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Optimal control problem of a discrete spatiotemporal prey–predator three-species fishery model

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In this work, we discuss a spatiotemporal discrete prey–predator model. It consists of three compartments: prey, predator, and super-predator. The proposed model describes the interaction between prey, predator, and super-predator in a region with a discrete displacement. We also provide research on appropriate regional control strategies. The controls are applied to the predator and the super-predator, respectively; they represent catching these in measured quantities in a space and a time chosen. The aim is to increase the number of prey and reduce the number of predators, restore the food chain system, and ensure its sustainability. Finally, we provide graphical visuals and numerical simulations to support our analytical findings.

Keywords: *prey–predator; spatiotemporal discrete model; optimal control, Pontryagin's maximum principle.*

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1. Introduction

For years, scientists have been warning of the coming catastrophe due to overfishing, which is harvesting sea bounties at rates greater than the ability of marine life to reproduce and restore its natural numbers. These warnings accompany efforts by international leaders.

In response to the decline in desirable fish populations, commercial fishing fleets are beginning to reach farther and catch other species of fish below the food pyramid, with disastrous effects on the nutritional balance. For instance, a decrease in the number of fish that consume algae, which keeps corals clean and healthy, has an impact on coral reefs. Fishing for vegetarian fish, directly or indirectly, weakens coral reefs and reduces their resilience to climate change. Fishing debris directly destroys coral reefs. Trawling scoops everything that falls in the net's path, not just shrimp, tuna, dolphins, turtles, sharks, and birds. These species are in danger of extinction due to accidental fishing. Illustrations of overfishing are found in places like the North Sea, Newfoundland's Grand Pax, and the East China Sea. Overfishing has proven to be catastrophic for fish stocks at these sites as well as for product-dependent fishing communities. Residents of the Upper Adriatic Sea have been hunting for millennia, just like other extractive industries [1,2]. The increase in fishing blocks big groups of sulphur fish from leaving the Gulf of Trieste. Fishermen in Santa Croce, Contofilo, and Barcola caught the last big tuna catch in 1954. Peruvian coastal balmia (anchovies) fisheries collapsed in the 1970s after overfishing, and the El Nino season greatly depleted the balmia because of its waterways. Balmia was a great natural treasure in Peru. In fact, in 1971, 10.2 million metric tons of balmia were produced. Anyhow, the Peruvian fleet's catch did not exceed four million tons in the following five years, which constituted a significant loss for the Peruvian economy [3]. Fisheries in the Irish Sea and other sites are overfishing, causing such an actual breakdown, according to the official Biodiversity Action Plan of the UK Government. The UK has formed elements in this plan to try to renovate fisheries, but the expansion of the worldwide population and increasing requests for fish have linked to a point where food demand jeopardizes the stability, if not the viability, of these fisheries, of the species [4] etc.

Mathematical modeling has proven its significant utility when it comes to describing the dynamics of natural phenomena, particularly in ecology [5]. As an important and essential tool for handling many real-life problems and for understanding ecological systems, it gives us the opportunity to predict the conduct of samples of the investigated population. This is the primary reason for the growing advantage of many researchers in this field. As a consequence, a huge improvement has been observed in ecological mathematical models. In this context, precisely in population dynamics, particularly in prey–predator dynamics interplay in aquatic creatures, modeling has become increasingly important. Predators consume prey, while the prey is the organism that is being hunted. Predators rely on their prey to survive, failing to get food, leading to their demise.

The renowned Lotka–Volterra model is the first prey predator model, whose mathematical formulation is based on a seminal work by Lotka in 1925 and Volterra in 1926 [6,7]. It is a nonlinear differential equation system that represents the changes in population size over time. This model has been embraced and expanded upon in multiple works [1,5,6,8–10]. Applying bifurcation theory and differential algebra, the authors investigated the dynamic behaviors of a biological economic predator–prey model in which the prey is gaining. In [11], the authors studied prey predator dynamics, developed a model that extends the functional responses of Holling, and demonstrated outcomes relating to both local and global features as well as oscillations. The authors of [12] provided a mathematical fishing model with the goal of optimizing human fishermen's interests while maintaining biological equilibrium. The authors of [13] investigated the global stability of a system of "Holling–Tanner" in a confined environment. In [14], the authors developed a bio-economics model that applies to a number of regions where populations are mobilized. Their quantitative analysis assisted them in determining the best fishing effort, the best way to distribute resources, and a set of functional management measures. In [15], to better understand the consequences of juvenile predator predation on immature prey, the authors have developed a prey predator model of "the Beddington–DeAngelis" type functional response. The dynamics of a prey–predator model with disease in a super-predator were examined by the authors in [16]. The concept of a super-predator was examined in [17], wherein the authors examined a prey-predator model from an economic standpoint.

In this paper, we aim to extend the study of the optimal control problem of the discrete spatiotemporal for the prey-predator three-species fishery, the statistical data are composed at discrete time and discrete space, as well as being highly immediate, highly adequate, and highly precise to describe the fish population. By using discrete spatiotemporal modeling, we may avoid some mathematical complexities, like the existence of solutions.

This work is arranged as seen below: In Section 2, we show you a discrete prey–predatory mathematical model. In Section 3, we give you the optimal control problem about the suggested model. Numerical results to support the proposed model are provided in Section 4. Finally, we discuss and show a conclusion in Section 5.

2. Mathematical model

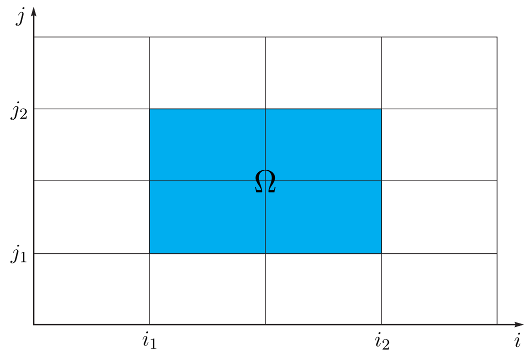
The following system describes a spatiotemporal prey–predator discrete model of three compartements: $x_t^{i,j}$ prey, $y_t^{i,j}$ predator and $z_t^{i,j}$ super-predator in a position (i,j) at time t,

$$x_{t+1}^{i,j} = x_t^{i,j} + r x_t^{i,j}\left(1 - \frac{x_t^{i,j}}{k}\right) - \frac{\alpha x_t^{i,j} y_t^{i,j}}{a + x_t^{i,j}} - \frac{m (x_t^{i,j})^2 z_t^{i,j}}{b + (x_t^{i,j})^2} + \alpha_1 \nabla^2 x_t^{i,j}, \tag{1}$$

$$y_{t+1}^{i,j} = y_t^{i,j} + \frac{\beta x_t^{i,j} y_t^{i,j}}{a + x_t^{i,j}} - \frac{n (y_t^{i,j})^2 z_t^{i,j}}{b + (y_t^{i,j})^2} - d_1 y_t^{i,j} + \alpha_2 \nabla^2 y_t^{i,j}, \tag{2}$$

$$z_{t+1}^{i,j} = z_t^{i,j} + \frac{n_1 (y_t^{i,j})^2 z_t^{i,j}}{b + (y_t^{i,j})^2} + \frac{m_1 (x_t^{i,j})^2 z_t^{i,j}}{b_1 + (x_t^{i,j})^2} - d_2 z_t^{i,j} + \alpha_3 \nabla^2 z_t^{i,j} \tag{3}$$

in the location $\Omega = \{(i,j) | I_1 \leqslant i \leqslant I_2 \text{ and } J_1 \leqslant j \leqslant J_2\}$ as follow



where $N_t^{i,j} = x_t^{i,j} + y_t^{i,j} + z_t^{i,j}$ and the discrete diffusion is described [18]:

$$\nabla^2 x_k^{i,j} = x_k^{i-1,j} + x_k^{i+1,j} + x_k^{i,j-1} + x_k^{i,j+1} - 4x_k^{i,j},$$
$$\nabla^2 y_k^{i,j} = y_k^{i-1,j} + y_k^{i+1,j} + y_k^{i,j-1} + y_k^{i,j+1} - 4y_k^{i,j},$$
$$\nabla^2 z_k^{i,j} = z_k^{i-1,j} + z_k^{i+1,j} + z_k^{i,j-1} + z_k^{i,j+1} - 4z_k^{i,j},$$

with initial conditions $x_0^{i_0,j_0} \geqslant 0$, $y_0^{i_0,j_0} \geqslant 0$ and $z_0^{i_0,j_0} \geqslant 0$. Here, we look at b and b_1 as saturation constants in the relative functional reactions of the predator and prey in a generalist predator. In order to simplify computation, we take into consideration $b_1 = b$, then $n_1 < n$, $m_1 < m$ and $\beta < \alpha$. None of the parameters are negative and defined in (Table 1).

In terms of prey density $x_t^{i,j}$: we suppose the prey increases surly, with a hauling capacity of k and a steady rate of expansion of r. Interactions with predators and super-predators reduce the availability of this prey. Furthermore, prey is a preferred diet of predators, as shown by a holing type (II) functional reaction (furthermore known as a Michaelis Menten functional reaction) defined by $\frac{ax_t^{i,j}y_t^{i,j}}{a+x_t^{i,j}}$ and the super-predator meals on the prey using a holling type (III) functional reaction, as indicated by $\frac{(x_t^{i,j})^2 z_t^{i,j}}{b+(x_t^{i,j})^2}$.

In terms of predator density $y_t^{i,j}$: this population grows by feeding on prey that has a holing type (II). The predator, otherwise, is predated by a super-predator with holing type (III) functional reaction, as shown by $\frac{(x_t^{i,j})^2 z_t^{i,j}}{b+(x_t^{i,j})^2}$ as well as a drop in the natural death rate d_1. The term α_2 denoted the rate of predator entering from different sides.

In terms of super-predators density $z_t^{i,j}$: this population expands as a result of predation on the prey, which is illustrated by $\frac{(mx_t^{i,j})^2 z_t^{i,j}}{b+(x_t^{i,j})^2}$ and predating the predator with the term $\frac{(mx_t^{i,j})^2 z_t^{i,j}}{b+(x_t^{i,j})^2}$. This population is decreasing due to natural mortality, at a rate of d_2, and by harvesting. The term α_2 denoted the rate of super-predator entering from different sides.

Table 1. The description of parameters used for the definition of discrete time systems 1–3.

Parameter	Description
r	The prey's intrinsic growth rate.
k	The prey's environmental carrying capacity.
α	Predator-to-prey capture rate.
m	Rate at which the super-predator captures its victim.
n	The super-predator's capture rate per predator.
a, b	Constants for half saturation.
d_1	Natural predator Death rate.
d_2	Super predator Death rate in the wild.
β	Rate of consumption by predators on prey.
n_1	Exceptional rate of ingestion by predators.

3. An optimal control problem

The optimal control used in the model's aim is to reduce the number of predators and super-predators to return the spaces on which we focus to a normal situation, and for this, we introduce the following control variables: targeting the interaction's original location between the species, a regional fishing management effort has been chosen as the approach. Furthermore, to keep the differential chain structure and guarantee ecological durability, we employ a harvesting function technique. As a result, based on time, we present two regional control mechanisms, $t \in [0;T]$ and location $\omega \subset \Omega$.

The regional controls $u_{1,k}$, $u_{2,k}$ describe the increased emphasis on catchability q to gather predators (super-predators), respectively, while reducing the total number of predators and super-predators, but concentrating on the conservation of predators and super-predators that pose a threat to prey over time, in order to preserve a distinct chain system while ensuring environmental sustainability.

As a result, the mathematical model with controls is described below multi-patches system of difference equations:

$$x^{i,j}_{t+1} = x^{i,j}_t + r x^{i,j}_t \left(1 - \frac{x^{i,j}_t}{k}\right) - \frac{\alpha x^{i,j}_t y^{i,j}_t}{a + x^{i,j}_t} - \frac{m (x^{i,j}_t)^2 z^{i,j}_t}{b + (x^{i,j}_t)^2} + \alpha_1 \nabla^2 x^{i,j}_t, \tag{4}$$

$$y^{i,j}_{t+1} = y^{i,j}_t + \frac{\beta x^{i,j}_t y^{i,j}_t}{a + x^{i,j}_t} - \frac{n (y^{i,j}_t)^2 z^{i,j}_t}{b + (y^{i,j}_t)^2} - d_1 y^{i,j}_t + \alpha_2 \nabla^2 y^{i,j}_t - \mathbb{1}_w(i,j) q E u^{i,j}_{1,t} y^{i,j}_t, \tag{5}$$

$$z^{i,j}_{t+1} = z^{i,j}_t + \frac{n_1 (y^{i,j}_t)^2 z^{i,j}_t}{b + (y^{i,j}_t)^2} + \frac{m_1 (x^{i,j}_t)^2 z^{i,j}_t}{b + (x^{i,j}_t)^2} - d_2 z^{i,j}_t + \alpha_3 \nabla^2 z^{i,j}_t - \mathbb{1}_w(i,j) q E u^{i,j}_{2,t} z^{i,j}_t, \tag{6}$$

with $w = [I'_1, I'_2] \times [J'_1, J'_2]$ the space chosen to apply the controls.

We are interested in controlling the predators and super-predators. The challenge is to minimize the objective functional $J(u_1, u_2, t)$ specified by

$$J(u_1, u_2, t) = \sum_{t=0}^{T} \sum_{i=I_1}^{I_2} \sum_{j=J_1}^{J_2} A x^{i,j}_k - B y^{i,j}_k - C z^{i,j}_k + \frac{\tau_1}{2} (u^{i,j}_{1,t})^2 + \frac{\tau_2}{2} (u^{i,j}_{2,t})^2$$

subject to system. Here A, B and C are positive constants to keep a balance in the size of $x^{i,j}_k$, $y^{i,j}_k$ and $z^{i,j}_k$ respectively. The positive weight parameters linked with the controls in the objective functional are τ_1 and τ_2, $u^{i,j}_{1,t}$ and $u^{i,j}_{2,t}$.

Our aim is to minimize the predators and super-predators and maximize the prey. In other words, we are looking for the best possible control. $(u^{i,j}_{1,t})^*$ and $(u^{i,j}_{2,t})^*$ such that

$$J\big((u^{i,j}_{1,t})^*; (u^{i,j}_{2,t})^*\big) = \min \big\{ J(u^{i,j}_{1,t}; u^{i,j}_{2,t}) | (u^{i,j}_{1,t}; u^{i,j}_{2,t}) \in U \big\}.$$

where U denotes the set of allowable controls defined by $U = \big\{ (u_1, u_2) = ((u^{i,j}_{1,t})^*, (u^{i,j}_{2,t})^*) | 0 \leqslant u_1^{\min} \leqslant u_1 \leqslant u_1^{\max} \leqslant 1, 0 \leqslant u_2^{\min} \leqslant u_2 \leqslant u_2^{\max} \leqslant 1; t \in \{0, \dots, T-1\}, i \in \{0, \dots, I-1\} \text{ and } j \in \{0, \dots, J-1\} \big\}$.

The following theorem proves that the presence of an optimal control for the issue is a necessary condition.

Theorem 1. *For the state equations linked with the optimal control issue in the system, there exists a controls* $((u^{i,j}_{1,t})^*; (u^{i,j}_{2,t})^*) \in U$ *such that*

$$J\big((u^{i,j}_{1,t})^*; (u^{i,j}_{2,t})^*\big) = \min \big\{ J(u^{i,j}_{1,t}; u^{i,j}_{2,t}) | (u^{i,j}_{1,t}; u^{i,j}_{2,t}) \in U \big\}.$$

Proof. There are just a specific number of time steps, $x^{i,j}_t = (x^{0,0}_t, x^{1,1}_t, \dots, x^{I,J}_t)$, $y^{i,j}_t = (y^{0,0}_t, y^{1,1}_t, \dots, y^{I,J}_t)$ and $z^{i,j}_t = (z^{0,0}_t, z^{1,1}_t, \dots, z^{I,J}_t)$ are all evenly bounded $(u_1, u_2) \in U$. Thus $J(u_1, u_2)$ is uniformly bounded for all (u_1, u_2) in the control set U. Since $J(u_1, u_2)$ is bounded, $\inf\limits_{(u_1,u_2) \in U} J(u_1, u_2)$ is finite, and there exists a sequence $(u_{1,t}, u_{2,t})$ in the set of control such that $\lim\limits_{t \to \infty} J(u_{1,t}, u_{2,t}) = \inf\limits_{(u_1,u_2) \in U} J(u_1, u_2)$ and corresponding sequences of states $(x^{i,j}_t, y^{i,j}_t, z^{i,j}_t)$. Since there is a finite number of uniformly bounded sequences, there exist $((u^{i,j}_{1,t})^*; (u^{i,j}_{2,t})^*) \in U$ and $x^{i,j}_t, y^{i,j}_t, z^{i,j}_t \in \mathbb{R}^{T+1}$ such that on a subsequence, $u_{1,t} \to (u^{i,j}_{1,t})^*$, $u_{2,t} \to (u^{i,j}_{2,t})^*$, $x^{i,j}_t \to (x^{i,j}_t)^*$,

$y_t^{i,j} \to (y_t^{i,j})^*$ and $z_t^{i,j} \to (z_t^{i,j})^*$. Finally, because of the finite dimension structure of the system with the control and the objective function $J(u_1,u_2)$, $((u_{1,t}^{i,j})^*;(u_{2,t}^{i,j})^*)$ is an optimal corresponding states $(x_t^{i,j})^*,(y_t^{i,j})^*,(z_t^{i,j})^*$. Therefore in $\inf\limits_{(u_1,u_2)\in U} J(u_1,u_2)$ is achieved. ■

We develop required criteria for our optimal control using Pontryagin's Maximum Principle [8]. To this end, the Hamiltonian is defined as:

$$
\begin{aligned}
\mathcal{H} &= \sum_{t=0}^{T}\sum_{i=I1}^{I2}\sum_{j=J1}^{J2}\mathcal{H}_t^{i,j}\\
&= \sum_{t=0}^{T}\sum_{i=I1}^{I2}\sum_{j=J1}^{J2}\Bigg(Ax_t^{i,j} - By_t^{i,j} - Cz_t^{i,j} + \frac{\tau_1}{2}(u_{1,t}^{i,j})^2 + \frac{\tau_2}{2}(u_{2,t}^{i,j})^2\\
&\quad + \lambda_{1,t+1}^{i,j}(x_t^{i,j} + rx_t^{i,j}\left(1-\frac{x_t^{i,j}}{k}\right) - \frac{\alpha x_t^{i,j}y_t^{i,j}}{a+x_t^{i,j}} - \frac{m(x_t^{i,j})^2z_t^{i,j}}{b+(x_t^{i,j})^2} + \alpha_1\nabla^2x_t^{i,j})\\
&\quad + \lambda_{2,t+1}^{i,j}\left(y_t^{i,j} + \frac{\beta x_t^{i,j}y_t^{i,j}}{a+x_t^{i,j}} - \frac{n(y_t^{i,j})^2z_t^{i,j}}{b+(y_t^{i,j})^2} - d_1y_t^{i,j} + \alpha_2\nabla^2y_t^{i,j} - \mathbb{1}_w(i,j)qEu_{1,t}^{i,j}y_t^{i,j}\right)\\
&\quad + \lambda_{3,t+1}^{i,j}\left(z_t^{i,j} + \frac{n_1(y_t^{i,j})^2z_t^{i,j}}{b+(y_t^{i,j})^2} + \frac{m_1(x_t^{i,j})^2z_t^{i,j}}{b+(x_t^{i,j})^2} - d_2z_t^{i,j} + \alpha_3\nabla^2z_t^{i,j} - \mathbb{1}_w(i,j)qEu_{2,t}^{i,j}z_t^{i,j}\right)\Bigg).
\end{aligned}
$$

Theorem 2 (Necessary Conditions). *Given an optimal controls $\left((u_{1,t}^{i,j})^*,(u_{2,t}^{i,j})^*\right)$ and solutions $(x_t^{i,j})^*$, $(y_t^{i,j})^*$ and $(z_t^{i,j})^*$, there exists $(\zeta_k)_{t+1}^{i,j}$, $k=1,2,3$ the adjoint variables meets this requirements*

$$
\begin{aligned}
(\zeta_1)_{t+1}^{i,j} &= -A - \lambda_{1,t+1}^{i,j}\left(\left(1-\frac{x_t^{i,j}}{k}\right) + r\frac{2x_t^{i,j}}{k} + \frac{\alpha(y_t^{i,j}b_1 - (x_t^{i,j})^2y_t^{i,j})}{(b_1+(x_t^{i,j})^2)^2}\right)\\
&\quad - \lambda_{2,t+1}^{i,j}\frac{\beta y_t^{i,j}(b_1+(x_t^{i,j}) - 2\beta(x_t^{i,j})^2y_t^{i,j}}{\left(b_1+(x_t^{i,j})^2\right)^2} - \lambda_{3,t+1}^{i,j}\frac{2m_1x_t^{i,j}z_t^{i,j}(b+(x_t^{i,j})^2) - 2m_1(x_t^{i,j})^3z_t^{i,j}}{\left(b+(x_t^{i,j})^2\right)^2}\\
&\quad - \alpha_1\left(\lambda_{1,t+1}^{i-1,j} + \lambda_{1,t+1}^{i+1,j} + \lambda_{1,t+1}^{i-1,j} + \lambda_{1,t+1}^{i+1,j} - 4\lambda_{1,t+1}^{i,j}\right),\\
(\zeta_2)_{t+1}^{i,j} &= B + \lambda_{1,t+1}^{i,j}\frac{\alpha x_t^{i,j}}{(b_1+(x_t^{i,j})^2)} - \lambda_{2,t+1}^{i,j}\left(1 + \frac{\beta x_t^{i,j}}{a+(x_t^{i,j})} - d - \frac{2nb_1y_t^{i,j}z_t^{i,j}}{\left(b_1+(y_t^{i,j})^2\right)^2}\right) - \lambda_{3,t+1}^{i,j}\frac{2n_1by_t^{i,j}z_t^{i,j}}{\left(b+(y_t^{i,j})^2\right)^2}\\
&\quad - \alpha_2\left(\lambda_{2,t+1}^{i-1,j} + \lambda_{2,t+1}^{i+1,j} + \lambda_{2,t+1}^{i-1,j} + \lambda_{2,t+1}^{i+1,j} - 4\lambda_{2,t+1}^{i,j}\right) + \lambda_{2,t+1}^{i,j}\mathbb{1}_w(i,j)qEu_{1,t}^{i,j}y_t^{i,j}\\
(\zeta_3)_{t+1}^{i,j} &= C + \lambda_{2,t+1}^{i,j}\frac{n(y_t^{i,j})^2z_t^{i,j}}{b_1+(y_t^{i,j})^2} - \lambda_{3,t+1}^{i,j}\left(1 + \frac{n_1(y_t^{i,j})^2}{(b+(y_t^{i,j})^2} + \frac{m_1(x_t^{i,j})^2z_t^{i,j}}{(b+(x_t^{i,j})^2} - d_2\right)\\
&\quad - \alpha_3\left(\lambda_{3,t+1}^{i-1,j} + \lambda_{3,t+1}^{i+1,j} + \lambda_{3,t+1}^{i-1,j} + \lambda_{3,t+1}^{i+1,j} - 4\lambda_{3,t+1}^{i,j}\right) + \lambda_{3,t+1}^{i,j}\mathbb{1}_w(i,j)qEu_{2,t}^{i,j}z_t^{i,j}.
\end{aligned}
$$

With transversality conditions $(\zeta_1)_{t+1}^{i,j} = A$, $(\zeta_2)_{t+1}^{i,j} = -B$ and $(\zeta_3)_{t+1}^{i,j} = -C$.

Furthermore, the optimal control $\left((u_{1,t}^{i,j})^*,(u_{2,t}^{i,j})^*\right)$ is given for $(i,j)\in w = [I_1',I_2']\times[J_1',J_2']$ and $t=1,\ldots,T$ by

$$
(u_{1,t}^{i,j})^* = \min\left\{\max\left(u_1^{\min}, -\frac{2qEy_t^{i,j}}{\tau_1}\right), u_1^{\max}\right\} \text{ and } (u_{2,t}^{i,j})^* = \min\left\{\max\left(u_2^{\min}, -\frac{2qEz_t^{i,j}}{\tau_2}\right), u_2^{\max}\right\},
$$

and $u_{1,t}^{i,j} = u_{1,t}^{i,j} = 0$, if $(i,j)\in\omega$.

Proof. Using the Maximum Principle of Pontryagin [8], and setting $(x_t^{i,j})^*$, $(y_t^{i,j})^*$, $(z_t^{i,j})^*$ and $u_1 = (u_{1,t}^{i,j})^*$, $u_2 = (u_{2,t}^{i,j})^*$ we acquire the following adjoint equations

$$
\Delta(\zeta_1)_{t+1}^{i,j} = -\frac{\partial\mathcal{H}}{\partial x_t^{i,j}} = -\left[\frac{\partial\mathcal{H}_t^{i,j}}{\partial x_t^{i,j}} + \frac{\partial\mathcal{H}_t^{i+1,j}}{\partial x_t^{i,j}} + \frac{\partial\mathcal{H}_t^{i-1,j}}{\partial x_t^{i,j}} + \frac{\partial\mathcal{H}_t^{i,j+1}}{\partial x_t^{i,j}} + \frac{\partial\mathcal{H}_t^{i,j-1}}{\partial x_t^{i,j}}\right]
$$

$$
\begin{aligned}
&= -\Bigg(A + \lambda_{1,t+1}^{i,j}\Bigg(\Bigg(1-\frac{x_t^{i,j}}{k}\Bigg) - r\frac{x_t^{i,j}}{k} - \frac{\alpha y_t^{i,j}(b_1+(x_t^{i,j}) - 2(x_t^{i,j})^2\alpha y_t^{i,j}}{(b_1+(x_t^{i,j})^2)^2}\Bigg)\Bigg)\\
&\quad + \lambda_{2,t+1}^{i,j}\frac{\beta y_t^{i,j}(b_1+(x_t^{i,j}) - 2\beta(x_t^{i,j})^2 y_t^{i,j}}{(b_1+(x_t^{i,j})^2)^2} + \lambda_{3,t+1}^{i,j}\frac{2m_1 x_t^{i,j} z_t^{i,j}(b+(x_t^{i,j})^2) - 2m_1(x_t^{i,j})^3 z_t^{i,j}}{(b+(x_t^{i,j})^2)^2}\\
&\quad + \alpha_1\big(\lambda_{1,t+1}^{i-1,j} + \lambda_{1,t+1}^{i+1,j} + \lambda_{1,t+1}^{i-1,j} + \lambda_{1,t+1}^{i+1,j} - 4\lambda_{1,t+1}^{i,j}\big)\Bigg)\\
\Delta(\zeta_2)_{t+1}^{i,j} &= -\frac{\partial\mathcal{H}}{\partial y_t^{i,j}} = -\left[\frac{\partial\mathcal{H}_t^{i,j}}{\partial y_t^{i,j}} + \frac{\partial\mathcal{H}_t^{i+1,j}}{\partial y_t^{i,j}} + \frac{\partial\mathcal{H}_t^{i-1,j}}{\partial y_t^{i,j}} + \frac{\partial\mathcal{H}_t^{i,j+1}}{\partial y_t^{i,j}} + \frac{\partial\mathcal{H}_t^{i,j-1}}{\partial y_t^{i,j}}\right]\\
&= -\Bigg(-B - \lambda_{1,t+1}^{i,j}\frac{\alpha x_t^{i,j}}{(b_1+(x_t^{i,j})^2)} + \lambda_{2,t+1}^{i,j}\Bigg(1 + \frac{\beta x_t^{i,j}}{(a+(x_t^{i,j})} - d\\
&\quad - \frac{2n y_t^{i,j} z_t^{i,j}(b_1+(y_t^{i,j})^2)^2) - 2n(y_t^{i,j})^2)^2 z_t^{i,j}}{(b_1+(y_t^{i,j})^2)^2}\Bigg)\\
&\quad + \lambda_{3,t+1}^{i,j}\frac{2n_1 y_t^{i,j} z_t^{i,j}(b+(y_t^{i,j})^2)^2) - 2n_1(y_t^{i,j})^2)^2 z_t^{i,j}}{(b+(y_t^{i,j})^2)^2}\\
&\quad + \alpha_2\big(\lambda_{2,t+1}^{i-1,j} + \lambda_{2,t+1}^{i+1,j} + \lambda_{2,t+1}^{i-1,j} + \lambda_{2,t+1}^{i+1,j} - 4\lambda_{2,t+1}^{i,j}\big) - \lambda_{2,t+1}^{i,j}\mathbb{1}_w(i,j)qEu_{1,t}^{i,j}y_t^{i,j}\Bigg),\\
\Delta(\zeta_3)_{t+1}^{i,j} &= -\frac{\partial\mathcal{H}}{\partial z_t^{i,j}} = -\left[\frac{\partial\mathcal{H}_t^{i,j}}{\partial z_t^{i,j}} + \frac{\partial\mathcal{H}_t^{i+1,j}}{\partial z_t^{i,j}} + \frac{\partial\mathcal{H}_t^{i-1,j}}{\partial z_t^{i,j}} + \frac{\partial\mathcal{H}_t^{i,j+1}}{\partial z_t^{i,j}} + \frac{\partial\mathcal{H}_t^{i,j-1}}{\partial z_t^{i,j}}\right]\\
&= -\Bigg(-C - \lambda_{2,t+1}^{i,j}\frac{n(y_t^{i,j})^2 z_t^{i,j}}{(b_1+(y_t^{i,j})^2} + \lambda_{3,t+1}^{i,j}\Bigg(1 + \frac{n_1(y_t^{i,j})^2}{(b+(y_t^{i,j})^2} + \frac{m_1(x_t^{i,j})^2 z_t^{i,j}}{(b+(x_t^{i,j})^2} - d_2\Bigg)\\
&\quad + \alpha_3\big(\lambda_{3,t+1}^{i-1,j} + \lambda_{3,t+1}^{i+1,j} + \lambda_{3,t+1}^{i-1,j} + \lambda_{3,t+1}^{i+1,j} - 4\lambda_{3,t+1}^{i,j}\big) - \lambda_{3,t+1}^{i,j}\mathbb{1}_w(i,j)qEu_{2,t}^{i,j}z_t^{i,j}\Bigg)
\end{aligned}
$$

with $(i,j) \in w = [I_1', I_2'] \times [J_1', J_2']$ and $t = 1, \ldots, T$.

Then

$$
\begin{aligned}
\Delta(\zeta_1)_{t+1}^{i,j} &= -A - \lambda_{1,t+1}^{i,j}\Bigg(\Bigg(1-\frac{x_t^{i,j}}{k}\Bigg) + r\frac{2x_t^{i,j}}{k} + \frac{\alpha\big(y_t^{i,j}b_1 - (x_t^{i,j})^2 y_t^{i,j}\big)}{\big(b_1+(x_t^{i,j})^2\big)^2}\Bigg)\\
&\quad - \lambda_{2,t+1}^{i,j}\frac{\beta y_t^{i,j}(b_1+(x_t^{i,j}) - 2\beta(x_t^{i,j})^2 y_t^{i,j}}{\big(b_1+(x_t^{i,j})^2\big)^2} - \lambda_{3,t+1}^{i,j}\frac{2m_1 x_t^{i,j} z_t^{i,j}(b+(x_t^{i,j})^2) - 2m_1(x_t^{i,j})^3 z_t^{i,j}}{\big(b+(x_t^{i,j})^2\big)^2}\\
&\quad - \alpha_1\big(\lambda_{1,t+1}^{i-1,j} + \lambda_{1,t+1}^{i+1,j} + \lambda_{1,t+1}^{i-1,j} + \lambda_{1,t+1}^{i+1,j} - 4\lambda_{1,t+1}^{i,j}\big),\\
\Delta(\zeta_2)_{t+1}^{i,j} &= B + \lambda_{1,t+1}^{i,j}\frac{\alpha x_t^{i,j}}{(b_1+(x_t^{i,j})^2)} - \lambda_{2,t+1}^{i,j}\Bigg(1 + \frac{\beta x_t^{i,j}}{a+(x_t^{i,j})} - d - \frac{2nb_1 y_t^{i,j} z_t^{i,j}}{\big(b_1+(y_t^{i,j})^2\big)^2}\Bigg)\\
&\quad - \lambda_{3,t+1}^{i,j}\frac{2n_1 b y_t^{i,j} z_t^{i,j}}{(b+(y_t^{i,j})^2)^2} - \alpha_2\big(\lambda_{2,t+1}^{i-1,j} + \lambda_{2,t+1}^{i+1,j} + \lambda_{2,t+1}^{i-1,j} + \lambda_{2,t+1}^{i+1,j} - 4\lambda_{2,t+1}^{i,j}\big)\\
&\quad + \lambda_{2,t+1}^{i,j}\mathbb{1}_w(i,j)qEu_{1,t}^{i,j}y_t^{i,j},\\
\Delta(\zeta_3)_{t+1}^{i,j} &= C + \lambda_{2,t+1}^{i,j}\frac{n(y_t^{i,j})^2 z_t^{i,j}}{b_1+(y_t^{i,j})^2} - \lambda_{3,t+1}^{i,j}\Bigg(1 + \frac{n_1(y_t^{i,j})^2}{b+(y_t^{i,j})^2} + \frac{m_1(x_t^{i,j})^2 z_t^{i,j}}{b+(x_t^{i,j})^2} - d_2\Bigg)\\
&\quad - \alpha_3\big(\lambda_{3,t+1}^{i-1,j} + \lambda_{3,t+1}^{i+1,j} + \lambda_{3,t+1}^{i-1,j} + \lambda_{3,t+1}^{i+1,j} - 4\lambda_{3,t+1}^{i,j}\big) + \lambda_{3,t+1}^{i,j}\mathbb{1}_w(i,j)qEu_{2,t}^{i,j}z_t^{i,j}.
\end{aligned}
$$

With $(i,j) \in w = [I_1', I_2'] \times [J_1', J_2']$, $t = 1, \ldots, T$ and with transversality conditions:

$$(\zeta_1)_{t+1}^{i,j} = A, \quad (\zeta_2)_{t+1}^{i,j} = -B, \quad (\zeta_3)_{t+1}^{i,j} = -C.$$

To acquire the ideal conditions, we consider the variation in relation to control $u_{1,t}^{i,j}$, $u_{2,t}^{i,j}$ and set it equal to zero, we get

$$\frac{\partial \mathcal{H}}{\partial u_{1,t}^{i,j}} = u_{1,t}^{i,j} - \frac{\lambda_{2,t+1}^{i,j} \mathbb{1}_w(i,j) q E y_t^{i,j}}{\tau_1} = 0 \quad \text{and} \quad \frac{\partial \mathcal{H}}{\partial u_{2,t}^{i,j}} = u_{2,t}^{i,j} - \frac{\lambda_{3,t+1}^{i,j} \mathbb{1}_w(i,j) q E z_t^{i,j}}{\tau_2} = 0.$$

Thereby, for $(i,j) \in w = [I_1', I_2'] \times [J_1', J_2']$

$$\frac{\partial \mathcal{H}}{\partial u_{1,t}^{i,j}} = u_{1,t}^{i,j} - \frac{\lambda_{2,t+1}^{i,j} q E y_t^{i,j}}{\tau_1} = 0 \quad \text{and} \quad \frac{\partial \mathcal{H}}{\partial u_{2,t}^{i,j}} = u_{2,t}^{i,j} - \frac{\lambda_{3,t+1}^{i,j} q E z_t^{i,j}}{\tau_2} = 0.$$

Then we obtain the optimal controls:

$$u_{1,t}^{i,j} = \frac{\lambda_{2,t+1}^{i,j} q E y_t^{i,j}}{\tau_1} \quad \text{and} \quad u_{2,t}^{i,j} = \frac{\lambda_{3,t+1}^{i,j} q E z_t^{i,j}}{\tau_2},$$

with $(i,j) \in w = [I_1', I_2'] \times [J_1', J_2']$.

It is simple to obtain by the boundaries in U of the control, $(u_{1,t}^{i,j})^*$, $(u_{2,t}^{i,j})^*$ for $(i,j) \in w = [I_1', I_2'] \times [J_1', J_2']$ in the following form

$$(u_{1,t}^{i,j})^* = \min\left\{\max\left(u_1^{\min}, \frac{\lambda_{2,t+1}^{i,j} q E y_t^{k,l}}{\tau_1}\right), u_1^{\max}\right\}, \; (u_{2,t}^{i,j})^* = \min\left\{\max\left(u_2^{\min}, \frac{\lambda_{3,t+1}^{i,j} q E z_t^{k,l}}{\tau_2}\right), u_2^{\max}\right\}.$$

■

4. Numerical simulation

We present in this part a set of numerical simulations to explain the theoretical findings of our work.

The numerical scheme used in our simulation is based on the forward backward sweep method [19].

Utilizing a cyclic approach, we resolve the optimality. First, because of the transversality conditions, the case system using preliminary estimation is solved forward in time, and the adjoint system is resolved backward in time. Following that, we used the estimations of the state and variable costs acquired in the preceding steps to update the optimal control values. Finally, we run the algorithm until we reach a tolerance criterion, and we use the following parameters [9]: $x_1^{i,j} = 160$, $y_1^{i,j} = 90$, $z_1^{i,j} = 70$, with different density in $i \in \{0, 1, \ldots, I\}$ and $j \in \{0, 1, \ldots, J\}$ with $I = 10\,\text{km}$ and $J = 10\,\text{km}$, $r = 0.058$, $k = 280$, $a = 102$, $\beta = 0.02$, $\alpha = 0.022$, $m = 0.005$, $m_1 = 0.005$, $n = 0.005$, $n_1 = 0.005$, $b = 61$, $b_1 = 0.616$, $b_2 = 0.13$, $d_1 = 0.0002$, $d_2 = 0.0001$, $\alpha_1 = 0.01$, $\alpha_2 = 0.05$, $\alpha_3 = 0.09$, $q = 0.162$, $E = 0.211$.

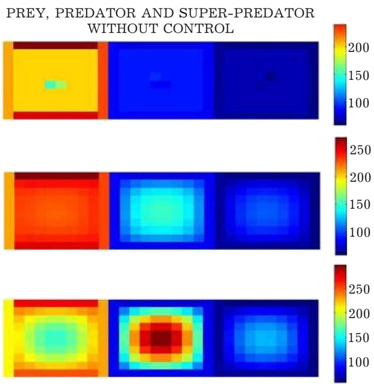


Fig. 1. Evolution of prey, predators and super-predator before applying the controls.

In our discussion of both systems without controls and with controls. Over the course of a year for the variation of prey, predator and super-predator.

The experience focused on a space chose already ω, the Figure 1 shows that in the day 1 we had the prey all over the space, the predator start with 90 and the super-predator with 70 in the space chosen for the experiences. After the first day, the change began to be noticeable and below is a detailed explanation of the change that occurred over time.

The number of predator about 90 in the first day, after one year the predator increase fast to 380. The same for the super-predator.

4.1. The coexistence of fish without any interference from humans

In the region chosen, Figure 1 shows that the prey number is 160 at $t = 0$, the number of the prey decreases to be 70 after one year but we can very clearly see every four months the prey is decreases because Consumed by predators and super-predators.

Not the same thing for the predator it is because we have 90 at $t = 0$, the number of the predator increasing to be 375 after one year and the number of the super-predator start with 70 at $t = 0$, the number of the super-predator increasing to 140 after one year.

We can see in Figure 1 after 360 days in the absence of control, i.e. in the normal condition. The decline in prey numbers is due to consumption by predators and super-predators, so the predator spreads rapidly throughout space as a force of the super-predator.

4.2. Strategy 1: predatory fishing i.e. the first control application u_1

After we applied the cell control $(2, 2)$ and $(4, 4)$, in the first stage, we focused on applying the control on the predator, and the result is clear, for the prey, we started with 160, it increases to 270 but after a while, we notice a decrease in the prey to reach to 125, for the predator we mentioned as 90 at $t = 0$ and after one year we can see the predator drop below then 5, and for the super-predator we have no control over for that super-predator it slowly increases from 70 and after one year to 120.

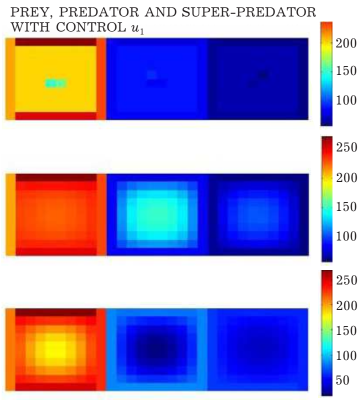


Fig. 2. The evolution of the prey, predator and super-predator with the control u_1.

4.3. Strategy 2: super-predatory fishing i.e. the second control application u_2

We apply the control in cell $(2, 2)$ and $(4, 4)$, in the second stage, this time we focused on applying the control to the super-predator, the result was as follows, for the prey we started with 160 at $t = 0$, and after one month later, we notice a slow increase to 220 and after that the prey go back to decrease to be 110, for the predator we started with 90, and after one year we can see the predator rising to 600 because we don't have any controls applied to the predator, and for the super-predator we have apply the controls, so for this super-predator, the rest is pretty stable: we start at 70 and a year later we are at 60.

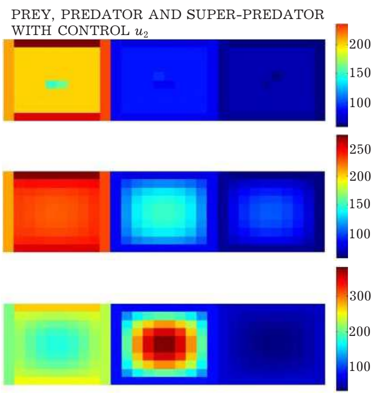


Fig. 3. The evolution of the prey, predator and super-predator with the control u_2.

4.4. Strategy 3: predatory and super-predatory fishing i.e. controls application u_1 and u_2

In the same cell we apply controls over the predator and the super-predator, and this is reflected in the whole system. We start with the prey, as noted in the document, we started with 160 and after a one month it goes to 230 and after another ten months the prey increase to 110, for the predator and the super-predator we started with 90 and 70 respectively, and after a year we notice that two compartments go back to 12 and 40 respectively. These were the goals to be reached to return to the virgo nature.

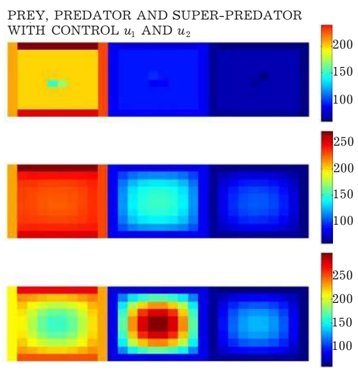


Fig. 4. The evolution of the prey, predator and super-predator with the controls u_1 and u_2.

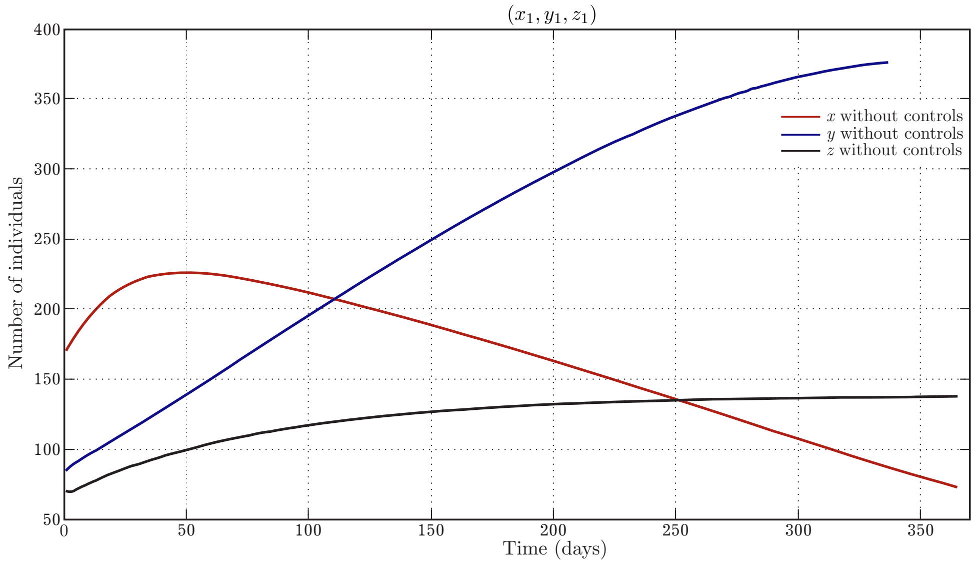


Fig. 5. The evolution of the prey without controls.

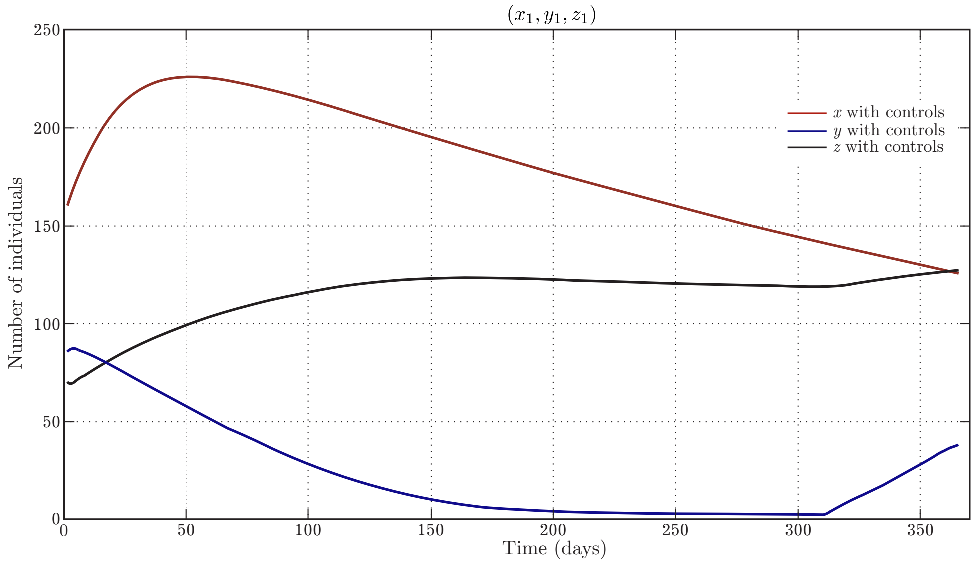


Fig. 6. The evolution of the predator without controls.

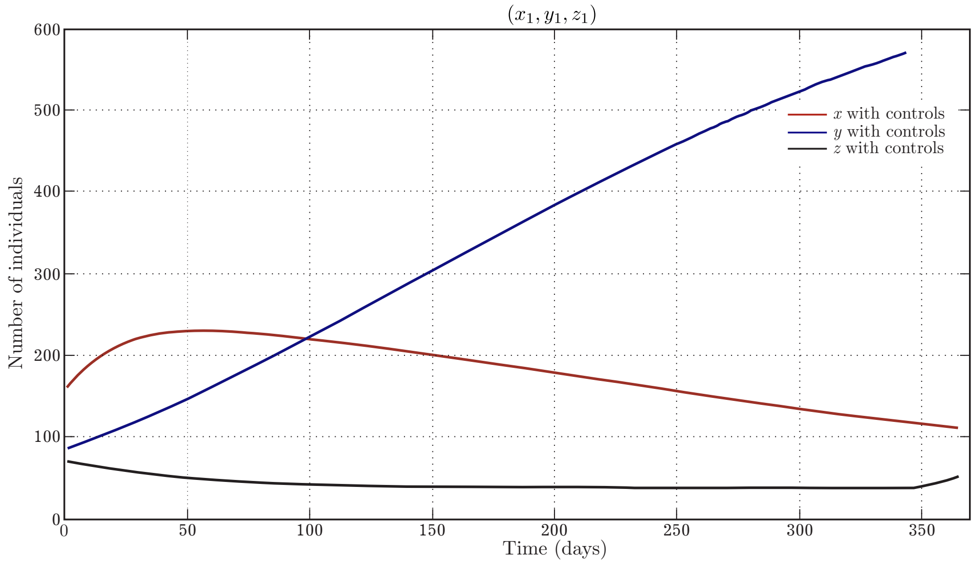


Fig. 7. The evolution of the predator without controls.

The last figure brief everything we discussed in four scenarios and from it we conclude that the control u_1 and u_2 has been applied and gives all the required results in that, after we controlled the predator and super-predator. The second and third scenarios do not give satisfactory results because we get a disruption in the food chain.

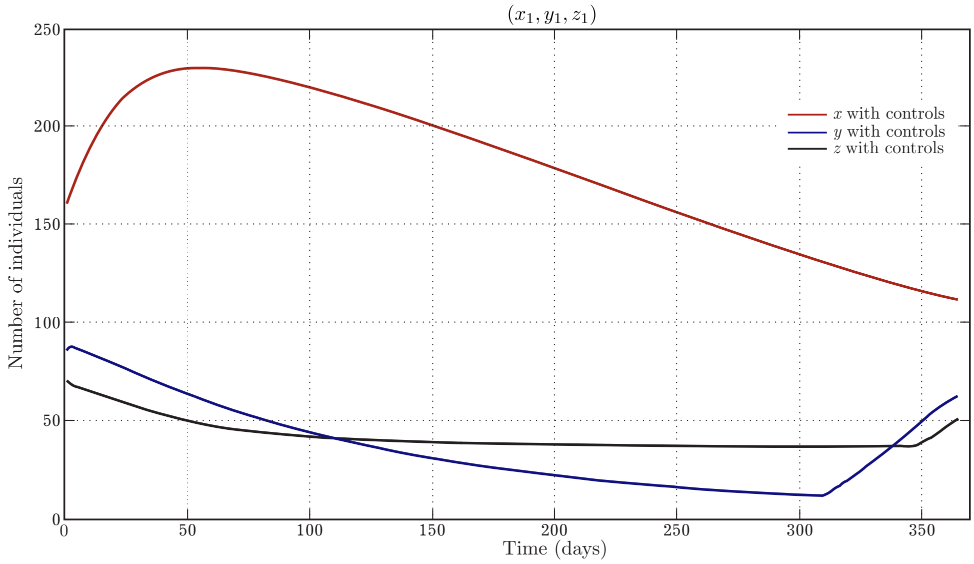


Fig. 8. The evolution of the super-predator without controls.

5. Conclusion

In the sea, as nature, we have prey and predators, but the changes caused by humans put the sea food chain in a state of great turmoil, so we propose this work to return the sea food chain to its natural state. On the other hand, we discuss our model in three scenarios: we controlled the predator, the super-predator, and we controlled both at once. The system we developed has produced important results. After seeing the effect of the applied controls, we obtained the expected results. The increase and decrease of predators and prey occurs periodically after a while it becomes normal, meaning we will be sure the marine food chain returns to normal. This effort is of particular importance in the marine environment, as human management of marine resources in an irrational manner may lead to the extinction of some species, which is an assault on the environment and its stability, as well as harm to the entire regional economy and a fault in nature.

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Проблема оптимального керування дискретною просторово-часовою моделлю рибальства "жертва–хижак" трьох видів

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У цій статті обговорюємо просторово-часову дискретну модель "хижака–жертва". Вона складається з трьох складових: здобич, хижак і суперхижак. Запропонована модель описує взаємодію між жертвою, хижаком і суперхижаком в області з дискретним переміщенням. Також проводиться дослідження відповідних регіональних стратегій керування. Елементи керування застосовуються відповідно до хижака та суперхижака; вони представляють вилов їх у виміряних кількостях у вибраному просторі та часі. Мета — збільшити кількість здобичі та зменшити чисельність хижаків, відновити систему харчового ланцюга та забезпечити її стійкість. Нарешті, подано графічну візуалізацію та чисельне моделювання для підтвердження наших аналітичних висновків.

Ключові слова: *здобич–хижак; просторово-часова дискретна модель; оптимальне керування, принцип максимуму Понтрягіна.*