MATHEMATICAL MODELING AND COMPUTING, Vol. 11, No. 2, pp. 492–504 (2024)

Analysis and optimal control problem for a fractional mathematical model of tuberculosis with smoking consideration

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(Received 27 April 2023; Revised 3 November 2023; Accepted 8 November 2023)

This article studies a mathematical model of the fractional order of tuberculosis (TB). It describes the dynamics of the spread of tuberculosis among smokers. The purpose of this research is to protect vulnerable people against the virus. According to the survey results, the required model has an equilibrium point: the disease-free equilibrium point E_f. We also analyze the local stability of this equilibrium point of the model, using the basic reproduction number $\mathcal{R}_0$ calculated according to the new generation method. In our model, we include three controls that represent: restricting individual contact, treatment, and sensitization. This article aims at reducing the number of infected smokers and non-smokers using an optimal control strategy and a fractional derivation. The maximum principle of Pontryagin is used to describe optimal controls with Caputo-derived fractional over time and the optimal system is resolved iteratively. The numerical simulation is presented according to the method presented by Matlab.

Keywords: *Caputo fractional derivative; optimal control; tuberculosis; smoking; contagious virus; local stability; dynamic system; infectious diseases; stability; free equilibrium; Pontryagin maximum.*

2010 MSC: 34A12, 34D20, 37M05, 92-08, 92-10 **DOI:** 10.23939/mmc2024.02.492

1. Introduction

Smoking and tuberculosis (TB) are the most common health problems in the world. Smoking is widely recognized by the medical community and the general public as a serious public health problem. It is the greatest preventable threat to human health in developed countries and the leading cause of premature death worldwide. The annual tobacco-related death toll in 1985 was about 1.7 million, 2.0 million in 1990 and is projected to reach 8.4 million by 2020. Smoking contributes to mortality from lung disease, which accounts for 87% of deaths from lung cancer and 82% of deaths from chronic obstructive pulmonary disease and cardiovascular disease. It is responsible for 21% of deaths from coronary artery disease and 18% of strokes. The risk associated with tobacco smoke affects smokers and those around them. The environmental hazards of tobacco exposure are increasingly recognized. Second-hand tobacco smoke (ETS) increases the risk of lung cancer by 30% and contributes to non-smoking adult absenteeism from work due to respiratory disease [1].

In contrast, tuberculosis, the most common infectious disease with a high human mortality rate, continues to cause 3 million deaths per year, or about 5 deaths per minute. This pathogen infects 8 to 10 million people every year. About 3.9 million cases were sputum-positive, the most convincing type of disease. Although most TB sufferers live in the more populous countries of Asia, the estimated incidence rate is highest in Africa (356 cases per 100 000 annually). Half come from Bangladesh, China, India, Indonesia and Pakistan. In the 22 most populous countries in the world, approximately 80% are diagnosed with the disease each year [2, 3].

The link between smoking and tuberculosis has been known for almost a century. The effect of smoking on tuberculosis has pleasantly been set up with-inside the closing decade: active and passive exposure to smoke are independent risk factors for tuberculosis infection, progression of tuberculosis infection to disease, increased disease severity, and increased risk of post-treatment recurrence and mortality [4].

A study conducted by the Department of Epidemiology of the Tuberculosis Research Center established a connection between smoking and the development of tuberculosis [5]. In addition, several systematic reviews and analyses of experimental studies have highlighted the association between smoking and tuberculosis-related mortality and morbidity, with smokers dying nine times more often than those who have never smoked. However, this risk was reduced by 66% with smoking cessation, suggesting that smoking cessation is an important criterion for reducing TB-related mortality. Another excellent study conducted in China in 2017 found a direct link between smoking history and a higher risk of latent tuberculosis. A recent mathematical modeling study estimates that smoking can potentially increase TB cases worldwide by 18 million between 2010 and 2050 and significantly increase secondary mortality if smoking continues [6].

Fractional derivatives and integrals and their ability to make use of them have received first-rate importance, especially due to the fact they have come to be effective equipment with extra accurate, efficient, and successful results in the mathematical modeling of several complex phenomena in numerous, seemingly diverse, and widespread areas of science, engineering, and finance.

This research is structured as follows: Section 2 gives some basic definitions and properties of fractional order integrals and derivatives. For fractional order differentiation, we will use the Caputo definition since it is suitable for the initial conditions of the differential equations. Section 3 presents the mathematical model in terms of fractional differential equations. In Section 4, the positivity and boundedness of the solution are studied. The primary reproduction number and equilibrium points are given in Section 5. The local stability analysis is proved in Section 6. Numerical simulations are performed in Section 7 to verify the theoretical results. The fractional order model with three controls is based on the characterization of the optimal control terms using Pontryagin's maximum principle and is described in Section 8. Numerical simulations are presented in Section 9. Finally, Section 10 concludes the paper.

2. Preliminary results

Let us now recall the definitions of the Mittag–Leffler function and the fractional temporal derivative of Caputo. Firstly, the Mittag–Leffler function, $E_\alpha(y)$, is defined as the family of integer functions of y given as:

$$E_\alpha(y) = \sum_{t=0}^{\infty} \frac{y^t}{\Gamma(t\alpha + 1)}, \quad \alpha > 0, \quad y \in \mathbb{C},$$

when the series converges [7], where $\Gamma(\cdot)$ is the Gamma function. Observe that the Mittag–Leffler function generalizes the exponential function: $E_1(y) = \exp(y)$.

Consider the following commensurate fractional-order system

$$\begin{cases} D^\alpha x(t) = g(x), \\ x(t_0) = x_0. \end{cases} \tag{1}$$

Let $g\colon \mathbb{R}^+ \to \mathbb{R}$, with $n \geqslant 1$. Where $0 < \alpha \leqslant 1$, $t_0 \in \mathbb{R}$. For the global existence of the system solution (1), we need the following theorem.

Theorem 1 (Ref. [8]). *Suppose that g satisfies the subsequent conditions*

- *$g(t,x)$ and $\frac{\partial g}{\partial x}$ are continuous respectively $x \in \mathbb{R}^n$;*
- *$\|g(t,x)\| \leqslant \mu + \Lambda\|x\|$ $\forall x \in \mathbb{R}^n$, for nearly each $t \in \mathbb{R}$ and for all $x \in \mathbb{R}^d$, where μ and Λ are two positive constants.*

Under these conditions, there exists a unique solution on $[0, +\infty)$ solving the system (1).

Lemma 1 (Ref. [9]). *Let $v(t) \in C([0,+\infty))$. If $v(t)$ satisfies $D_*^\beta v(t) \leqslant -\Lambda v(t) + n$, $v(0) = v_0 \in \mathbb{R}$, where $\beta \in (0,1]$, $\Lambda, n \in \mathbb{R}$ and $\Lambda \neq 0$, then $v(t) \leqslant \left(v_0 - \frac{n}{\Lambda}\right) E_\beta\left[-\Lambda t^\beta\right] + \frac{n}{\Lambda}$.*

Lemma 2 (Ref. [10]). *Let $v(t) \in \mathrm{C}(\mathbb{R}_+)$ and its fractional derivatives of order β exist for any $\beta \in (0,1]$. Then, for any $t > 0$ we have $D_*^\beta \left[v(t) - v^* - v^* \ln \frac{v(t)}{v^*}\right] \leqslant \left(1 - \frac{v^*}{v(t)}\right) D_*^\beta v(t)$, $v^* \in \mathbb{R}_+$.*

Theorem 2 (Ref. [11]). *The autonomous system:* $D^{\beta}y(t) = Hy(t)$ *is asymptotically stable if and only if* $|\arg(\mathrm{spec}(H))| > \frac{\beta\pi}{2}$, *where* $\beta \in [0,1)$, $\arg(\cdot)$ *is the principal argument of a given complex number and* $\mathrm{spec}(H)$ *is the spectrum (set of all eigenvalues) of* H *and* $y(t_0) = y_0$.

3. Presentation of the model

We consider the mathematical model that considers smokers and non-smokers in the SIR model of tuberculosis infection [12–15]. Then, we obtain the following model

$$\begin{cases} D^{\alpha}S(t) = \Lambda - \theta_1 S - \theta_2 S - m S, \\ D^{\alpha}S_s(t) = -\gamma_1 \dfrac{S_s I_s}{\mathcal{N}} - \gamma_2 \dfrac{S_s I_a}{\mathcal{N}} + \theta_1 S - m S_s, \\ D^{\alpha}S_a(t) = -\gamma_3 \dfrac{S_a I_s}{\mathcal{N}} - \gamma_4 \dfrac{S_a I_a}{\mathcal{N}} + \theta_2 S - m S_a, \\ D^{\alpha}I_s(t) = \gamma_1 \dfrac{S_s I_s}{\mathcal{N}} + \gamma_2 \dfrac{S_s I_a}{\mathcal{N}} - (m + \delta_1 + \sigma_1) I_s, \\ D^{\alpha}I_a(t) = \gamma_3 \dfrac{S_a I_s}{\mathcal{N}} + \gamma_4 \dfrac{S_a I_a}{\mathcal{N}} - (m + \delta_2 + \sigma_2) I_a, \\ D^{\alpha}R(t) = \sigma_1 I_s + \sigma_2 I_a - m R. \end{cases} \tag{2}$$

With the subsequent non-negative preliminary conditions:

$$S(0) > 0, \quad S_s(0) > 0, \quad S_a > 0, \quad I_s > 0, \quad I_a > 0, \quad R > 0,$$

and $\mathcal{N}(t) = S(t) + S_s(t) + S_a(t) + I_s(t) + I_a(t) + R(t)$ indicate the whole population at time $t > 0$.

The biological characterization of the model parameters is provided in Tables 1 and 2.

Table 1. Compartments meaning.

Compartment	Meaning
S_s	Non-smokers susceptible to tuberculosis
S_a	Smokers susceptible to tuberculosis
I_s	Individual non-smokers infected by tuberculosis
I_a	individual smokers infected by tuberculosis
R	Recovered individuals

Table 2. Parameters meaning.

Parameter	Meaning
Λ	The incidence of susceptible
m	The natural death rate
γ_1	The level of infected non-smokers in contact with infected non-smokers
γ_2	The rate of non-smokers infected by contact with an infected smoker
γ_3	The level of smokers infected by contact with infected nonsmokers
γ_4	The level of infected smokers through an infected smoker
δ_1	The mortality rate of non-smokers infected with tuberculosis
δ_2	The mortality rate of smokers infected with tuberculosis
θ_1	The level of non-smokers susceptible in the study citizenry
θ_2	The level of smokers susceptible in the study citizenry
σ_1	The rate of recovery of the virus in non-smokers from tuberculosis
σ_2	The rate of recovery of the virus in smokers from tuberculosis

4. Positivity and boundedness of solution

Theorem 3. *All solutions* $S(t)$, $S_s(t)$, $S_a(t)$, $I_s(t)$, $I_a(t)$, *and* $R(t)$ *of system equation (2) are bounded by the region* $\mathcal{O} = \left\{(S, S_s, S_a, I_s, I_a, R) \in \mathbb{R}^6_+ / \mathcal{N}(t) \leqslant \frac{\Lambda}{m}\right\}$, *and* $\mathcal{O}$ *is positively invariant under the system (2) with the initial conditions* $S(0) > 0$, $S_w(0) > 0$, $S_m(0) > 0$, $I_w(0) > 0$, $I_a(0) > 0$, $I_m(0) > 0$, *and* $R(0) > 0$.

Proof. We put

$$f_1(I_s, I_a) = \gamma_1 \frac{I_s}{\mathcal{N}} + \gamma_2 \frac{I_a}{\mathcal{N}} \quad \text{and} \quad f_2(I_s, I_a) = \gamma_3 \frac{I_s}{\mathcal{N}} + \gamma_4 \frac{I_a}{\mathcal{N}}.$$

Let $\mathcal{X}(t) = (S, S_s, S_a, I_s, I_a, R)^T$ involved in $\mathbb{R}^6_+$. The system (2) is capable of being reformulated as

$$D^\alpha \mathcal{X}(t) = \mathcal{F}(\mathcal{X}(t)),$$

where

$$\mathcal{F}(\mathcal{X}) = \begin{pmatrix} \Lambda - (\theta_1 + \theta_2 + m)\, S \\ -f_1(I_s, I_a) S_s + \theta_1 S - m S_s \\ -f_2(I_s, I_a) S_a + \theta_2 S - m S_a \\ f_1(I_s, I_a) S_s - (m + \delta_1 + \sigma_1)\, I_s \\ f_2(I_s, I_a) S_a - (m + \delta_2 + \sigma_2)\, I_a \\ \sigma_1 I_s + \sigma_2 I_a - m R \end{pmatrix}$$

obviously $\mathcal{F}$ fulfilled the first requirement of Theorem 1.

As proof of the second, we denounce

$$\mathcal{Z} = \begin{pmatrix} \Lambda \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \mathcal{N}_1 = \begin{pmatrix} -(m + \theta_1 + \theta_2) & 0 & 0 & 0 & 0 & 0 \\ \theta_1 & -m & 0 & 0 & 0 & 0 \\ \theta_2 & 0 & -m & 0 & 0 & 0 \\ 0 & 0 & 0 & -(m + \sigma_1 + \delta_1) & 0 & 0 \\ 0 & 0 & 0 & 0 & -(m + \sigma_2 + \delta_2) & 0 \\ 0 & 0 & 0 & \sigma_1 & \sigma_2 & -m \end{pmatrix},$$

$$\mathcal{N}_2 = \begin{pmatrix} 0 & -f_1(I_s, I_a) & 0 & 0 & 0 & 0 \\ 0 & 0 & -f_2(I_s, I_a) & 0 & 0 & 0 \\ 0 & f_1(I_s, I_a) & 0 & 0 & 0 & 0 \\ 0 & 0 & f_2(I_s, I_a) & 0 & 0 & 0 \end{pmatrix}.$$

Then, we have

$$\mathcal{F}(\mathcal{X}(t)) = \mathcal{N}_1 \mathcal{X}(t) + \mathcal{N}_2 \mathcal{X}(t) + \mathcal{Z}.$$

Thus,

$$\begin{aligned} \|\mathcal{F}(\mathcal{X})\| &= \|\mathcal{N}_1 \mathcal{X}(t) + \mathcal{N}_2 \mathcal{X}(t) + \mathcal{Z}\| \\ &\leqslant \|\mathcal{Z}\| + \|\mathcal{N}_1\| \, \|\mathcal{X}(t)\| + \|\mathcal{N}_2\| \, \|\mathcal{X}(t)\| \\ &\leqslant \|\mathcal{Z}\| + (\|\mathcal{N}_1\| + \|\mathcal{N}_2\|) \, \|\mathcal{X}(t)\|, \end{aligned}$$

we find that:

$$\|\mathcal{F}(\mathcal{X})\| \leqslant \mu + \Lambda_1 \, \|\mathcal{X}(t)\|,$$

where $\mu = \|\mathcal{Z}\|$ and $\Lambda_1 = (\|\mathcal{N}_1\| + \|\mathcal{N}_2\|)$. According to the Theorem 1, the system (2) has a unique solution on $[0, +\infty[$.

For positively, we get

$$\begin{cases} D^\alpha S(t)|_{S=0} = \Lambda \geqslant 0, \\ D^\alpha S_s(t)|_{S_s=0} = \theta_1 S \geqslant 0, \\ D^\alpha S_a(t)|_{S_a=0} = \theta_2 S \geqslant 0, \\ D^\alpha I_s(t)|_{I_s=0} = \gamma_2 \dfrac{S_s I_a}{\mathcal{N}} \geqslant 0, \\ D^\alpha I_a(t)|_{I_a=0} = \gamma_3 \dfrac{S_a I_s}{\mathcal{N}} \geqslant 0, \\ D^\alpha R(t)|_{R=0} = \sigma_1 I_s + \sigma_2 I_a \geqslant 0. \end{cases}$$

Thus, with the initial condition, the solution of (2) remains non-negative for every $t \geqslant 0$. To determine the scope of the solution, we apply,

$$\mathcal{N}(t) = S(t) + S_s(t) + S_a(t) + I_s(t) + I_a(t) + R(t),$$

therefore

$$\begin{aligned} D^\alpha \mathcal{N}(t) &= D^\alpha S(t) + D^\alpha S_s(t) + D^\alpha S_a(t) + D^\alpha I_s(t) + D^\alpha I_a(t) + D^\alpha R(t) \\ &= \Lambda - mS - mS_s - mS_a - (m+\delta_1)\, I_s - (m+\delta_2)\, I_a - mR \\ &\leqslant \Lambda - m\,(S + S_s + S_a + I_s + I_a + R)\,, \\ D^\alpha \mathcal{N}(t) &\leqslant \Lambda - m\mathcal{N}(t). \end{aligned}$$

According to Lemma 1, we have

$$\mathcal{N}(t) \leqslant \left(\zeta(0) - \frac{\Lambda}{m}\right) E_\alpha(mt^\alpha) + \frac{\Lambda}{m},$$

where E_α is the Mittag–Leffler function.

Since $E_\alpha(-\delta t^\alpha) \to 0$ as $t \to \infty$, we have

$$\mathcal{N}(t) \leqslant \frac{\Lambda}{m}.$$

Consequently, all (2) system solutions that begin in $\mathbb{R}^6_+$ are limited to the $\mathcal{O}$ area where:

$$\mathcal{O} = \left\{(S, S_s, S_a, I_s, I_a, R) \in \mathbb{R}^6_+ / \mathcal{N}(t) \leqslant \frac{\Lambda}{m}\right\}.$$

In this way, all (2) fractional ordering system solutions are uniformly limited. ∎

The first equation of the system (2) does not hang on the R, then we can disregard the last equation of the system (2).

The problem can therefore be alleviated to:

$$\begin{cases} D^\alpha S(t) = \Lambda - \theta_1\, S - \theta_2\, S - m\, S, \\ D^\alpha S_s(t) = -\gamma_1 \dfrac{S_s\, I_s}{\mathcal{N}} - \gamma_2 \dfrac{S_s\, I_a}{\mathcal{N}} + \theta_1\, S - m\, S_w, \\ D^\alpha S_a(t) = -\gamma_3 \dfrac{S_a\, I_s}{\mathcal{N}} - \gamma_4 \dfrac{S_a\, I_a}{\mathcal{N}} + \theta_2\, S - m\, S_m, \\ D^\alpha I_s(t) = \gamma_1 \dfrac{S_s\, I_s}{\mathcal{N}} + \gamma_2 \dfrac{S_s\, I_a}{\mathcal{N}} - (m + \delta_1 + \sigma_1)\, I_s, \\ D^\alpha I_a(t) = \gamma_3 \dfrac{S_a\, I_s}{\mathcal{N}} + \gamma_4 \dfrac{S_a\, I_a}{\mathcal{N}} - (m + \delta_2 + \sigma_2)\, I_a. \end{cases} \tag{3}$$

5. Basic reproduction number $\mathcal{R}_0$ calculation and disease-free equilibrium

5.1. The basic reproduction number $\mathcal{R}_0$

Theorem 4. *The basic reproduction number $\mathcal{R}_0$ is given by $\mathcal{R}_0 = \max\left(\mathcal{R}_0^1, \mathcal{R}_0^2\right)$, where $\mathcal{R}_0^1 = \frac{\gamma_1\, \theta_1}{(m+\delta_1+\sigma_1)\,(m+\theta_1+\theta_2)}$ and $\mathcal{R}_0^2 = \frac{\gamma_4\, \theta_2}{(m+\delta_2+\sigma_2)(m+\theta_1+\theta_2)}$.*

Proof. To identify the baseline reproduction number, we are using the next generation matrix technology created by Bani–Yaghoub et al. [16–19]. ∎

5.2. The disease-free equilibrium

To find the sickness-free balance point, we set the right side of the (2) model to zero, stocktaking at $I_s = I_a = 0$ and tackling for the uninfected and non-carrier state variables. Therefore, the point of equilibrium without sickness is $E_f\left(\frac{\Lambda}{(m+\theta_1+\theta_2)}, \frac{\theta_1\, \Lambda}{m\,(m+\theta_1+\theta_2)}, \frac{\theta_2\, \Lambda}{m\,(m+\theta_1+\theta_2)}, 0, 0, 0\right)$.

6. Stability of the free equilibrium

6.1. Local stability

Theorem 5. *The free equilibrium point E_f is stable if $\mathcal{R}_0 < 1$ and unstable if $\mathcal{R}_0 > 1$.*

Proof. The Jacobian matrix associated with the system (3) at the free equilibrium point E_f is granted by

$$J(E_f) = \begin{pmatrix} -m-\theta_1-\theta_2 & 0 & 0 & 0 & 0 \\ \dfrac{\theta_1\Lambda}{m+\theta_1+\theta_2} & -m & 0 & -\dfrac{\gamma_1\theta_1}{m+\theta_1+\theta_2} & -\dfrac{\gamma_2\theta_1}{m+\theta_1+\theta_2} \\ \theta_2 & 0 & -m & -\dfrac{\gamma_3\alpha_2}{m+\theta_1+\theta_2} & -\dfrac{\gamma_4\theta_2}{m+\theta_1+\theta_2} \\ 0 & 0 & 0 & \dfrac{\gamma_1\theta_1}{m+\theta_1+\theta_2} - m - \delta_1 - \sigma_1 & \dfrac{\gamma_2\theta_1}{m+\theta_1+\theta_2} \\ 0 & 0 & 0 & \dfrac{\gamma_3\theta_2}{m+\alpha_1+\theta_2} & \dfrac{\gamma_4\theta_2}{m+\theta_1+\alpha_2} - m - \delta_2 - \sigma_2 \end{pmatrix}.$$

We are only interested in the sign of the eigenvalues of the matrix $\tilde{J}(E_f)$ defined below because the eigenvalues $\lambda_1 = -m-\theta_1-\theta_2$ and $\lambda_2 = -m$ are satisfied to the following conditions $|\arg(\lambda_1)| > \frac{\alpha\pi}{2}$ and $|\arg(\lambda_2)| > \frac{\alpha\pi}{2}$. Then, we consider the following matrix

$$\tilde{J}(E_f) = \begin{pmatrix} \frac{\gamma_1\theta_1}{m+\alpha_1+\theta_2} - m - \delta_1 - \sigma_1 & \frac{\gamma_2\theta_1}{m+\theta_1+\theta_2} \\ \frac{\gamma_3\theta_2}{m+\theta_1+\theta_2} & \frac{\gamma_4\theta_2}{m+\theta_1+\theta_2} - m - \delta_2 - \sigma_2 \end{pmatrix}.$$

The characteristic polynomial of the matrix $\tilde{J}(E_f)$ is given by

$$\mathcal{Q}(\lambda) = \lambda^2 - \left[(m+\delta_1+\sigma_1)(\mathcal{R}_0^1-1) + (m+\delta_2+\sigma_2)(\mathcal{R}_0^2-1)\right]\lambda + (m+\delta_1+\sigma_1)(\mathcal{R}_0^1-1)(m+\delta_2+\sigma_2)(\mathcal{R}_0^2-1).$$

Therefore, we have $\lambda_3 = (m+\delta_1+\sigma_1)(\mathcal{R}_0^1-1)$ and $\lambda_4 = (m+\delta_2+\sigma_2)(\mathcal{R}_0^2-1)$, we observe that, if $\mathcal{R}_0 < 1$, then $|\arg(\lambda_3)| > \frac{\alpha\pi}{2}$ and if $\mathcal{R}_0 > 1$ then $|\arg(\lambda_4)| < \frac{\alpha\pi}{2}$. Therefore E_f is locally asymptotically stable if $\mathcal{R}_0 < 1$ and is a saddle point if $\mathcal{R}_0 > 1$. ∎

7. Simulation without control

In this section, we validate our theoretical outcomes through numerical simulations.

Let Δt be the scale of the temporal step. Such as $t_n = n \cdot \Delta t$ for $n \in \mathbb{N}$. The fractional derivative of Caputo can be approximated by

$${}^{C}D_t^{\alpha} X(t_n) \approx \frac{1}{\Delta t^{\alpha}} \sum_{j=0}^{n} \zeta_j^{\alpha} X(t_{n-j}) - \bar{X}_n,$$

where $\bar{X} = \frac{X(0)t_n^{-\alpha}}{\Gamma(1-\alpha)}$ and ζ_j^{α} is the coefficient binomial fractional with recursive formula

$$\zeta_j^{\alpha} = \left(1 - \frac{1+\alpha}{j}\right)\zeta_{j-1}^{\alpha}, \quad \zeta_0^{\alpha} = 1.$$

For the numerical illustrations, we choose in all this section $\Lambda = 0.001$, $m = 0.0001$, $\gamma_1 = 0.121$, $\gamma_2 = 0.141$, $\gamma_3 = 0.21$, $\gamma_4 = 0.29$, $\delta_1 = 0.0019$, $\delta_2 = 0.005$, $\sigma_1 = 0.04$, $\sigma_2 = 0.010$, $\theta_1 = 0.66$, and $\theta_2 = 0.34$. The initial conditions used are: $S(0) = 0.4$, $S_s(0) = 0.4$, $S_a(0) = 0.05$, $I_s(0) = 0.1$, $I_a(0) = 0.05$, and $R(0) = 0.01$. For $\alpha = 0.5$, $\theta_1 = 0.7$, and $\alpha_3 = 0.3$ that displayed respectively in Figure 1. We also calculate $\mathcal{R}_0 = 6.5291 > 0$. Hence, system (2) has a unique equilibrium $E_f = (9.99, 6.59, 3.39, 0, 0)$. Because fractional derivatives accurately describe the situation, it can be said that the outbreak takes longer to be stable. That is crucial when referring to the economy and looking at control strategies. Presently, we heed $\gamma_1 = 0.0121$, $\gamma_2 = 0.0141$, $\gamma_3 = 0.021$, $\gamma_4 = 0.029$, $\sigma_1 = 0.4$, and let the same previous set of parameters. Then, $\mathcal{R}_0 = 0.6529$, Figure 2 elucidates this result for diverse values of α.

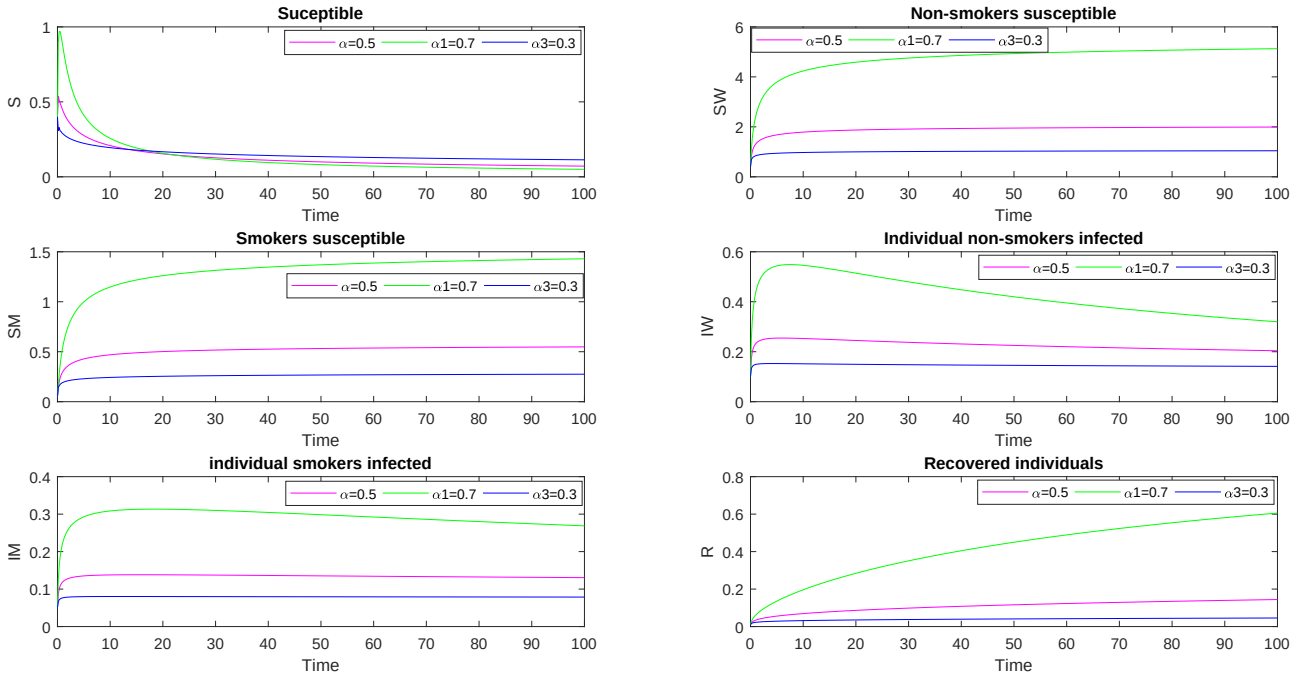

Fig. 1. In the case where $R_0 = 6.5291 > 1$, the stability asymptotic of the infection-free equilibrium E_f

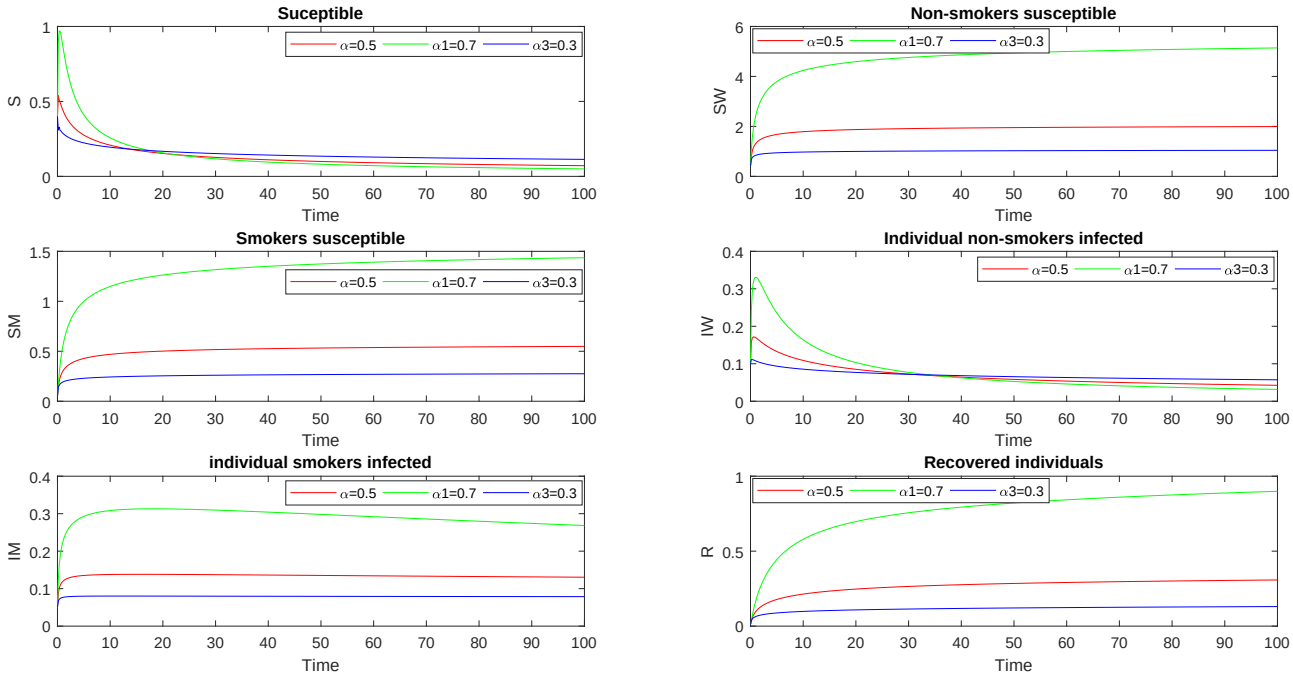

Fig. 2. In the case where $R_0 = 0.6529 < 1$, the stability asymptotic of the infection-free equilibrium E_f

8. Problem of optimal control

8.1. Presentation of the controls

The control strategy is a program designed to minimize the proliferation of the TB virus. Our primary goal is to minimize the total of people affected. In the current model, we have embedded three $n(t)$, $o(t)$, and $h(t)$ controls for $t \in [0, t_f]$.

The primary control n constitutes the proportion to be sensitized and prevented by media and education, and the proportion to be prevented from gathering by security campaigns at a certain instant t. The next control o represented the treatment at a certain instant t. The last control h constitutes the sensitization of the population about the dangers of smoking on health in general and especially in the case of Tuberculosis at a certain instant t

$$\begin{cases} D^{\alpha}S(t) = \Lambda - \theta_1\, S - \theta_2\, S - mS, \\ D^{\alpha}S_s(t) = -\gamma_1(1-n(t))\dfrac{S_s\, I_s}{\mathcal{N}} - \gamma_2(1-n(t))\dfrac{S_s\, I_a}{\mathcal{N}} + \theta_1\, S - m\, S_s + h(t)S_a, \\ D^{\alpha}S_a(t) = -\gamma_3(1-n(t))\dfrac{S_a\, I_s}{\mathcal{N}} - (1-n(t))\gamma_4\dfrac{S_a\, I_a}{\mathcal{N}} + \theta_2\, S - m\, S_a - h(t)S_a, \\ D^{\alpha}I_s(t) = \gamma_1(1-n(t))\dfrac{S_s\, I_s}{\mathcal{N}} + \gamma_2(1-n(t))\dfrac{S_s\, I_a}{\mathcal{N}} - (m+\delta_1+\sigma_1)\, I_s - o(t)I_s, \\ D^{\alpha}I_a(t) = \gamma_3(1-n(t))\dfrac{S_a\, I_s}{\mathcal{N}} + \gamma_4(1-n(t))\dfrac{S_a\, I_a}{\mathcal{N}} - (m+\delta_2+\sigma_2)\, I_a - o(t)I_a, \\ D^{\alpha}R(t) = \sigma_1\, I_s + \sigma_2\, I_a - m\, R + o(t)(I_s + I_a), \end{cases} \tag{4}$$

where $S(0) > 0$, $S_s(0) > 0$, $S_a > 0$, $I_s > 0$, $I_a > 0$, and $R(0) > 0$.

8.2. Objective functional

The objective function $\mathcal{K}$ is set out below

$$\mathcal{K}(n,o,h) = \int_0^{t_f} \left(I_s(t) + I_a(t) + \frac{1}{2}\mathcal{A}n^2(t) + \frac{1}{2}\mathcal{C}o^2(t) + \frac{1}{2}\mathcal{G}h^2(t) \right) dt, \tag{5}$$

where $\mathcal{A} > 0$, $\mathcal{C} > 0$, and $\mathcal{G} > 0$ are the fee factors, they are chosen to assess the materiality of $n(t)$, $o(t)$, and $h(t)$ at time t. t_f is the latest.

In short, we search for optimum controls (n^*, o^*, h^*) such that

$$\mathcal{K}\,(n^*, o^*, h^*) = min\{\mathcal{K}(n,o,h)/(n,o,h) \in \mathcal{U}_{n,o,h}\} \tag{6}$$

with $\mathcal{U}_{n,o,h}$ is the range of controls specified in

$$\mathcal{U}_{n,o,h} = \big\{(n,o,h)/0 \leqslant n_{\min} \leqslant n(t) \leqslant n_{\max} \leqslant 1, 0 \leqslant o_{\min} \leqslant o(t) \leqslant o_{\max} \leqslant 1, \\ 0 \leqslant h_{\min} \leqslant h(t) \leqslant h_{\max} \leqslant 1/t \in [0, t_f]\big\}.$$

8.3. Sufficient conditions

The existence of optimal control can be derived using a result of Fleming and Rishel [17, 20], and Lukes [21, 22].

Theorem 6. *Contemplate the control problem of system (4). There is an optimal control* $(n^*, o^*, h^*) \in \mathcal{U}_{n,o,h}$ *such that* $\mathcal{K}(n^*, o^*, h^*) = \min\{\mathcal{K}(n,o,h)/(n,o,h) \in \mathcal{U}_{n,o,h}\}$.

8.4. Necessary conditions

The main point of this project is to retrieve the optimal control (n, o, h) which Pontryagin's maximum principle [23–26] transforms (4)–(5), and (6) in an issue of minimizing a Hamiltonian, $\mathcal{H}$:

$$\mathcal{H}(t) = I_s(t) + I_a(t) + \frac{1}{2}\mathcal{A}n^2(t) + \frac{1}{2}\mathcal{C}o^2(t) + \frac{1}{2}\mathcal{G}h^2(t) + \sum_{i=1}^{6} \zeta_i(t) y_i(S, S_s, S_a, I_s, I_a, R), \tag{7}$$

where y_i is the corner of the differential equation of the state variable (4).

In the subsequent theorem, we present the vital conditions for the existence of optimal control.

Theorem 7. *Provided optimal controls* n^*, o^*, h^* *and solutions* S^*, S_s^*, S_a^*, I_s^*, I_a^* *and* R^* *of suitable state system (4), there exists* ζ_i, $i = 1, \ldots, 6$ *the adjoint parameters that indulge the subsequent*

equations

$$
\begin{aligned}
&D^{\alpha}\zeta_1(t) = -[(-m-\theta_1-\theta_2)\zeta_1+\zeta_2\theta_1+\zeta_3\theta_2],\\
&D^{\alpha}\zeta_2(t) = -\left[\zeta_2\left(-\frac{\gamma_1(1-n)I_s}{\mathcal{N}}-\frac{\gamma_2(1-n)I_a}{\mathcal{N}}-m\right)+\zeta_4\left(\frac{\gamma_1(1-n)I_s}{\mathcal{N}}+\frac{\gamma_2(1-n)I_a}{\mathcal{N}}\right)\right],\\
&D^{\alpha}\zeta_3(t) = -\left[\zeta_2 h+\zeta_3\left(-\frac{\gamma_3(1-n)I_s}{\mathcal{N}}-\frac{\gamma_4(1-n)I_a}{\mathcal{N}}-m-h\right)+\zeta_5\left(\frac{\gamma_3(1-n)I_s}{\mathcal{N}}+\frac{\gamma_4(1-n)I_a}{\mathcal{N}}\right)\right],\\
&D^{\alpha}\zeta_4(t) = -\left[1-\frac{\zeta_2\gamma_1(1-n)S_s}{\mathcal{N}}-\frac{\zeta_3\gamma_3(1-n)S_a}{\mathcal{N}}+\zeta_4\left(\frac{\gamma_1(1-n)S_s}{\mathcal{N}}-m-\delta_1-\sigma_1-o\right)\right.\\
&\qquad\left.+\frac{\zeta_5\gamma_3(1-n)S_a}{\mathcal{N}}+\zeta_6(\sigma_1+o)\right],\\
&D^{\alpha}\zeta_5(t) = -\left[1-\frac{\zeta_2\gamma_2(1-n)S_s}{\mathcal{N}}-\frac{\zeta_3\gamma_4(1-n)S_a}{\mathcal{N}}+\frac{\zeta_4\gamma_2(1-n)S_s}{\mathcal{N}}\right.\\
&\qquad\left.+\zeta_5\left(\frac{\gamma_4(1-n)S_a}{\mathcal{N}}-m-\delta_2-\sigma_2-o\right)+\zeta_6(\sigma_2+o)\right],\\
&D^{\alpha}\zeta_6(t) = \zeta_6\, m
\end{aligned}
$$

with the conditions of transversality at time t_f

$$\zeta_1(t_f)=0,\quad \zeta_2(t_f)=0,\quad \zeta_3(t_f)=0,\quad \zeta_4(t_f)=1,\quad \zeta_5(t_f)=1,\quad \zeta_6(t_f)=-1. \tag{8}$$

Moreover, we acquire the optimal control (n^*, o^*, h^*) *as*

$$n^*(t)=\min\left\{\max\left\{n_{\min},\frac{X(t)}{\mathcal{A}}\right\},n_{\max}\right\} \tag{9}$$

with

$$
\begin{aligned}
X(t) = &-\frac{I_aS_a\gamma_4\zeta_3}{\mathcal{N}}+\frac{I_aS_a\gamma_4\zeta_5}{\mathcal{N}}-\frac{I_aS_s\gamma_2\zeta_2}{\mathcal{N}}+\frac{I_aS_s\gamma_2\zeta_4}{\mathcal{N}}-\frac{I_sS_a\gamma_3\zeta_3}{\mathcal{N}}+\frac{I_sS_a\gamma_3\zeta_5}{\mathcal{N}}\\
&-\frac{I_sS_s\gamma_1\zeta_2}{\mathcal{N}}+\frac{I_sS_s\gamma_1\zeta_4}{\mathcal{N}},\\
o^*(t) = &\min\left\{\max\left\{o_{\min},\frac{\zeta_5I_a-I_a\zeta_6+\zeta_4I_s-I_s\zeta_6}{\mathcal{C}}\right\},o_{\max}\right\},
\end{aligned} \tag{10}
$$

$$h^*(t)=\min\left\{\max\left\{h_{\min},\frac{(\zeta_2-\zeta_3)\,S_a}{\mathcal{G}}\right\},h_{\max}\right\}. \tag{11}$$

Proof. For $t\in[0,t_f]$, the principle of maximum Pontryagin makes it possible to obtain at the same time additional equations and transversal conditions. Refs. [23, 24, 27, 28] such that

$$
\begin{aligned}
&D^{\alpha}\zeta_1(t)=-\frac{\partial\mathcal{H}}{\partial S},\quad D^{\alpha}\zeta_2(t)=-\frac{\partial\mathcal{H}}{\partial S_s},\\
&D^{\alpha}\zeta_3(t)=-\frac{\partial\mathcal{H}}{\partial S_a},\quad D^{\alpha}\zeta_4(t)=-\frac{\partial\mathcal{H}}{\partial I_s},\\
&D^{\alpha}\zeta_5(t)=-\frac{\partial\mathcal{H}}{\partial I_a},\quad D^{\alpha}\zeta_6(t)=-\frac{\partial\mathcal{H}}{\partial R}.
\end{aligned} \tag{12}
$$

Equations (12) and (8) describe the conditions required according to a Hamiltonian defined above.

These conditions produce an array of fractional differential equations, built on variables S, S_s, S_a, I_s, I_a, controls n, o, h and Lagrange multiplying ζ_i, to solve analytically, numerically, or even both.

In addition, the optimal controls (n^*, o^*, h^*) can be determined from the optimal conditions

$$
\begin{aligned}
\frac{\partial\mathcal{H}}{\partial n}=0\quad\Rightarrow\quad &\mathcal{A}n+\zeta_2\left(\frac{\gamma_1S_sI_s}{\mathcal{N}}+\frac{\gamma_2S_sI_a}{\mathcal{N}}\right)+\zeta_3\left(\frac{\gamma_3S_aI_s}{\mathcal{N}}+\frac{\gamma_4S_aI_a}{\mathcal{N}}\right)+\zeta_4\left(-\frac{\gamma_1S_sI_s}{\mathcal{N}}-\frac{\gamma_2S_sI_a}{\mathcal{N}}\right)\\
&+\zeta_5\left(-\frac{\gamma_3S_aI_s}{\mathcal{N}}-\frac{\gamma_4S_aI_a}{\mathcal{N}}\right)=0,
\end{aligned}
$$

$$\frac{\partial \mathcal{H}}{\partial o} = 0 \quad \Rightarrow \quad \mathcal{C}o - \zeta_4 I_s - \zeta_5 I_a + \zeta_6(I_a + I_s) = 0,$$
$$\frac{\partial \mathcal{H}}{\partial h} = 0 \quad \Rightarrow \quad \mathcal{G}h + \zeta_2 S_a - \zeta_3 S_a = 0,$$

we obtain

$$n^*(t) = \frac{1}{\mathcal{A}}\left(-\frac{I_a S_a \gamma_4 \zeta_3}{\mathcal{N}} + \frac{I_a S_a \gamma_4 \zeta_5}{\mathcal{N}} - \frac{I_a S_s \gamma_2 \zeta_2}{\mathcal{N}} + \frac{I_a S_s \gamma_2 \zeta_4}{\mathcal{N}} - \frac{I_s S_a \gamma_3 \zeta_3}{\mathcal{N}} + \frac{I_s S_a \gamma_3 \zeta_5}{\mathcal{N}} - \frac{I_s S_s \gamma_1 \zeta_2}{\mathcal{N}} + \frac{I_s S_s \gamma_1 \zeta_4}{\mathcal{N}}\right),$$
$$o^*(t) = \frac{\zeta_5 I_a - I_a \zeta_6 + \zeta_4 I_s - I_s \zeta_6}{\mathcal{C}},$$
$$h^*(t) = -\frac{(\zeta_2 - \zeta_3)\, S_a}{\mathcal{G}}.$$

By the bounds in $\mathcal{U}_{n,o,h}$ of the controls, we can easily obtain n^*, o^*, and h^* are given by (9), (10), and (11) in the form of system (12). ■

9. Numerical simulation

In this section, we are going to solve numerically the optimum control problem for our $SS_sS_aI_sI_aR$ as a fractional order model. We analyze and compare the numerical outcome of the control strategy below. The technique consists of combining all the aforementioned controls. In other words, we are going to combine awareness and prevention through media and education and treat and raise public awareness of the health risks of smoking in general, particularly in the case of TB.

The adjoint system is solved by using the method of finite differences over time, with $T = 100$ days and final conditions $\zeta_1(n) = 0$, $\zeta_2(n) = 0$, $\zeta_3(n) = 0$, $\zeta_4(n) = 1$, $\zeta_5(n) = 1$, $\zeta_6(0) = -1$. The n, o, and h controls are considered bounded and weightings within the objective function are estimated at $\mathcal{A} = 10$, $\mathcal{C} = 1$, and $\mathcal{G} = 1$. In this article, all graphs for condition variables are in logarithmic form. The code is created and compiled into Matlab using the data below.

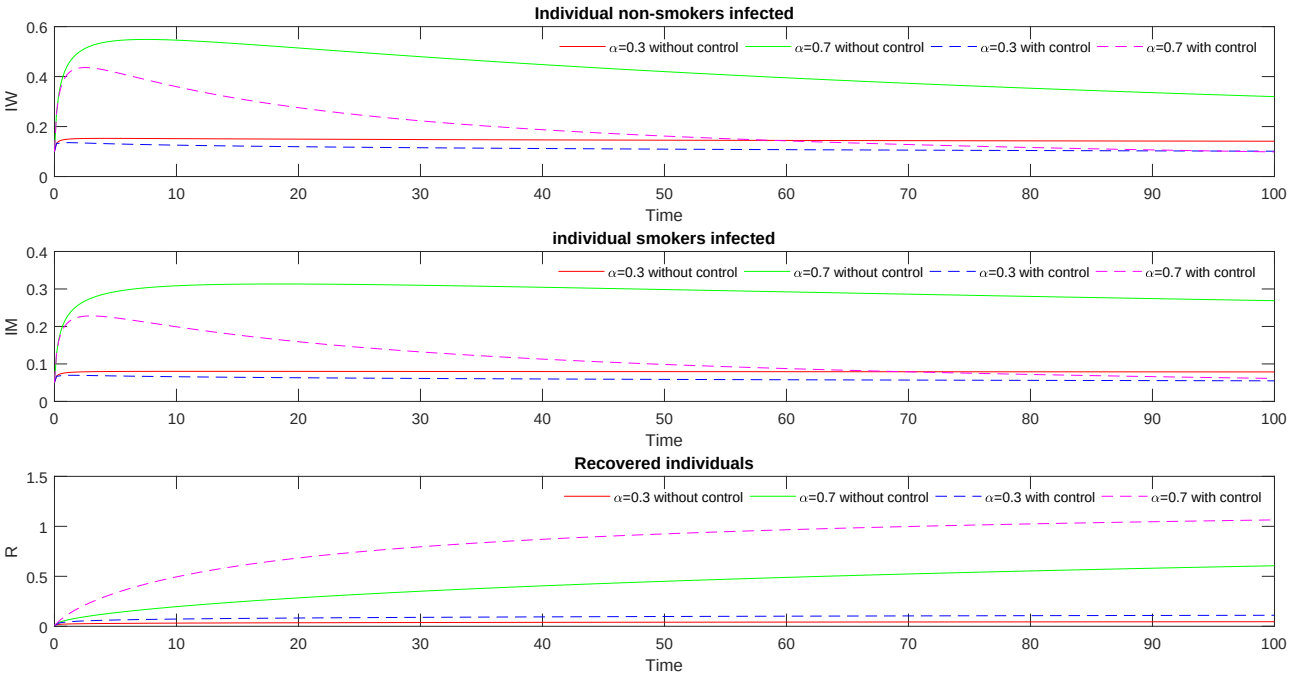

Fig. 3. The evolution of the number of smokers and non-smokers infected and recovered individuals with and without all controls.

Figure 3 shows the behavior of approximate solutions $I_s(t)$, $I_a(t)$, and $R(t)$ at $\alpha = 0.3$ and $\alpha = 0.7$ with and without controls. Figure 4 shows the approximate solutions $S_s(t)$ and $S_a(t)$ at $\alpha = 0.3$ and $\alpha = 0.7$ in the two cases with and without controls.

In this strategy, we used all $n(t)$, $o(t)$ and $h(t)$ controls simultaneously to improve statistical performance. According to Figure 3, once we applied this strategy, we noted a decrease in the number of smokers, and non-smokers infected with Tuberculosis which led to a relative balance of both smokers and non-smokers infected after post-awareness and treatment, thereby reducing the relative spread of the virus and a sharp rise in the number of recovered individuals, that after 100 days, the number of smokers infected without control is 0.09 ($\alpha = 0.3$) to 0.05 ($\alpha = 0.3$) with control, and the number of smokers infected without control is 0.29 ($\alpha = 0.7$) to 0.05 ($\alpha = 0.7$) with control. The number of non-smokers infected without control is 0.15 ($\alpha = 0.3$) to 0.1 ($\alpha = 0.3$) with control, and the number of non-smokers infected without control is 0.35 ($\alpha = 0.7$) to 0.1 ($\alpha = 0.7$) with control. Concerning the number of recovered individuals, we notice that the number of this last is 0.08 ($\alpha = 0.3$) to 0.1 ($\alpha = 0.3$) with control, and the number of recovered without control is 0.6 ($\alpha = 0.7$) to 1.1 ($\alpha = 0.7$) with control.

In addition, fractional derivatives play an important role in the description of memory effects in dynamic systems. As α limits to 1, memory effects are decreased. Moreover, fractional-derived order α plays a part in the delay in ordinary differential models. Figures 3 and 4 show that when the derivative order α is reduced by 1, the memory effect of the system increases. As a result, the number of infected smokers and non-smokers decreases for an extended period of time, and the same thing for susceptible smokers and non-smokers.

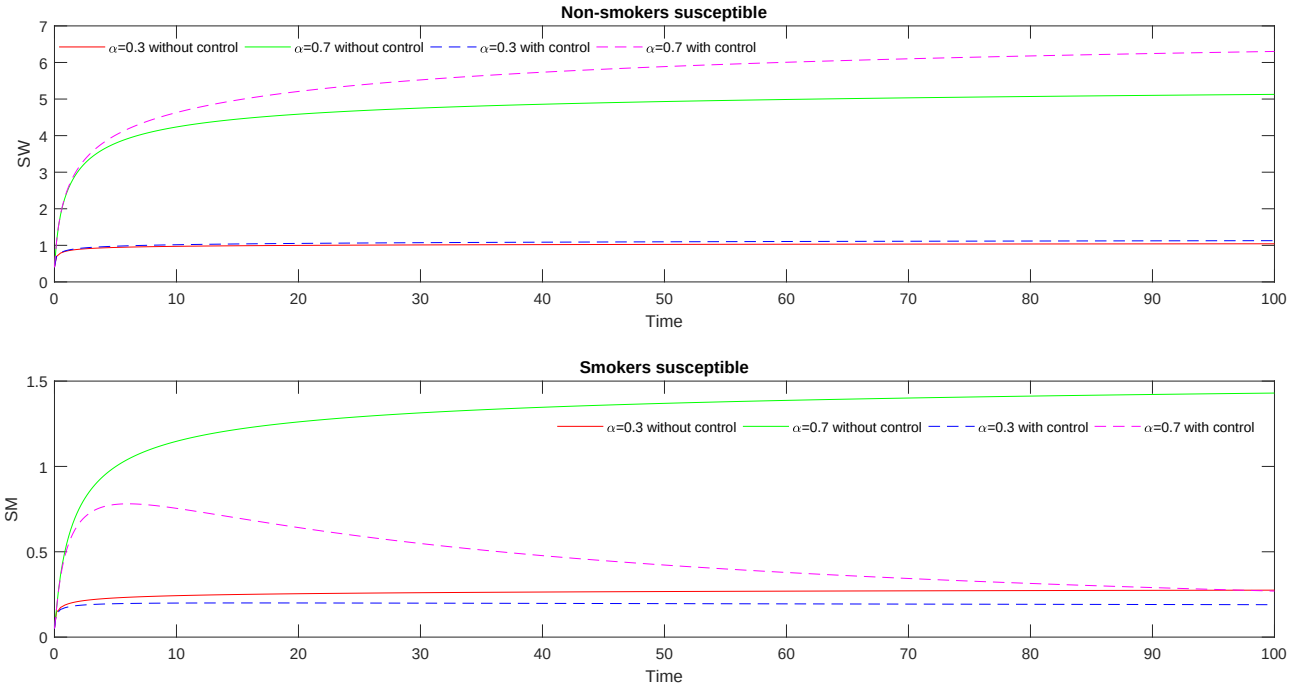

Fig. 4. The evolution of the number of smokers and non-smokers susceptible to tuberculosis and recovered individuals with and without all controls.

10. Conclusion

In conclusion, our research focused on a fractional-order mathematical model for tuberculosis (TB) dynamics among smokers. This model aimed to protect vulnerable populations from TB and identified a disease-free equilibrium point, E_f. By calculating the basic reproduction number, $\mathcal{R}_0$, we analyzed the local stability of this equilibrium. Our model incorporated three controls: individual contact restriction, treatment, and sensitization. Using the Pontryagin maximum principle, we determined

optimal control strategies and validated them through iterative numerical simulations in Matlab. This study advances TB management and suggests future research directions using various mathematical approaches.

Acknowledgements

The authors would like to thank the reviewer for taking the time to contribute to the improvement of this research.

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Задача аналізу та оптимального керування дробовою математичною моделлю туберкульозу з урахуванням паління

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Ця стаття присвячена дослідженню математичної моделі дробового порядку ТБ. Вона описує динаміку поширення туберкульозу серед курців. Метою цього дослідження є захист вразливих людей від вірусу. За результатами опитування шукана модель має точку рівноваги: точку рівноваги без захворювань E_f. Також досліджуємо локальну стійкість цієї точки рівноваги моделі, використовуючи базове число відтворення $\mathcal{R}_0$, яке розраховане згідно з методом нового покоління. У запропонованій моделі включено три елементи керування, які представляють: обмеження індивідуальних контактів, лікування та сенсибілізації. Ця стаття спрямована на зменшення кількості інфікованих курців і некурців за допомогою оптимальної стратегії контролю та дробової похідної. Принцип максимуму Понтрягіна використовується для опису оптимальних керувань із дробовими значеннями у часі, які отримані за Капуто, а оптимальна система розв'язується ітераційно. Чисельне моделювання представлено відповідно до методу, який представлений у Matlab.

Ключові слова: *дробова похідна Капуто; оптимальне керування; туберкульоз; куріння; заразний вірус; локальна стійкість; динамічна система; інфекційні захворювання; стійкість; вільна рівновага; максимум Понтрягіна.*