1,k} + r_{1,k})\\
&\quad - h_3^2(g_{2,k} + 2g_{0,k}) - (r_2 + h_{2,k})^2 g_{3,k}^2 + 2h_3 g_{3,k}(h_{1,k} + r_{1,k}) + 2h_3^2 g_{1,k} + 2(h_{2,k} + r_2)(h_{0,k}\\
&\quad + r_{0,k}) + 2g_{2,k}(h_{2,k} + r_2)^2,\\
v_8 &= 4h_3^2(r_{1,k} + h_{1,k})^2 - 4h_3^3(r_2 + h_{2,k})(r_{0,k} + h_{0,k}) + (h_{2,k} + r_2)^2 - 4h_3(r_2 + h_{2,k})^2(r_{1,k} + h_{1,k})\\
&\quad + 2g_{2,k}(r_{1,k} + h_{1,k})^2 + 2h_3(r_{1,k} + h_{1,k})(g_{2,k} + 2g_{0,k}) + 2g_{0,k}g_{2,k}h_3^2 + 2g_{3,k}^2(r_2 + h_{2,k})\\
&\quad (h_{0,k} + r_{0,k}) + 2g_{3,k}g_{1,k}(h_{1,k} + r_2)^2 - (h_{1,k} + r_{1,k})^2 g_{3,k}^2 - h_3(h_{1,k} + r_{1,k})g_{3,k}g_{1,k} - g_{1,k}^2 h_3^2\\
&\quad - (h_{0,k} + r_{0,k})^2 - 4g_{2,k}(h_{0,k} + r_{0,k})(r_2 + h_{2,k}) - (g_{2,k} + 2g_{0,k})(r_2 + h_{2,k})^2,\\
v_6 &= -4h_3(r_{1,k} + h_{1,k})^2 - (r_{1,k} + h_{1,k})^2(g_{2,k} + 2g_{0,k}) - 4h_3(r_{1,k} + h_{1,k})g_{0,k}g_{2,k} - h_3^2 g_{0,k}^2\\
&\quad - (r_{0,k} + h_{0,k})^2 g_{3,k} - 4g_{3,k}g_{1,k}(r_{1,k} + h_{0,k})(r_2 + h_{2,k}) - (h_{2,k} + r_{2,k})^2 g_{1,k} + 2(h_{1,k} + r_{1,k})^2\\
&\quad g_{3,k}g_{1,k} + 2h_3(h_{1,k} + r_{1,k})g_{1,k}^2 + 2g_{2,k}(h_{0,k} + r_{0,k})^2 + 2g_{2,k}(h_{0,k} + r_{0,k})(h_{2,k} + r_2) + 4g_{0,k}\\
&\quad (h_{0,k} + r_{0,k})(h_{2,k} + r_2) + g_{0,k}g_{2,k}(h_{0,k} + r_2)^2,\\
v_4 &= (r_{1,k} + h_{1,k}) - 2g_{0,k}g_{2,k}(r_{1,k} + h_{1,k})^2 + h_3(r_{1,k} + h_{1,k})g_{0,k}^2 + 2g_{3,k}g_{1,k}(h_{0,k} + r_{0,k})^2 + 2g_{1,k}\\
&\quad (h_{2,k} + r_2)(h_{0,k} + r_{0,k}) - (r_{1,k} + h_{1,k})^2 g_{1,k}^2 - (g_{2,k} + 2g_{0,k})(h_{0,k} + r_{0,k})^2 - 4g_{0,k}g_{2,k}\\
&\quad (r_{0,k} + h_{0,k})(h_{2,k} + r_2) - (h_{1,k} + r_2)g_{0,k}^2,\\
v_2 &= -g_{1,k}(r_{0,k} + h_{0,k})^2 + 2g_{0,k}g_{2,k}(r_{0,k} + h_{0,k})^2 + (r_{0,k} + h_{0,k})(h_{2,k} + r_2)\\
&\quad g_{0,k}g_{2,k} + 2(r_{0,k} + h_{0,k})(h_{2,k} + r_2)g_{0,k}^2 + 2(h_{0,k} + r_{0,k})^2(h_{2,k} + r_2),\\
v &= (h_{0,k} + r_{0,k})^4 + g_{0,k}(h_{0,k} + r_{0,k})^2 + (h_{1,k} + r_{1,k})^2 g_{0,k}^2.
\end{aligned}$$

Let $z = \omega^2$, Eq. (34) can be written as

$$d(z) := v_{14}z^7 + v_{12}z^6 + v_{10}z^5 + v_8 z^4 + v_6 z^3 + v_4 z^2 + v_2 z + v = 0. \tag{36}$$

Clearly, if $a < \frac{\gamma_0 g_1}{(g_1 Y^* + g_2)^2} - s(\delta_1 - 1) + \delta_1$ and $d(z) = 0$ has no positive roots, then E^* of (28) is locally asymptotically stable for all $\tau \geqslant 0$. Otherwise, for a certain $k_0 \in \mathbb{N}$, if Eq. (36) has positive roots, without loss of generality, we suppose that Eq. (36) with $k = k_0$ has seven positive roots, noted, z_n ($n = 1, \dots, 7$). Accordingly, Eq. (34) has seven positive roots $\omega_n = \sqrt{z_n}$ ($n = 1, \dots, 7$).

For $n = 1, \dots, 7$, one can extract the matching $\tau_n^j > 0$ from (30) such that (29) has a pair of purely imaginary roots $\pm \mathrm{i}\,\omega_n$ stated by

$$\tau_n^j = \frac{1}{\omega_n} \arccos \frac{T_2(\omega_n^3 g_{3,k} - \omega_n g_{1,k}) + T_1(-\omega_n^4 + \omega_n^2 g_{2,k} - g_{0,k})}{L(\omega_n)} + \frac{2\pi j}{\omega_n}, \quad j = 0, 1, \dots. \tag{37}$$

Let the root of Eq. (29) be $\lambda(\tau) = \xi(\tau) + \mathrm{i}\,\omega(\tau)$ satisfying $\xi(\tau_n^j) = 0$, $\omega(\tau_n^j) = \omega_n$. Now, differentiating both sides of (29) with respect to τ leads to

$$\begin{aligned}\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right)^{-1} = {} & \frac{4\lambda^3 + 3\lambda^2 g_{3,k} + 2\lambda g_{2,k} + g_{1,k}}{\lambda(\lambda^3 h_3 + (h_{2,k} + r_2)\lambda^2 + (r_{1,k} + h_{1,k})\lambda + r_{0,k} + h_{0,k})e^{-\lambda\tau}} \\ & + \frac{3\lambda^2 h_3 + 2(r_2 + h_{2,k})\lambda + h_{1,k} + r_{1,k}}{\lambda(\lambda^3 h_3 + (h_{2,k} + r_2)\lambda^2 + (r_{1,k} + h_{1,k})\lambda + r_{0,k} + h_{0,k})} - \frac{\tau}{\lambda}.\end{aligned}$$

Denote

$$\tau_0^* = \tau_{n_0}^{(0)} = \min_{n \in \{1,\dots,7\}} \left\{\tau_n^{(0)}\right\}, \quad \omega_0^* = \omega_{n_0}.$$

By a simple computation, one get that

$$\operatorname{sign}\left\{\frac{\mathrm{d}(\operatorname{Re}\lambda)}{\mathrm{d}\tau}\right\}_{\tau=\tau_0^*} = \operatorname{sign}\left\{\operatorname{Re}\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right)^{-1}\right\}_{\tau=\tau_{1,0}^*} = \operatorname{sign}\left\{\frac{d'(z_n^*)}{L(\omega_0^*)}\right\},$$

where $z_n^* = {\omega_0^*}^2$. The results from the discussion above are as follows.

Theorem 5. *Assume that $a < \frac{\gamma_0 g_1}{(g_1 Y^* + g_2)^2} - s(\delta_1 - 1) + \delta_1$.*

(i) The economic equilibrium E^* of system (28) is locally asymptotically stable for all $\tau \geqslant 0$, if $d(z) = 0$ has no positive roots.

(ii) If $\operatorname{sign}\{d'(z_n^*)/L(\omega_0^*)\} > 0$, then system (28) undergoes a Hopf bifurcation at E^* when $\tau = \tau_0^*$. Further, the economic equilibrium E^* of system (28) is unstable for $\tau > \tau_0^*$ and locally asymptotically stable for $\tau \in [0, \tau_0^*)$.

4. Conclusion

Despite the fact that there have been several publications exploring the stability and bifurcation in delayed IS-LM models, the majority of them do not take into account the diffusion effects that are unavoidable in economics and the money supply as endogenous variable in the same time. In this article, we have proposed and analyzed a diffusive IS-LM model with two temporal delays in gross product, capital stock and the endogenous money supply. We have firstly established the mathematical and economic soundness of the suggested model. In addition, we have studied the stability criteria for the equilibrium. By using the delay as a bifurcation parameter, it was shown that the economic equilibrium loses stability when the delay crosses a crucial point, leading to a Hopf bifurcation.

[1] Elkarmouchi M., Lasfar S., Hattaf Kh., Yousfi N. Spatiotemporal dynamics of a delayed IS-LM model with interest rate and general investment function. Journal of Mathematics and Computer Science. **31** (1), 70–80 (2023).

[2] Hattaf K., Riad D., Yousfi N. A generalized business cycle model with delays in gross product and capital stock. Chaos, Solitons & Fractals. **98**, 31–37 (2017).

[3] Hu W., Zhao H., Dong T. Dynamic Analysis for a Kaldor–Kalecki Model of Business Cycle with Time Delay and Diffusion Effect. Complexity. **2018**, 1263602 (2018).

[4] Riad D., Hattaf K., Yousfi N. Mathematical analysis of a delayed IS-LM model with general investment function. The Journal of Analysis. **27** (4), 1047–1064 (2019).

[5] Cai J. P. Hopf bifurcation in the IS-LM business cycle model with time delay. Electronic Journal of Differential Equations. **2005**, 15 (2005).

[6] Zhou L., Li Y. A generalized dynamic IS-LM model with delayed time in investment processes. Applied Mathematics and Computation. **196** (2), 774–781 (2008).

[7] Cesare L. D., Sportelli M. A dynamic IS-LM model with delayed taxation revenues. Chaos Solitons & Fractals. **25** (1), 233–244 (2005).

[8] Sportelli M., Cesare L. D., Binetti M. T. A dynamic IS-LM model with two time delays in public sector. Applied Mathematics and Computation. **243**, 728–739 (2014).

[9] Akanksha R., Bhatia S. K., Hiremath K. R. Inspecting the stability of non-linear IS-LM model with dual time delay. Chaos, Solitons & Fractals. **165** (2), 112821 (2022).

[10] Hattaf K., Yousfi N. A generalized HBV model with diffusion and two delays. Computers & Mathematics with Applications. **69** (1), 31–40 (2015).

[11] Hale J. Theory of Functional Differential Equations. Springer–Verlag, Heidelberg (1977).

[12] Ruan S., Wei J. On the zeros of transcendental functions with applications to stability of delay differential equations with two delays. Dynamics of Continuous Discrete and Impulsive Systems Series A. Mathematical Analysis. **10**, 863–874 (2003).

Узагальнена дифузійна модель бізнес-циклу IS-LM із затримками валового продукту та капіталу

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У статті пропонується модель бізнес-циклу IS-LM із дифузійним і відстроченим процесом із відсотковою ставкою, загальними інвестиціями та пропозицією грошей за однорідних граничних умов Неймана. Часові затримки відповідно включені до основного капіталу та валового продукту. Спочатку продемонстровано обґрунтування математичної та економічної моделі. Шляхом дослідження відповідного характеристичного рівняння доведено локальну стійкість економічної рівноваги та існування біфуркації Хопфа.

Ключові слова: *економіка; дифузія; стабільність; затримки часу; біфуркація Хопфа.*