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Numerical simulation by Deep Learning of a time periodic $p(x)$-Laplace equation

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The objective of this paper is to focus on the study of a periodic temporal parabolic equation involving a variable exponent $p(x)$. After proving the existence and uniqueness of the solution, we provide a method for its numerical simulation using emerging deep learning technologies.

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1. Introduction

In the early literature, various numerical methods were introduced to tackle the numerical analysis and simulation of periodic equations. Among these methods, the collocation method stands out as a powerful technique for finding periodic solutions. This method, extensively discussed in [1, 2]. In addition to the collocation method, other numerical techniques have also been proposed for solving periodic equations. For example, [3] has introduced a quasi-linear numerical method, which offers an alternative approach for calculating periodic solutions. Furthermore, the authors in [4] have developed the Lattice Boltzmann method, which provides a unique perspective on simulating periodic systems.

In the meantime, nowadays, numerical methods have become an indispensable tool in a wide range of scientific, technological and industrial fields, playing a crucial role in simplifying the study of complex differential and partial differential equations. Researchers are constantly striving to develop innovative methods and tools to tackle these equations, with the aim of simulating real-life phenomena with greater accuracy and efficiency. Among these digital methods, deep learning and machine learning techniques have made revolutionary advances. Machine learning is now ubiquitous in fields such as image recognition and finance. Neural networks, which are designed to mimic the functionality of the human brain, have become a popular class of machine learning models.

Neural network architectures like Convolutional Neural Networks (CNNs) [5], Recurrent Neural Networks (RNNs) [6], and Autoencoders [7] have seen significant advancements in the last 60 years. These architectures are inspired by the structure and function of the human brain.

In recent years, deep learning methods based on deep neural networks have made significant advances in a wide range of fields, including image classification [8], natural language processing [9] and error detection [10]. These advances have been driven by the success of deep neural networks in mimicking the intricate functions of the human brain, enabling them to tackle complex problems with remarkable accuracy and efficiency. Deep neural networks have shown tremendous performance in tasks such as computer vision, speech recognition and language translation. This is due to their ability to learn and extract patterns from large datasets.

The Universal Approximation Theorem states that a feed-forward neural network with a single hidden layer and a finite number of neurons, has the ability to approximate continuous functions on compact subsets of $\mathbb{R}^n$ under mild assumptions [8, 11]. This fundamental theorem highlights the power of neural networks to approximate complex functions, making them versatile tools for a wide range of applications. Indeed, neural networks offer significant advantages in terms of functional customization. Their ability to adapt, learn from data and process information efficiently makes them indispensable in a wide range of applications, transforming various fields and enabling new possibilities in computational science and engineering. These advantages are particularly evident in tasks such as: image classification, neural language processing, speech recognition and error detection.

Neural network-based methods are becoming increasingly important in the study of ordinary differential systems, exploiting the power of neural networks to efficiently solve systems of ordinary differential equations (ODEs). Scientists have developed innovative techniques to deal with the complexity of ODEs and improve numerical solutions by reformulating the problem using neural networks. These methods have shown promise in various applications, including: modeling and simulation, parameter identification, Data-driven Solutions.

Neural network techniques offer a promising approach to solving mathematical problems such as differential equations, showing their versatility and potential to improve computational methods for solving ODEs [12–14].

A number of studies have been carried out, including [15, 16]. The physical information neural network proposed by [15,16], demonstrates a novel approach to solving forward and inverse problems related to nonlinear partial differential equations. This innovative approach combines deep learning techniques with the principles of physics to address complex problems involving nonlinear PDEs. The authors in [15] aimed to provide efficient and accurate solutions by training neural networks to respect physical laws while solving supervised tasks. By integrating physics into the neural network training process, the framework offered a promising way to tackle challenging problems in various fields involving nonlinear partial differential equations. Reference [16] focused specifically on computing data-driven solutions to partial differential equations, highlighting the ability of physics-informed neural networks to infer solutions and generate physics-informed surrogate models that are fully differentiable with respect to all input coordinates and free parameters.

In [17–19], adaptive activation functions were introduced into physically informative deep neural networks (PINNs) to better approximate solutions of complex functions and partial differential equations. Based on previous research on PINNs, this study optimizes the method and constructs a physically informative neural network for the wave equation, the KdV-Burgers equation, the KdV equation, the rigid Brusselator raction-diffusion equation, and the generalized Burgers–Huxley equations.

In this study, we focus on the type of problem (1) that has arisen in the modeling of various models relevant not only in electro-rheology, thermorheology, but also in robotics, fluid dynamics and image processing [20]. Section 2 is devoted to the mathematical preliminaries, where we give the definition of the notion of weak periodic solution as well as the appropriate mathematical framework. Section 3 is devoted to the presentation of the DeepXDE library [21], which is dedicated to deep learning numerical simulation of ordinary differential equations and partial differential equations. In section 4, a numerical analysis is carried out to simulate these periodic solutions using deep learning.

The numerical simulations prove that the method based on deep learning is a promising approach to solve this type of highly nonlinear periodic problem, knowing that classical simulation methods are not efficient. Our numerical experiences show that this requires good knowledge for the correct choice of hyper-parameters and network architecture.

2. Preliminaries and problem position

Our focus in this paper is to provide a theoretical and numerical solution to a to a nonlinear periodic parabolic equation with $p(x)$-growth modeled as

$$\begin{cases} \dfrac{\partial u}{\partial t} - \operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u\right) = f(t,x) & \text{in } Q_T :=]0,T[\times\Omega, \\ u(0,\cdot) = u(T,\cdot) & \text{in } \Omega, \\ u(t,x) = 0 & \text{on } \Sigma_T :=]0,T[\times\partial\Omega. \end{cases} \tag{1}$$

Let Ω be a bounded open subset of $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $T > 0$ is the period, f designates a measurable T-periodic function with period T, belonging to a suitable Banach space. We define the

$p(x)$-Laplacian operator as $\Delta_{p(x)}u = \operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u\right)$, where the exponent $p(\cdot)$ is assumed to be either a continuous function: $p\colon \overline{\Omega} \to]1,+\infty[$.

We begin by giving the appropriate mathematical framework for analyzing this type of equation and then define the notion of periodic solution.

2.1. A quick reminder about variable exponent spaces

Here we review some definitions and properties of variable exponent Lebesgue and Sobolev spaces. Let $p\colon \overline{\Omega} \to]1,+\infty[$ be a continuous function satisfying

$$1 < p^- \leqslant p(x) \leqslant p^+ < \infty, \tag{2}$$

where

$$p^- = \inf_{x\in\overline{\Omega}} p(x) \quad \text{and} \quad p^+ = \sup_{x\in\overline{\Omega}} p(x).$$

The variable exponent Lebesgue space $L^{p(x)}(\Omega)$ is defined as follows

$$L^{p(x)}(\Omega) = \left\{u\colon \Omega \to \mathbb{R} \text{ measurable such that } \rho_{p(x)}(u) < \infty\right\},$$

where $\rho_{p(\cdot)}$ is the following convex modular

$$\rho_{p(x)}(u) = \int_\Omega |u(x)|^{p(x)}dx.$$

The space $L^{p(x)}(\Omega)$ is equipped by the Luxembourg norm

$$\|u\|_{L^{p(x)}(\Omega)} = \inf\left\{\alpha > 0, \rho_{p(x)}\left(\frac{u}{\alpha}\right) \leqslant 1\right\}.$$

The space $\left(L^{p(x)}(\Omega), \|\cdot\|_{p(x)}\right)$ is a separable and reflexive Banach space. Its dual space is defined by $L^{p'(x)}(\Omega)$, where $p'(x) = \frac{p(x)}{p(x)-1}$. Furthermore, for any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$, one can prove the following $p(x)$-Hölder inequality

$$\int_\Omega |uv|\,dx \leqslant \left(\frac{1}{p^-} + \frac{1}{p'^-}\right)\|u\|_{p(x)}\|v\|_{p'(x)} \leqslant 2\|u\|_{p(x)}\|v\|_{p'(x)}.$$

An interesting property for Lebesgue spaces with variable exponents is given by

$$\min\left\{\|u\|^{p^-}_{p(x)}, \|u\|^{p^+}_{p(x)}\right\} \leqslant \rho_{p(x)}(u) \leqslant \max\left\{\|u\|^{p^-}_{p(x)}, \|u\|^{p^+}_{p(x)}\right\},$$

$$\min\left\{\rho^{\frac{1}{p^-}}_{p(x)}(u), \rho^{\frac{1}{p^+}}_{p(x)}(u)\right\} \leqslant \|u\|_{p(x)} \leqslant \max\left\{\rho^{\frac{1}{p^-}}_{p(x)}(u), \rho^{\frac{1}{p^+}}_{p(x)}(u)\right\}.$$

Here we define the variable exponent Lebesgue space $L^{p(x)}(Q_T)$ by the following sense

$$L^{p(x)}(Q_T) = \left\{u\colon Q_T \to \mathbb{R} \text{ measurable with } \int_{Q_T}|u(t,x)|^{p(x)}dx\,dt < \infty\right\}.$$

It is equipped with the following norm

$$\|u\|_{p(x)} = \inf\left\{\alpha > 0, \int_{Q_T}\left|\frac{u(t,x)}{\alpha}\right|^{p(x)}dx\,dt \leqslant 1\right\},$$

the space $\left(L^{p(x)}(Q_T), \|\cdot\|_{p(x)}\right)$ is a separable, reflexive Banach.

The Sobolev space with variable exponent is defined as

$$W^{1,p(x)}(\Omega) = \left\{u \in L^{p(x)}(\Omega), |\nabla u| \in \left(L^{p(x)}(\Omega)\right)^N\right\}.$$

Its associated standard norm is given by

$$\|u\|_{1,p(x)} = \|u\|_{p(x)} + \|\nabla u\|_{p(x)}.$$

A popular equivalent norm is defined as

$$\|u\|_{1,p(x)} = \inf\left\{\alpha > 0, \int_\Omega\left(\left|\frac{\nabla u(x)}{\alpha}\right|^{p(x)} + \left|\frac{u(x)}{\alpha}\right|^{p(x)}\right)dx \leqslant 1\right\}.$$

Thereafter, if there exists a constant C, such that $p(x)$ satisfies the log-Hölder continuity condition:

$$|p(x_1) - p(x_2)| \leqslant \frac{C}{-\log|x_1 - x_2|}, \ \forall x_1, x_2 \in \Omega, \text{ with } |x_1 - x_2| < \frac{1}{2},$$

then $\mathcal{C}_c^\infty(\Omega)$ is dense in $W^{1,p(x)}(\Omega)$.

Now, one can define $W_0^{1,p(x)}(\Omega) := \overline{\mathcal{C}_c^\infty(\Omega)}^{W^{1,p(x)}(\Omega)}$ and denote $\big(W_0^{1,p(x)}(\Omega)\big)^*$ its dual space. The spaces $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$ are separable and reflexive Banach's. For any $u \in W_0^{1,p(x)}(\Omega)$, the following $p(x)$-Poincaré inequality holds

$$\|u\|_{p(x)} \leqslant C\|\nabla u\|_{p(x)},$$

where C is a constant depending only on $p(x)$ and Ω. After this, the following norm is a validated one on $W_0^{1,p(x)}(\Omega)$

$$\|u\|_{W_0^{1,p(x)}(\Omega)} = \|\nabla u\|_{p(x)}.$$

For a comprehensive reference on these spaces, readers are referred to Radulescu's work in [20].

2.2. Problem position

We introduce the appropriate functional framework for our problem (1). For $0 < T < +\infty$, consider the space

$$L^{p^-}\big(0,T;W_0^{1,p(x)}(\Omega)\big) = \left\{u \in L^{p(x)}(Q_T) \colon \int_0^T \|\nabla u\|_{p(x)}^{p^-} dt < \infty\right\}.$$

Let $L^{p^-}\big(0,T;W_0^{1,p(x)}(\Omega)\big)$ have the following norm

$$\|u\|_{L^{p^-}\left(0,T;W_0^{1,p(x)}(\Omega)\right)} = \left(\int_0^T \|\nabla u\|_{p(x)}^{p^-} dt\right)^{\frac{1}{p^-}}.$$

We also introduce the space $\mathcal{V}$ frequently used in this type of non-linear parabolic problem with growth $p(x)$. We set

$$\mathcal{V} = \left\{u \in L^{p^-}\big(0,T;W_0^{1,p(x)}(\Omega)\big) \colon |\nabla u| \in \big(L^{p(x)}(Q_T)\big)^N\right\}$$

with the following norm $\|u\|_{\mathcal{V}} = \|\nabla u\|_{L^{p(x)}(Q_T)}$.

It is easy to show that the $\|\cdot\|_{\mathcal{V}}$ norm is equivalent to the following usual norm

$$\|u\|_{\mathcal{V}} = \|u\|_{L^{p^-}\left(0,T;W_0^{1,p(x)}(\Omega)\right)} + \|\nabla u\|_{p(x)}.$$

The space $(\mathcal{V}, \|\cdot\|_{\mathcal{V}})$ is a separable and reflexive Banach space. Let $\mathcal{V}^*$ be the dual space of $\mathcal{V}$. In the following result, we summarize some properties of the space $\mathcal{V}$.

Lemma 1 (Ref. [22]). *Let $\mathcal{V}$ denotes the space defined as above. Then,*
i)

$$L^{p+}\big(0,T;W_0^{1,p(x)}(\Omega)\big) \hookrightarrow \mathcal{V} \hookrightarrow L^{p^-}\big(0,T;W_0^{1,p(x)}(\Omega)\big),$$

$\mathcal{C}_c^\infty(Q_T)$ is dense in $L^{p+}\big(0,T;W_0^{1,p(x)}(\Omega)\big)$, and in $\mathcal{V}$.
For the corresponding dual spaces we have

$$L^{(p^-)'}\big(0,T;(W_0^{1,p(x)}(\Omega))^*\big) \hookrightarrow \mathcal{V}^* \hookrightarrow L^{(p+)'}\big(0,T;(W_0^{1,p(x)}(\Omega))^*\big).$$

ii) Moreover, the elements of $\mathcal{V}^*$ can be represented as follow: for all $\zeta \in \mathcal{V}^*$, there exists $\xi = (\xi_1, \ldots, \xi_N) \in \big(L^{p'(x)}(Q_T)\big)^N$ such that: $\zeta = \operatorname{div}(\xi)$ and

$$\langle \zeta, \varphi\rangle_{\mathcal{V}^*,\mathcal{V}} = \int_{Q_T} \xi\, \nabla\varphi\, dx\, dt$$

for any $\varphi \in \mathcal{V}$. Then,

$$\|\zeta\|_{\mathcal{V}^*} = \max\left\{\|\xi_i\|_{L^{p(x)}(Q_T)}, i = 1, \ldots, N\right\}.$$

iii) For any $u \in \mathcal{V}$ the following equation is true

$$\min\left\{\|u\|_{\mathcal{V}}^{p^-}, \|u\|_{\mathcal{V}}^{p^+}\right\} \leqslant \int_{Q_T} |\nabla u|^{p(x)}\, dx\, dt \leqslant \max\left\{\|u\|_{\mathcal{V}}^{p^-}, \|u\|_{\mathcal{V}}^{p^+}\right\} \tag{3}$$

To conclude this paragraph, we recall the following result.

Theorem 1 (Ref. [23]). *If $\mathcal{X}$ is a reflexive Banach space, $\mathcal{L}\colon \mathcal{D}(\mathcal{L}) \subseteq \mathcal{X} \to \mathcal{X}^*$ is a linear maximal monotone operator and $\mathcal{A}\colon \mathcal{X} \to \mathcal{X}^*$ is a bounded, pseudo-monotone, coercive operator (i.e. $\frac{\langle \mathcal{A}(u),u\rangle_{\mathcal{X}^*,\mathcal{X}}}{\|u\|_{\mathcal{X}}} \to +\infty$ as $\|u\|_{\mathcal{X}} \to \infty$) then $\mathcal{L}+\mathcal{A}$ is surjective.*

2.3. Existence and uniqueness result

In this section, we give an existence and uniqueness result for 1. We begin with the notion of a weak periodic solution.

Definition 1. *A measurable function $u\colon Q_T \to \mathbb{R}$ is a weak T-periodic solution to* (1) *if the following conditions are satisfied*

$$u \in \mathcal{V}, \quad \frac{\partial u}{\partial t} \in \mathcal{V}^*, \quad u(0,x) = u(T,x),$$

$$\int_0^T \left\langle \frac{\partial u}{\partial t}, \varphi \right\rangle dt + \int_{Q_T} |\nabla u|^{p(x)-2} \nabla u \nabla \varphi \, dx\, dt = \int_0^T \langle f, \varphi\rangle \, dt$$

for all test function $\varphi \in \mathcal{V}$.

We can now state the main result of this section.

Theorem 2. *Let $p \in C(\overline{\Omega})$ satisfy (2) and $f \in L^2\big(0,T;\big(W_0^{1,p(x)}(\Omega)\big)^*\big)$, T-periodic. Then, problem (1) has a unique weak T-periodic solution.*

Proof. To show the existence of a periodic weak solution. We use the result of Theorem 1.

We start by setting

$$\mathcal{D}(\mathcal{L}) := \left\{ u \in \mathcal{V}, \frac{\partial u}{\partial t} \in \mathcal{V}^* \text{ and } u(0) = u(T) \right\}.$$

Based on the density property of $\mathcal{C}_c^\infty(Q_T)$ in $\mathcal{V}$ and by employing the fact that $\mathcal{C}_c^\infty(Q_T) \subset \mathcal{D}(\mathcal{L})$, we conclude that $\mathcal{D}(\mathcal{L})$ is dense in $\mathcal{V}$.

We introduce the operator $\mathcal{L}\colon \mathcal{D}(\mathcal{L}) \longrightarrow \mathcal{V}^*$ such that

$$\langle \mathcal{L}u, \varphi\rangle := \int_0^T \left\langle \frac{\partial u}{\partial t}, \varphi \right\rangle dt, \quad \text{for all } \varphi \in \mathcal{V}.$$

According to the result of [23, Lemma 1.1, p. 313], we have $\mathcal{L}$ is a maximal closed, asymmetric and monotone operator. Consider the operator $\mathcal{A}\colon \mathcal{V} \longrightarrow \mathcal{V}^*$ such that

$$\langle \mathcal{A}u, \varphi\rangle := \int_{Q_T} |\nabla u|^{p(x)-2} \nabla u \nabla \varphi \, dx\, dt, \quad \text{for all } \varphi \in \mathcal{V}.$$

We can verify that the existence of a weak periodic solution to (1) is equivalent to the existence of a solution to the following abstract equation

$$\mathcal{L}u + \mathcal{A}u = \mathcal{F}, \tag{4}$$

where $\mathcal{F}$ is an element of $\mathcal{V}^*$ defined as

$$\langle \mathcal{F}, \varphi\rangle := \int_0^T \langle f, \varphi\rangle \, dt, \quad \text{for all } \varphi \in \mathcal{V}.$$

Let us start by showing that the operator $\mathcal{A}$ is bounded.

Let $u, v \in \mathcal{V}$, by the help of $p(x)$-Höder's inequality, one gets

$$|\langle \mathcal{A}u, v\rangle| \leqslant \int_{Q_T} |\nabla u|^{p(x)-1} |\nabla v| \, dx\, dt \leqslant 2 \big\| |\nabla u|^{p(x)-1} \big\|_{p'(x)} \|\nabla v\|_{p(x)}. \tag{5}$$

On the other hand, the min-max properties of $L^{p(x)}$ spaces (see (1)) ensure that

$$\big\| |\nabla u|^{p(x)-1} \big\|_{p'(x)} \leqslant \max \left\{ \left(\int_{Q_T} |\nabla u|^{p(x)} dx\, dt \right)^{\frac{1}{(p')^-}}, \left(\int_{Q_T} |\nabla u|^{p(x)} dx\, dt \right)^{\frac{1}{(p')^+}} \right\}$$

$$
\begin{aligned}
&\leqslant \max\left\{\|\nabla u\|_{p(x)}^{\frac{p^+}{(p')}}, \|\nabla u\|_{p(x)}^{\frac{p^+}{(p')^+}}\right\} \\
&\leqslant \max\left\{\|u\|_{\mathcal{V}}^{\frac{p^+}{(p')^-}}, \|u\|_{\mathcal{V}}^{\frac{p^+}{(p')^+}}\right\}.
\end{aligned}
$$

Using the last estimate in (5), we obtain

$$
\|\mathcal{A}u\|_{\mathcal{V}^*} \leqslant C \max\left\{\|u\|_{\mathcal{V}}^{\frac{p^+}{(p')^-}}, \|u\|_{\mathcal{V}}^{\frac{p^+}{(p')^+}}\right\}.
$$

This means that $\mathcal{A}$ is a bounded operator.

Let us prove that the operator $\mathcal{A}$ is coercive. Thanks to inequality (3),

$$
\langle \mathcal{A}u, u\rangle := \int_{Q_T} |\nabla u|^{p(x)} dx\, dt \geqslant \min\left\{\|u\|_{\mathcal{V}}^2, \|u\|_{\mathcal{V}}^{p^+}\right\}.
$$

Thus

$$
\lim_{\|u\|_{\mathcal{V}}\to\infty} \frac{\langle \mathcal{A}u, u\rangle}{\|u\|_{\mathcal{V}}} \geqslant \lim_{\|u\|_{\mathcal{V}}\to\infty} \min\left\{\|u\|_{\mathcal{V}}^{2-1}, \|u\|_{\mathcal{V}}^{p^+-1}\right\} = \infty.
$$

Hence the result.

Finally, we will prove that the operator $\mathcal{A}$ is pseudo-monotone. Consider a sequence (u_n) in $\mathcal{V}$ such that (u_n) converges weakly to u in $\mathcal{V}$ and that

$$
\lim_{n\to\infty} \sup \langle \mathcal{A}u_n, u_n - u\rangle \leqslant 0. \tag{6}
$$

Proving that $\mathcal{A}$ is a pseudo-monotone operator comes to show that

$$
\lim_{n\to\infty} \inf \langle \mathcal{A}u_n, u_n - v\rangle \geqslant \langle \mathcal{A}u, u - v\rangle \text{ for all } v \in \mathcal{V}. \tag{7}
$$

First, to deduce that (7) is true, we will prove that the sequence (u_n) converges strongly to u in $\mathcal{V}$. Let us then consider

$$
E(n) = \langle \mathcal{A}u_n - \mathcal{A}u, u_n - u\rangle = \int_{Q_T} \big(|\nabla u_n|^{p(x)-2}\nabla u_n - |\nabla u|^{p(x)-2}\nabla u\big)(\nabla u_n - \nabla u)\, dx\, dt.
$$

From the weak convergence of (u_n) in $\mathcal{V}$ and (6), we have

$$
\lim_{n\to\infty} E(n) \leqslant 0. \tag{8}
$$

Let us recall the following well-known inequalities, for any $\eta, \xi \in \mathbb{R}^N$,

$$
\big(|\eta|^{p(x)-2}\eta - |\xi|^{p(x)-2}\xi\big)\cdot(\eta - \xi) \geqslant \begin{cases} 2^{2-p^+}|\eta-\xi|^{p(x)}, & \text{if } p(x) \geqslant 2, \\ (p^- - 1)\dfrac{|\eta-\xi|^2}{(|\eta|+|\xi|)^{2-p(x)}}, & \text{if } 1 < p(x) < 2. \end{cases} \tag{9}
$$

Then, we write

$$
\begin{aligned}
\int_{Q_T} |\nabla u_n - \nabla u|^{p(x)} dx\, dt &= \int_0^T\!\!\int_{\{x\in\Omega; p(x)\geqslant 2\}} |\nabla u_n - \nabla u|^{p(x)} dx\, dt + \int_0^T\!\!\int_{\{x\in\Omega; 1<p(x)<2\}} |\nabla u_n - \nabla u|^{p(x)} dx\, dt \\
&= I_1^n + I_2^n.
\end{aligned} \tag{10}
$$

Inequality (9) implies that the first integral I_1^n satisfies

$$
\begin{aligned}
I_1^n &\leqslant \frac{1}{2^{2-p^+}} \int_0^T \int_{\{x\in\Omega; p(x)\geqslant 2\}} \big(|\nabla u_n|^{p(x)-2}\nabla u_n - |\nabla u|^{p(x)-2}\nabla u\big)\cdot(\nabla u_n - \nabla u)\, dx\, dt \\
&\leqslant 2^{p^+-2} E(n).
\end{aligned} \tag{11}
$$

As for the second integral, we use $p(x)$-Hölder's inequality.

$$
I_2^n = \int_0^T \int_{\{x\in\Omega; 1<p(x)<2\}} \frac{|\nabla u_n - \nabla u|^{p(x)}}{\big(|\nabla u_n| + |\nabla u|\big)^{\frac{p(x)}{2}(2-p(x))}} \big(|\nabla u_n| + |\nabla u|\big)^{\frac{p(x)}{2}(2-p(x))} dx\, dt
$$

$$\leqslant 2\left\|\frac{|\nabla u_n-\nabla u|^{p(x)}}{(|\nabla u_n|+|\nabla u|)^{\frac{p(x)}{2}(2-p(x))}}\right\|_{L^{\frac{2}{p(x)}}(Q_T)}\left\|(|\nabla u_n|+|\nabla u|)^{\frac{p(x)}{2}(2-p(x))}\right\|_{L^{\frac{2}{2-p(x)}}(Q_T)}$$

$$\leqslant 2\max\left\{\left(\int_{Q_T}\frac{|\nabla u_n-\nabla u|^2}{(|\nabla u_n|+|\nabla u|)^{2-p(x)}}\,dx\,dt\right)^{\frac{p^\pm}{2}}\right\}\max\left\{\left(\int_{Q_T}(|\nabla u_n|+|\nabla u|)^{p(x)}dx\,dt\right)^{\frac{2-p^\pm}{2}}\right\}$$

$$\leqslant C\max\left\{\left(\frac{1}{p^- -1}\right)^{\frac{p^\pm}{2}}(E(n))^{\frac{p^\pm}{2}}\right\}\max\left\{\left(\int_{Q_T}\left(|\nabla u_n|^{p(x)}+|\nabla u|^{p(x)}\right)dx\,dt\right)^{\frac{2-p^\pm}{2}}\right\}.$$

From the weak convergence of (u_n) in $\mathcal{V}$, it comes that (∇u_n) is bounded in $\left(L^{p(x)}(Q_T)\right)^N$. Thus

$$I_2^n \leqslant C\max\left\{\left(\frac{1}{p^- -1}\right)^{\frac{p^\pm}{2}}(E(n))^{\frac{p^\pm}{2}}\right\}. \tag{12}$$

From inequalities (8), (10), (11), and (12), we establish

$$\lim_{n\to\infty}\int_{Q_T}|\nabla u_n-\nabla u|^{p(x)}dx\,dt=0.$$

Thus (u_n) converges strongly to u in $\mathcal{V}$. Hence, it can be deduced that $\mathcal{A}$ is a pseudo-monotone operator. The existence of $u\in\mathcal{D}(\mathcal{L})$, a solution of the abstract problem (4), can be deduced from the result of proposition 1. This implies the existence of a weak periodic solution to (1) which satisfies the weak formulation.

Finally, it is necessary to prove the uniqueness of the weak periodic solutions. Consider u_1 and u_2, two weak periodic solutions of (1), and take the difference between their associated weak formulations. Then, for all $\varphi\in\mathcal{V}$

$$\int_0^T\left\langle\frac{\partial(u_1-u_2)}{\partial t},\varphi\right\rangle dt+\int_{Q_T}\left(|\nabla u_1|^{p(x)-2}\nabla u_1-|\nabla u_2|^{p(x)-2}\nabla u_2\right)\nabla\varphi\,dx\,dt=0.$$

By choosing $\varphi=u_1-u_2$ as a test function, one gets

$$\int_0^T\left\langle\frac{\partial(u_1-u_2)}{\partial t},u_1-u_2\right\rangle dt+\int_{Q_T}\left(|\nabla u_1|^{p(x)-2}\nabla u_1-|\nabla u_2|^{p(x)-2}\nabla u_2\right)(\nabla u_1-\nabla u_2)\,dx\,dt=0.$$

Since u_1 and u_2 are periodic in time, we know that

$$\int_0^T\left\langle\frac{\partial(u_1-u_2)}{\partial t},u_1-u_2\right\rangle dt=0.$$

Therefore

$$\int_{Q_T}\left(|\nabla u_1|^{p(x)-2}\nabla u_1-|\nabla u_2|^{p(x)-2}\nabla u_2\right)(\nabla u_1-\nabla u_2)\,dx\,dt=0.$$

Inequality (9) ensures that $\nabla u_1=\nabla u_2$ a.e. in Q_T. On the other hand, Dirichlet boundary condition implies that $u_1=u_2$ a.e. in Q_T. ■

3. Physics informed neural networks (PINN)

PINNs are based on neural network training, that is achieved through an iterative process in which the neural network is automatically differentiated across boundaries and domains, minimizing learning loss and ensuring the network is adapted to the physics imposed by the specific problem.

In PINN, we exploit the density properties of neural networks [24, 25] to approximate the solution of boundary value problems such as

$$\begin{cases}\mathcal{B}_{t,x}[u]=f(x,t), & (x,t)\in\Omega\times]0,T[,\\ u(x,0)=h(x), & x\in\Omega\\ u(x,t)=g(x,t), & (x,t)\in\partial\Omega\times]0,T[,\end{cases}$$

where $\mathcal{B}_{t,x}$ is a general differential operator, $x \in \Omega$ and t are the spatial and temporal coordinates, respectively, Ω and $\partial\Omega$ denote the computational domain and the boundary. The function $f(x,t)$ represents the source term, and $u(x,t)$ designates the solution of the PDE with initial condition $h(x)$ and boundary condition $g(x,t)$.

The procedure is given as follows:

- Initialize a neural network $\hat{u}(t,x,W,b)$, where W and b are the weights and biases.
- Specify the training data $\tau_i = (x_i, t_i)$ such that $\mathcal{T} = \tau_1, \tau_2, \ldots, \tau_{|\mathcal{T}|}$ of size $|\mathcal{T}|$, which is comprised of scattered points inside the space-time domain $\mathcal{T}_f$, and on the time and space boundaries $\mathcal{T}_b$ as well as $\mathcal{T}_I$.
- Compute $\mathcal{L}_f$ the loss between the network and the PDE constraints inside the domain
$$\mathcal{L}_f(\theta, \mathcal{T}_f) = \frac{1}{|\mathcal{T}_f|} \sum_{x \in \mathcal{T}_f} \|\mathcal{B}_{t,x}[\hat{u}](\tau_i) - f(\tau_i)\|_2^2.$$
- Compute $\mathcal{L}_b$ the boundary condition loss
$$\mathcal{L}_b(\theta, \mathcal{T}_b) = \frac{1}{|\mathcal{T}_b|} \sum_{\tau_i \in \mathcal{T}_b} \|\hat{u}(\tau_i) - g(\tau_i)\|_2^2.$$
- Compute $\mathcal{L}_I$ the initial condition loss
$$\mathcal{L}_I(\theta, \mathcal{T}_I) = \frac{1}{|\mathcal{T}_I|} \sum_{\tau_i \in \mathcal{T}_I} \|\hat{u}(\tau_i) - h(\tau_i)\|_2^2.$$

Here, $\theta = \{W_l, b_l\}_{1 \leqslant L}$ is the set of all the weight matrices and bias vectors of the $\hat{u}$ neural network.

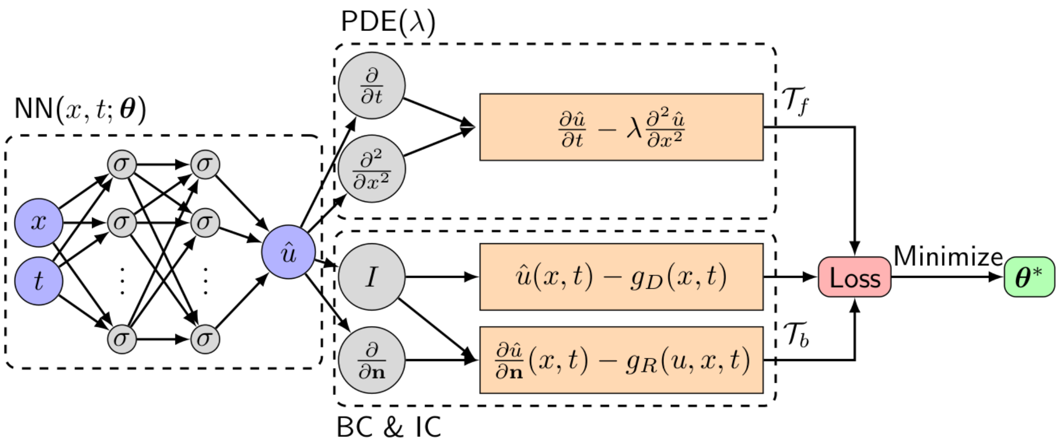

Fig. 1. Schematic of a PINN for solving a heat equation based PDE problem [21].

Algorithm 1 PINN solver procedure.

Let Ω, $\partial\Omega$, h, g, choose the number of points on domain and boundary, tolerance ε, maximal number of iterations $\max\limits_{it}$.
Construct a neural network $\hat{u}(\theta)$ with parameters θ
if $\mathcal{L} \geqslant \varepsilon$ or $k < \max\limits_{it}$
Optimize the general loss term $\mathcal{L} = \mathcal{L}_f + \mathcal{L}_b + \mathcal{L}_I$ w.r.t. W and b
Update θ by passing them back to the network (backpropagation)
$k = k + 1$

The final output is an optimized network $\hat{u}_\star$ that can be used to predict the solution inside the domain and on the boundaries such that $\hat{u}_\star(t,x) \approx u(t,x)$.

Here we are concerned with a periodic problem in time, to use a simulation by the Deep Learning method requires some additional adjustments, in particular:

- Snake' activation function: as suggested in [26], using the function $x^2 + \sin^2(x)$ (called the snake activation function) provides the desired periodic inductive bias to learn a periodic function, namely the solution to the proposed equation.
- 'Hard constraints': Transforming the architecture of the NN to fit periodic boundary conditions is another method of solving periodic problems using PINNs. This method has been demonstrated in [27, 28], where it was shown that it is more optimal to decompose the P-periodic input t_j into a weighted sum of the Fourier basis functions $(1, \cos\frac{2\pi t_j}{P}, \sin\frac{2\pi t_j}{P}, \cos\frac{4\pi t_j}{P}, \sin\frac{4\pi t_j}{P}, \ldots)$ to impose the periodicity in the t_j direction. This is more accurate than loss optimization and saves computation.

4. Results and discussion

In this section we implement the algorithm for solving the following periodic equation

$$\begin{cases} \dfrac{\partial u}{\partial t} - \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) = f & \text{in } Q :=]0,1[\times\Omega, \\ u(0,\cdot) = u(1,\cdot) & \text{in } \Omega, \\ u = 0 & \text{on } \Sigma :=]0,1[\times\partial\Omega, \end{cases} \tag{13}$$

where

$$p(x,y) = 2 + \frac{1}{1+x^2+y^2}, \quad f(t,x,y) = 1 + \sin(2\pi t) + x^2 + y^2,$$

and $\Omega =]0,1\times]0,1[$ is a unit square of $\mathbb{R}^2$.

Simulation by Deep Learning. We use the DeepXDE framework [21] due to its status as the most advanced Physically Informed Neural Network (PINN) framework [29]. Using DeepXDE guarantees access to the most advanced tools and a dynamic support network that improves the efficiency and reliability of our computational research.

The training is conducted in two phases: first through different iterations N_{rp} of the 'rmsprop' optimizer [30], which is a stochastic gradient descent method, followed by the iterative optimizer 'L-BFGS-B' [31, 32], which is a quasi-Newton method.

The parameters and model structure have a significant impact on the result and cannot be chosen arbitrarily, as the following results show.

After several tests, we varied the size (length $\times$ depth) of the hidden layers and the number of training points in the domain $\mathcal{T}_f|$, and noticed that they have an effect on the result. As long as the other parameters (activation function, number of iterations, initialization of weights, etc.) have no significant effect.

It can be concluded that hidden layers of size 40×6 are sufficient to compute the solution reliably and consistently over a sufficient number of training points. Smaller sizes regularly risk not converging, and larger sizes give results that are not sufficient because they require a longer execution time. We also noticed that *L-FGS-B* is the most important optimization method, on the other hand *RMSprop* is quite slow to converge; this is a consistent observation in most other applications in PINNs.

Our simulations give $\|u(1,x)-u(0,x)\|^2_{L^2(\Omega)} = 3.9745\cdot10^{-15}$ and Figures 2–4 show that the solutions at times 0 and 1 are identical. This confirms that the solution obtained is periodic.

Finally, as mentioned above, one of the great advantages of using Deep Learning for the numerical simulation of this type of system is that it allows the solution to be predicted outside the period using only the model trained over the period $[0, 1]$. The following Figure 5 shows the solution obtained by Deep Learning prediction at $t = 4$.

5. Conclusion

In this paper, we focus on the mathematical and numerical analysis of a parabolic periodic equation governed by the $p(x)$-laplacian operator. After showing a result for the existence and iniquity of weak

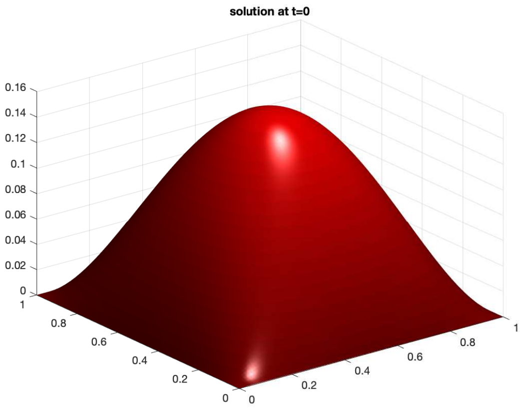

Fig. 2. The predicted solution of the $p(x)$-Laplacian equation (13) by Deep Learning for $t = 0$.

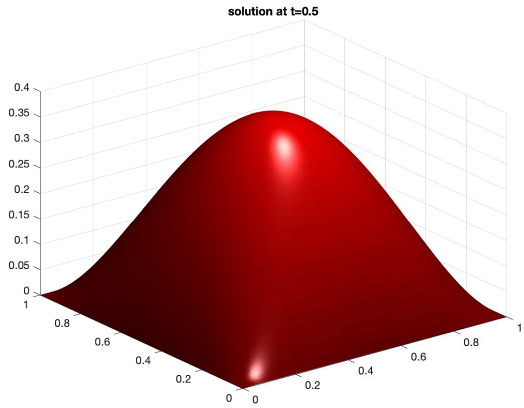

Fig. 3. The predicted solution of the $p(x)$-Laplacian equation (13) by Deep Learning for $t = 0.5$.

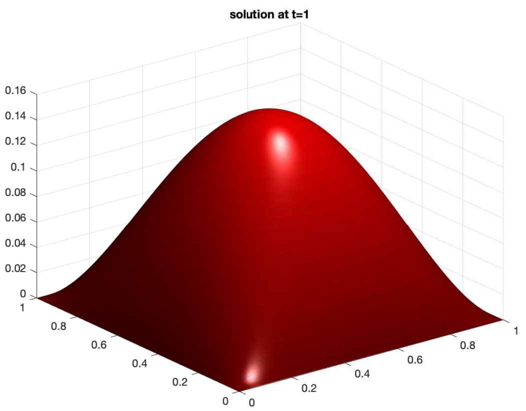

Fig. 4. The predicted solution of the $p(x)$-Laplacian equation (13) by Deep Learning for $t = 1$.

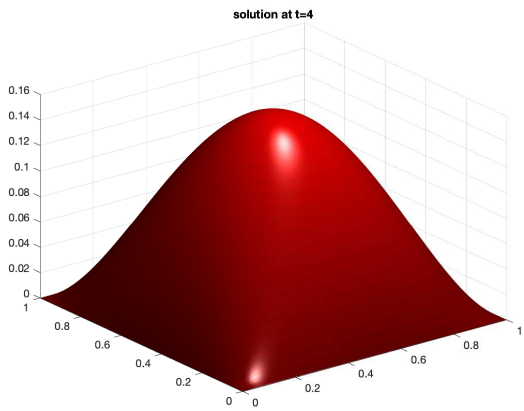

Fig. 5. Solution of problem (13) by Deep Learning for $t = 4$.

periodic solutions is given. Using the performance of the Deepxde library, a numerical code based on Deep Learning has been developed to numerically simulate the periodic solutions.

Overall, deep learning is a promising approach for solving highly nonlinear periodic PDEs. It should be noted that this type of problem cannot be simulated by conventional methods such as Finite Difference, Fine Elements or Finite Volume. However, further modifications are still required, depending on the problem to be solved.

Appendix

Simulation programs are available by contacting one of the authors.

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Чисельне моделювання за допомогою глибокого навчання періодичного $p(x)$-рівняння Лапласа

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Метою цієї роботи є вивчення періодичного часового параболічного рівняння зі змінним показником $p(x)$. Довівши існування та унікальність розв'язку, пропонується метод його чисельного моделювання з використанням нових технологій глибокого навчання.

Ключові слова: *періодичний розв'язок; $p(x)$-оператор Лапласа; глибоке навчання.*