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Stress state modeling of non-circular orthotropic hollow cylinders under different types of loading

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Based on a spatial model of the linear theory of elasticity, using an unconventional approach of the reduction of the original three-dimensional boundary value problem described by a system of partial differential equations with variable coefficients to a one-dimensional boundary value problem for a system of ordinary differential equations with constant coefficients, the problem of finding the dimensional stress of hollow elliptic orthotropic cylinders under the influence of various types of loading has been solved under certain boundary conditions at the orientation plane. Reducing the dimensionality of the original problem is carried out using analytical methods of separating variables in two coordinate directions in combination with the method of approximating functions by discrete Fourier series. The one-dimensional boundary value problem is solved by the stable numerical method of discrete orthogonalization.

Keywords: *discrete Fourier series; method of discrete orthogonalization; dimensional stress; orthotropic body; orientation plane; elliptic orthotropic cylinders.*

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1. Introduction

A significant number of structural elements of modern technology are made in the form of shells of complex shape and structure and are under the action of distributed, local, uneven loads due to various types of fastening of the orientation plane [1–3]. The extensive use of shell elements is explained by the desire to meet the requirements that take place in the difficult operating conditions of machines, aircraft, various structures and other units. Complications of structural forms and structures of shell systems require the development of approaches and methods for solving static and dynamic problems for shells made of anisotropic heterogeneous materials [4–7]. A characteristic feature of the development of approaches and methods for solving this class of problems is the correlation between the process of building a mathematical model and the development of a method used to solve the class of problems described by this model [8].

Together with universal approaches to solving problems of mechanics and mathematical physics using finite-difference schemes, finite-element and other discrete methods, approaches are used for a certain class of problems that allow reducing the original three-dimensional labor to ordinary differential equations based on the approximation of solutions relation to other variables using analytical tools.

This article uses an unconventional approach [9], which is based on the application of the analytical method of separating variables in two coordinate directions in combination with the approximation of functions by discrete series. This procedure makes it possible to reduce the dimensionality of the problem by transiting from a three-dimensional boundary value problem for a system of partial differential equations with variable coefficients to a one-dimensional boundary value problem for a system of ordinary differential equations with constant coefficients of a higher order. The last one-dimensional boundary value problem is solved by the stable numerical method of discrete orthogonalization.

2. Problem statement and problem-solving method

The statics problem of the linear theory of elasticity in the spatial formulation is solved for the stress state of orthotropic hollow cylinders with an elliptical cross-section, subjected to tangential stresses applied to their outer surface. Referring the cylinders to the curvilinear coordinate system, s, t, γ, which is constructed as follows: in the orthogonal curvilinear coordinate system, s, t a cylindrical surface is chosen as the reference surface, and the coordinate is placed along the normal to this surface. The first quadratic form in the selected coordinate system has the form

$$dS^2 = H_1^2 ds^2 + H_2^2 dt^2 + H_3^2 d\gamma^2,$$

where $H_1 = H_3 = 1$, $H_2 = (1+\gamma/R_t)^{-1}$ are Lame parameters, $R_t = R_t(t)$ is the radius of curvature of the transverse section. The middle surface equidistant from the side surfaces is chosen as the reference surface.

The initial equations are those of the linear theory of elasticity for an orthotropic body [10, 11]. The solution to the problem is sought in the domain $(0 \leqslant s \leqslant l,\ t_0 \leqslant t \leqslant t_1,\ -h/2 \leqslant \gamma \leqslant h/2)$, where l is the length of the cylinder, h is its thickness. The elliptical cross-section of the reference surface of the considered cylinders is given in a parametric form

$$x = b\cos\theta, \quad y = a\sin\theta \quad (0 \leqslant \theta \leqslant 2\pi), \tag{1}$$

here a, b are semi-axes of the ellipse, θ ші angle in cross-section.

We will consider cylinders whose cross-section perimeter of the reference surface is equal to the length of the circle of radius R, then the equality holds

$$\pi(a+b)\left(1+\frac{\Delta^2}{4}+\frac{\Delta^4}{64}+\frac{\Delta^6}{256}+\ldots\right) = 2\pi R. \tag{2}$$

Considering that $\Delta = (b-a)/(b+a)$ and including these terms up to Δ^6, in the balance equation (2), we have:

$$a = \frac{R}{f}(1-\Delta), \quad b = \frac{R}{f}(1+\Delta), \quad f = \left(1+\frac{\Delta^2}{4}+\frac{\Delta^4}{64}+\frac{\Delta^6}{256}\right), \quad \frac{a}{b} = \frac{1-\Delta}{1+\Delta}.$$

The value Δ (degree of ellipticity) characterizes the deviation of the shape of the cross-section of the reference surface from the circular one ($\Delta = 0$) cylinder of circular canonical form. The considered cylinders are closed along the guide, therefore the conditions of periodicity of all factors of their stress state are met. There are axial stresses σ_s at the ends, distributed in such a way that the end remains flat. The boundary conditions $s = 0$, $s = l$ at the ends are

$$\tau_{s\gamma} = \tau_{st} = u_s = 0. \tag{3}$$

Cylinders are under the influence of tangential stresses $\tau_{s\gamma} = \tau_0 \sin(\lambda_n s)\cos(\lambda_k t)$ ($\tau_0 = \text{const}$, $\lambda_n = \frac{\pi n}{l}$, $\lambda_k = \frac{2\pi k}{P}$), acting on the outer surface (P is the cross-sectional perimeter of the reference surface). The boundary conditions on the lateral surfaces have the form

$$\sigma_\gamma = 0, \quad \tau_{s\gamma} = \tau_0 \sin(\lambda_n s)\cos(\lambda_k t), \quad \tau_{t\gamma} = 0 \text{ at } \gamma = h/2,$$
$$\sigma_\gamma = 0, \quad \tau_{s\gamma} = 0, \quad \tau_{t\gamma} = 0 \text{ at } \gamma = -h/2. \tag{4}$$

When constructing a solving system of differential equations, the solving functions are chosen that make it possible to satisfy the boundary conditions on the limiting lateral surfaces (4) in the simplest way. Thus, three stress components σ_γ, $\tau_{s\gamma}$, $\tau_{t\gamma}$ and three displacement components are chosen as the solving functions u_γ, u_s, u_ψ. After some transformations, the basic equations can be used to solve a system of partial differential equations of the sixth order with variable coefficients [9]. Since the initial information regarding the shape of the cross-section (1) is specified using a parameter θ (different from the coordinate of the guide t), the coefficient of transition to coordinate t must be taken into account in the final calculations θ

$$\frac{dt}{d\theta} = \sqrt{\left(\frac{\partial x}{\partial\theta}\right)^2 + \left(\frac{\partial z}{\partial\theta}\right)^2} = \omega(\theta),$$

then the expression holds for an arbitrary function $V = V(\gamma, t(\theta))$

$$\frac{\partial V}{\partial \theta} = \frac{\partial V}{\partial t}\frac{\partial t}{\partial \theta}, \quad \frac{\partial V}{\partial t} = \frac{\partial V}{\partial \theta}\frac{1}{\omega(\theta)}.$$

Boundary conditions at the ends (3) allow us to reduce the dimensions of the problem due to the application of the method of separation of variables along the generating cylinder. For this, the solving functions are presented in the form of expansions in Fourier series along the coordinate s

$$X(s,t,\gamma) = \sum_{n=1}^{N} X_n(t,\gamma)\sin\lambda_n s, \quad Y(s,t,\gamma) = \sum_{n=0}^{N} Y_n(t,\gamma)\cos\lambda_n s, \tag{5}$$

$$X = \{\tau_{s\gamma}, u_s\}, \quad Y = \{\sigma_\gamma, \tau_{t\gamma}, u_\gamma, u_t\}.$$

After substituting the series (5) into the solving system of differential equations and boundary conditions (4), and separating the variables, for each $n = \overline{0, N}$ we arrive at a two-dimensional boundary value problem with respect to the amplitude values of the solving functions

$$\begin{aligned}
\frac{\partial \sigma_{\gamma,n}}{\partial \gamma} &= (c_2 - 1)\frac{1}{H_2}\frac{\partial H_2}{\partial \gamma}\sigma_{\gamma,n} - \lambda_n \tau_{s\gamma,n} - \frac{1}{H_2}\frac{\partial \tau_{t\gamma,n}}{\partial t} \\
&\quad + b_{22}\left(\frac{1}{H_2}\frac{\partial H_2}{\partial \gamma}\right)^2 u_{\gamma,n} + b_{12}\lambda_n \frac{1}{H_2}\frac{\partial H_2}{\partial \gamma}u_{s,n} + b_{22}\frac{1}{H_2^2}\frac{\partial H_2}{\partial \gamma}\frac{\partial u_{t,n}}{\partial t}, \\
\frac{\partial \tau_{s\gamma,n}}{\partial \gamma} &= c_1\lambda_n\sigma_{\gamma,n} - \frac{1}{H_2}\frac{\partial H_2}{\partial \gamma}\tau_{s\gamma,n} + b_{12}\lambda_n\frac{1}{H_2}\frac{\partial H_2}{\partial \gamma}u_{\gamma,n} + b_{11}\lambda_n^2 u_{s,n} \\
&\quad - b_{66}\frac{1}{H_2}\frac{\partial}{\partial t}\left(\frac{1}{H_2}\frac{\partial u_{s,n}}{\partial t}\right) + (b_{12} + b_{66})\lambda_n\frac{1}{H_2}\frac{\partial u_{t,n}}{\partial t}, \\
\frac{\partial \tau_{t\gamma,n}}{\partial \gamma} &= -c_2\frac{1}{H_2}\frac{\partial \sigma_{\gamma,n}}{\partial t} - \frac{2}{H_2}\frac{\partial H_2}{\partial \gamma}\tau_{t\gamma,n} - b_{22}\frac{1}{H_2}\frac{\partial}{\partial t}\left(\frac{1}{H_2}\frac{\partial H_2}{\partial \gamma}u_{\gamma,n}\right) \\
&\quad - (b_{12} - b_{66})\lambda_n\frac{1}{H_2}\frac{\partial u_{s,n}}{\partial t} - b_{22}\frac{1}{H_2}\frac{\partial}{\partial t}\left(\frac{1}{H_2}\frac{\partial u_{t,n}}{\partial t}\right) + b_{66}\lambda_n^2 u_{t,n}, \\
\frac{\partial u_{\gamma,n}}{\partial \gamma} &= c_4\sigma_{\gamma,n} - c_2\frac{1}{H_2}\frac{\partial H_2}{\partial \gamma}u_{\gamma,n} - c_1\lambda_n u_{s,n} - c_2\frac{1}{H_2}\frac{\partial u_{t,n}}{\partial t}, \\
\frac{\partial u_{s,n}}{\partial \gamma} &= a_{55}\tau_{s\gamma,n} + \lambda_n u_{\gamma,n}, \\
\frac{\partial u_{t,n}}{\partial \gamma} &= a_{44}\tau_{t\gamma,n} - \frac{1}{H_2}\frac{\partial u_{\gamma,n}}{\partial t} + \frac{1}{H_2}\frac{\partial H_2}{\partial \gamma}u_{t,n}
\end{aligned} \tag{6}$$

with boundary conditions

$$\begin{gathered}
\sigma_{\gamma,n} = 0, \quad \tau_{s\gamma,n} = \tau_0\cos(\lambda_k t), \quad \tau_{t\gamma,n} = 0 \quad \text{at} \quad \gamma = h/2, \\
\sigma_{\gamma,n} = 0, \quad \tau_{s\gamma,n} = 0, \quad \tau_{t\gamma,n} = 0 \quad \text{at} \quad \gamma = -h/2.
\end{gathered} \tag{7}$$

Further, we omit the index n in the notation of the solving functions for convenience.

The coefficients included in the system of equations (6) are determined due to the mechanical characteristics of the material of the cylinders and have the form

$$b_{11} = a_{22}a_{66}/\Omega, \quad b_{12} = -a_{12}a_{66}/\Omega, \quad b_{22} = a_{11}a_{66}/\Omega, \quad b_{66} = 1/a_{66}, \quad \Omega = (a_{11}a_{22} - a_{12})a_{66},$$

$$c_1 = -(b_{11}a_{13} + b_{12}a_{23}), \quad c_2 = -(b_{12}a_{13} + b_{22}a_{23}), \quad c_4 = a_{33} + c_1a_{13} + c_2a_{23},$$

where

$$a_{11} = \frac{1}{E_s}, \quad a_{12} = -\frac{\nu_{st}}{E_t} = -\frac{\nu_{ts}}{E_s}, \quad a_{13} = -\frac{\nu_{s\gamma}}{E_\gamma} = -\frac{\nu_{\gamma s}}{E_s},$$

$$a_{22} = \frac{1}{E_t}, \quad a_{23} = -\frac{\nu_{\gamma t}}{E_t} = -\frac{\nu_{t\gamma}}{E_\gamma}, \quad a_{33} = \frac{1}{E_\gamma}, \quad a_{44} = \frac{1}{G_{t\gamma}}, \quad a_{55} = \frac{1}{G_{s\gamma}}, \quad a_{66} = \frac{1}{G_{ts}}$$

(E_s, E_t, E_γ are moduli of elasticity in the direction of the coordinate axes, $G_{s\gamma}$, G_{st}, $G_{t\gamma}$ are shear moduli, $\nu_{s\gamma}$, ν_{st}, $\nu_{t\gamma}$ are corresponding Poisson ratios). Since the system of equations (6) includes Lame parameters $H_2(t,\gamma)$ dependent on two coordinates, it is impossible to separate the variables in the direction of the cylinder guide. To overcome this obstacle, we denote the product of the solving functions by the coefficients containing $H_2(t,\gamma)$, or their derivatives, by different complementary functions having

$$\begin{aligned}
&\varphi_1^j = \frac{1}{H_2}\frac{\partial H_2}{\partial \gamma}\left\{\sigma_\gamma, \tau_{s\gamma}, u_\gamma, u_s, \frac{1}{H_2}\frac{\partial H_2}{\partial \gamma}u_\gamma\right\} \quad (j=\overline{1,5}),\\
&\varphi_2^j = \frac{1}{H_2}\frac{\partial H_2}{\partial \gamma}\{\tau_{t\gamma}; u_t\} \quad (j=\overline{1,2}),\\
&\varphi_3^j = \frac{1}{H_2}\frac{\partial}{\partial t}\{\sigma_\gamma, u_\gamma, u_s\} \quad (j=\overline{1,3}),\\
&\varphi_4^j = \frac{1}{H_2}\left\{\frac{\partial \tau_{t\gamma}}{\partial t}, \frac{\partial u_t}{\partial t}, \frac{1}{H_2}\frac{\partial H_2}{\partial \gamma}\frac{\partial u_t}{\partial t}\right\} \quad (j=\overline{1,3}),\\
&\varphi_5 = \frac{1}{H_2}\frac{\partial \varphi_1^3}{\partial t},\\
&\varphi_6 = \frac{1}{H_2}\frac{\partial \varphi_3^3}{\partial t},\\
&\varphi_7 = \frac{1}{H_2}\frac{\partial \varphi_4^2}{\partial t}.
\end{aligned} \tag{8}$$

As a result of the introduction of additional functions (8), the system of partial differential equations with variable coefficients (6)formally will take the form of a system with constant coefficients, which will allow the separation of variables in the direction of the guide. Then, we present the solving and complementary functions in the form of expansions in Fourier series with respect to the coordinate t

$$X(t,\gamma) = \sum_{k=0}^{K} X_k(\gamma)\cos(\lambda_k t), \quad Y(t,\gamma) = \sum_{k=1}^{K} Y_k(\gamma)\sin(\lambda_k t), \tag{9}$$

$$X = \{\sigma_\gamma, \tau_{s\gamma}, u_\gamma, u_s, \varphi_1^j, \varphi_4^j, \varphi_6\}, \quad Y = \{\tau_{t\gamma}, u_t, \varphi_2^j, \varphi_3^j, \varphi_5, \varphi_7\}.$$

After substituting series (9) into the solving system of equations and boundary conditions and separating the variables, we obtain a one-dimensional boundary value problem described by a system of ordinary differential equations with constant coefficients in terms of the amplitude values of series (9) in the form

$$\begin{aligned}
&\frac{d\sigma_{\gamma,k}}{d\gamma} = (c_2-1)\varphi_{1,k}^1 - \lambda_n\tau_{s\gamma,k} - \varphi_{4,k}^1 + b_{22}\varphi_{1,k}^5 - b_{12}\lambda_n\varphi_{1,k}^4 + b_{22}\varphi_{4,k}^3,\\
&\frac{d\tau_{s\gamma,k}}{d\gamma} = c_1\lambda_n\sigma_{\gamma,k} - \varphi_{1,k}^2 + b_{12}\lambda_n\varphi_{1,k}^3 + b_{11}\lambda_n^2 u_{s,k} - b_{66}\varphi_{6,k} + (b_{12}+b_{66})\lambda_n\varphi_{4,k}^2,\\
&\frac{d\tau_{t\gamma,k}}{d\gamma} = -c_2\varphi_{3,k}^1 - 2\varphi_{2,k}^1 - b_{22}\varphi_{5,k} + (b_{12}+b_{66})\lambda_n\varphi_{3,k}^3 - b_{22}\varphi_{7,k} + b_{66}\lambda_n^2 u_{ti,k},\\
&\frac{du_{\gamma,k}}{d\gamma} = c_4\sigma_{\gamma,k} - c_2\varphi_{4,k}^2 - c_1\lambda_n u_{s,k} - c_2\varphi_{1,k}^3,\\
&\frac{du_{s,k}}{d\gamma} = a_{55}\tau_{s\gamma,k} + \lambda_n u_{\gamma,k},\\
&\frac{du_{t,k}}{d\gamma} = a_{44}\tau_{t\gamma,k} - \varphi_{3,k}^2 + \varphi_{2,k}^2
\end{aligned} \tag{10}$$

with the boundary conditions

$$\begin{gathered}
\sigma_{\gamma,k} = 0, \quad \tau_{s\gamma} = 0, \quad \tau_{t\gamma,k} = 0 \quad \text{at} \quad \gamma = -h/2,\\
\sigma_{\gamma,k} = 0, \quad \tau_{s\gamma,k} = \tau_0, \quad \tau_{t\gamma,k} = 0 \quad \text{at} \quad \gamma = h/2 \quad (k=\overline{0,K}).
\end{gathered} \tag{11}$$

The obtained one-dimensional boundary value problem (10), (2) is solved by the stable numerical method of discrete orthogonalization [12] simultaneously for all harmonics expanded into the Fourier series (9). Since, due to the formal introduction of complementary functions, the number of the unknowns in system (10) exceeds the number of equations. In the process of integration, at each step of applying the numerical method, the amplitude values of the complementary functions are calculated using their approximation by discrete Fourier series [13,14] with respect to the coordinate t using the standard procedure for finding Fourier coefficients for the functions given by the table of values.

Table 1. Evaluation of the accuracy of the results.

Δ	γ	$u_\gamma E_0/\tau_0$		σ_s/τ_0		σ_θ/τ_0		Error, %
		$\theta=0$	$\theta=\pi/2$	$\theta=0$	$\theta=\pi/2$	$\theta=0$	$\theta=\pi/2$	
0	$-h/2$	−6.9281	−6.9281	−2.6134	−2.6134	−0.7983	−0.7983	
	$h/2$	−5.8222	−5.8222	−4.6184	−4.6184	−0.7182	−0.7182	
0.001	$-h/2$	−6.8663	−6.9901	−2.6179	−2.6093	−0.7997	−0.7969	$\leqslant 1$
	$h/2$	−5.7617	−5.8829	−4.6124	−4.6243	−0.7152	−0.7212	
0.005	$-h/2$	−6.6204	−7.2394	−2.6348	−2.5919	−0.8051	−0.7917	$\leqslant 5.5$
	$h/2$	−5.5209	−6.1272	−4.5889	−4.6481	−0.7033	−0.7330	
0.01	$-h/2$	−6.3163	−7.5543	−2.6557	−2.5700	−0.8121	−0.7851	$\leqslant 10.5$
	$h/2$	−5.2233	−6.4359	−4.5595	−4.6781	−0.6884	−0.7478	

The reliability and accuracy of the results obtained on the basis of the considered technique is ensured by the corresponding variation in the number of integration and orthogonalization points when applying the numerical method and by the variation in the number of points chosen to construct tabular values of complementary functions and the number of retained terms of the discrete Fourier series [15]. It should be noted that in the latter case, as the retained terms of the discrete Fourier series increase, it increasingly approaches the usual ones.

In addition, the accuracy of the results is ensured on the basis of inductive means, in particular, the convergence of the solutions for non-circular hollow cylinders to the corresponding solutions in the case of circular cylinders, when the parameters characterizing the deviation of the shape of the cross-section from the circular one are zero.

In Table 1, the results of the convergence of the characteristics of the stress state of elliptical hollow cylinders to the solution are shown in the case of circular cylinders, when the degree of ellipticity is $\Delta \to 0$. The problem is solved using the following input data: $R = 30$, $l = 60$, $h = 6$, $\Delta = 0, 0.001, 0.005, 0.01$; mechanical parameters of the material [16] $E_s = 10E_0$, $E_\theta = 2.5E_0$, $E_\gamma = E_0$, $G_{s\gamma} = G_{\theta\gamma} = E_0$, $G_{s\theta} = 2E_0$, $\nu_{s\gamma} = 0.04$, $\nu_{s\theta} = 0.06$, $\nu_{\theta\gamma} = 0.1$; for tangential stresses applied on the outer surface ($\tau_{s\gamma} = \tau_0 \sin(\pi n/l)\cos(k\theta)$) putting $k = 0$, $n = 1$. Henceforth, all linear dimensions and displacements are normalized to a unit length, and stress is normalized to a unit load.

The values of normal displacements u_γ and normal stresses σ_s, σ_θ are given in the average cross-section of the length on the outer and inner surfaces for two values of the guide: in the zone of maximum $\theta = 0$ and minimum $\theta = \pi/2$ stiffness.

The values of normal displacements u_γ and normal stresses σ_s, and σ_θ are provided for an average cross-section along the length of the cylinder, both on the outer and inner surfaces. These values are given at two specific angular positions: $\theta = 0$, where the stiffness is maximum, and $\theta = \pi/2$, where the stiffness is minimum.

From the results given in Table 1, $\Delta = 0.01$ it can be seen that the relative error does not exceed 10.5%, $\Delta = 0.005 - 5.5\%$ and $\Delta = 0.001 - 1\%$. Thus, under the condition $\Delta \to 0$, the solution of the problem approaches the solution for a circular cylinder.

3. Numerical results and analysis

Consider the problem of the stress state of hollow elliptical cylinders with geometric parameters: $R = 30$, $l = 60$, $h = 6$, $\Delta = 0, 0.5, 0.1$; made of orthotropic material with mechanical parameters $E_s = 10E_0$, $E_\theta = 2.5E_0$, $E_\gamma = E_0$, $G_{s\gamma} = G_{\theta\gamma} = E_0$, $G_{s\theta} = 2E_0$, $\nu_{s\gamma} = 0.04$, $\nu_{s\theta} = 0.06$, $\nu_{\theta\gamma} = 0.1$, which are under the action of tangential stresses ($\tau_{s\gamma} = \tau_0 \sin(\pi n/l)\cos(k\theta)$), applied on the outer surface. We will consider the following variants of distribution of tangential stresses: (1) $k = 0$, $n = 1, 2, 5$; (2) $n = 1$, $k = 1, 3$.

The solution results are shown in Figures 1–5 at the average cross-section along the length: (***a***) — for the displacement fields ($-u_\gamma E_0/\tau_0$), (***b***) — for axial stress fields ($-\sigma_s/\tau_0$), (***c***) — for circular stress fields ($-\sigma_\theta/\tau_0$).

In Figures 1–4, the green color corresponds to the solution for a circular cylinder ($\Delta = 0$), red color is for the elliptical cylinder with $\Delta = 0.05$ degree of ellipticity, blue color is for the elliptical cylinder with $\Delta = 0.1$ degree of ellipticity. In Figure 5, characteristics of the stress state practically do not depend on the change in the shape of the cross-section, so they are marked in gray.

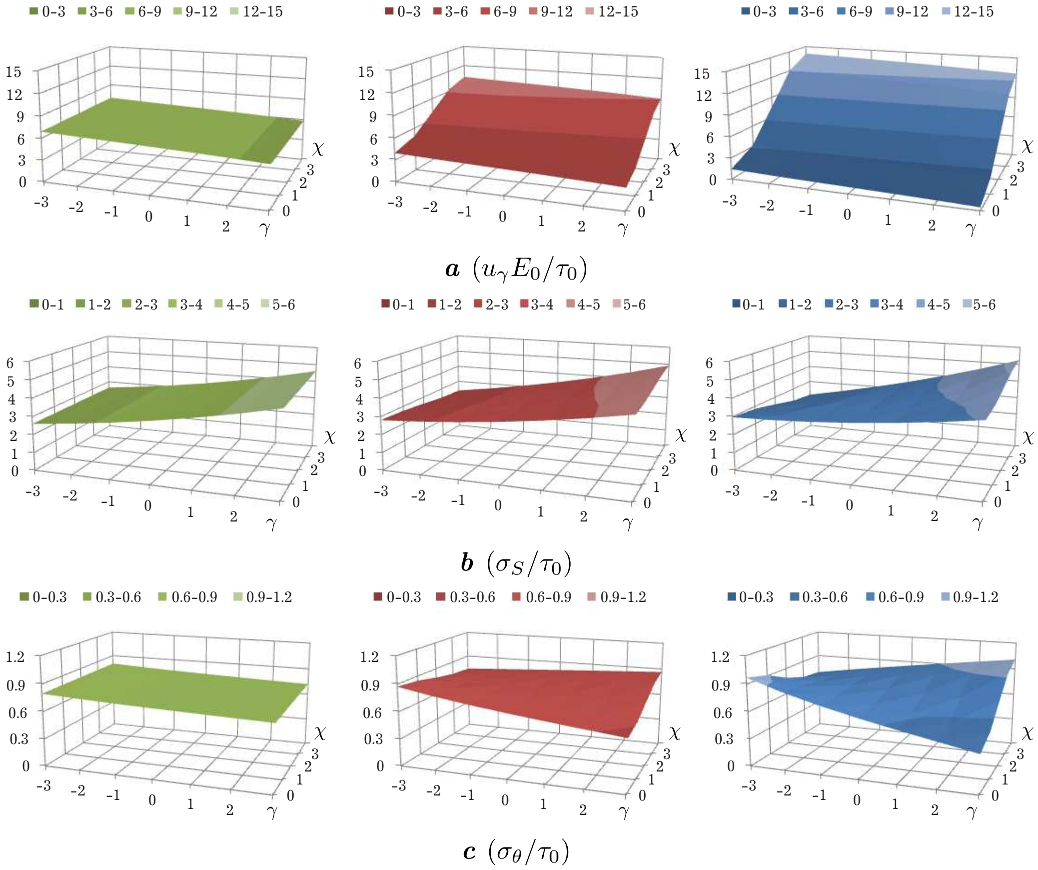


Fig. 1. Characteristics of the stress state of cylinders for loading $n = 1$, $k = 0$.

As can be seen from the above graphs, the distribution of displacements across the thickness is linear or nearly linear for almost all distributions of tangential stresses, except for $n = 5$, $k = 0$ (Figure 5a).

The maximum displacement amplitude value is obtained on the inner surface in the zone of minimum stiffness ($\theta = \pi/2$) for $n = 1, 2$, $k = 0$ (Figures 1, 3a); in the zone of maximum stiffness ($\theta = 0$) for $n = 1$, $k = 1$ (Figure 2a) and on the outer surface in the zone of minimum stiffness for $n = 5$, $k = 0$ (Figure 4a). In the case of non-uniform tangential stress along the direction for $n = 1$, $k = 3$ (Figure 5a), the displacements reach their maximum amplitude on the inner surface at two sections of the guide: in the zone of maximum stiffness ($\theta = 0$) and at the cross-section $\pi/4 \leqslant \theta \leqslant 3\pi/4$. At the same time, the effect of ellipticity is more pronounced in the case of uniform tangential stress ($n = 1$, $k = 0$ (Figure 1a)), uneven along the guide for $n = 1$, $k = 1$ (Figure 2a) and uneven along the generating line for $n = 2$, $k = 0$ (Figure 3a). Thus, with an increase in the degree of ellipticity,

Thus, with an increase in the degree of ellipticity, the amplitude values of the maximum displacements change as follows: increase by 1.5 times for $\Delta = 0.05$ and by 2 times for $\Delta = 0.1$ compared to a circular cylinder ($n = 1$, $k = 0$ (Figure 1a)); decrease by 1.2 times for $\Delta = 0.05$ and by 1.4 times for $\Delta = 0.1$ ($n = 1$, $k = 1$ (Figure 2a)); increase by 12% for $\Delta = 0.05$ and by 23% for $\Delta = 0.1$ ($n = 2$, $k = 0$ (Figure 3a)).

The distribution of displacements along the guide has a pronounced nonlinear character except for the case of non-uniform tangential stress along the ($n = 5$, $k = 0$ (Figure 4a)), for which the values of displacements are constant.

Axial stresses are predominant among normal ones σ_s. They acquire their maximum amplitude value on the outer loaded surface in the section $\theta = \pi/2$ for $n = 1, 2$, $k = 0$ (Figures 1, 3b), in cross-section $\theta = 0$ for $n = 1$, $k = 1$ (Figure 2b), in two sections $\theta = 0$ and $\pi/4 \leqslant \theta \leqslant 3\pi/4$ for $n = 1$, $k = 3$ (Figure 5b). In case $n = 5$, $k = 0$ (Figure 4b), axial stresses are distributed according to a constant linear law.

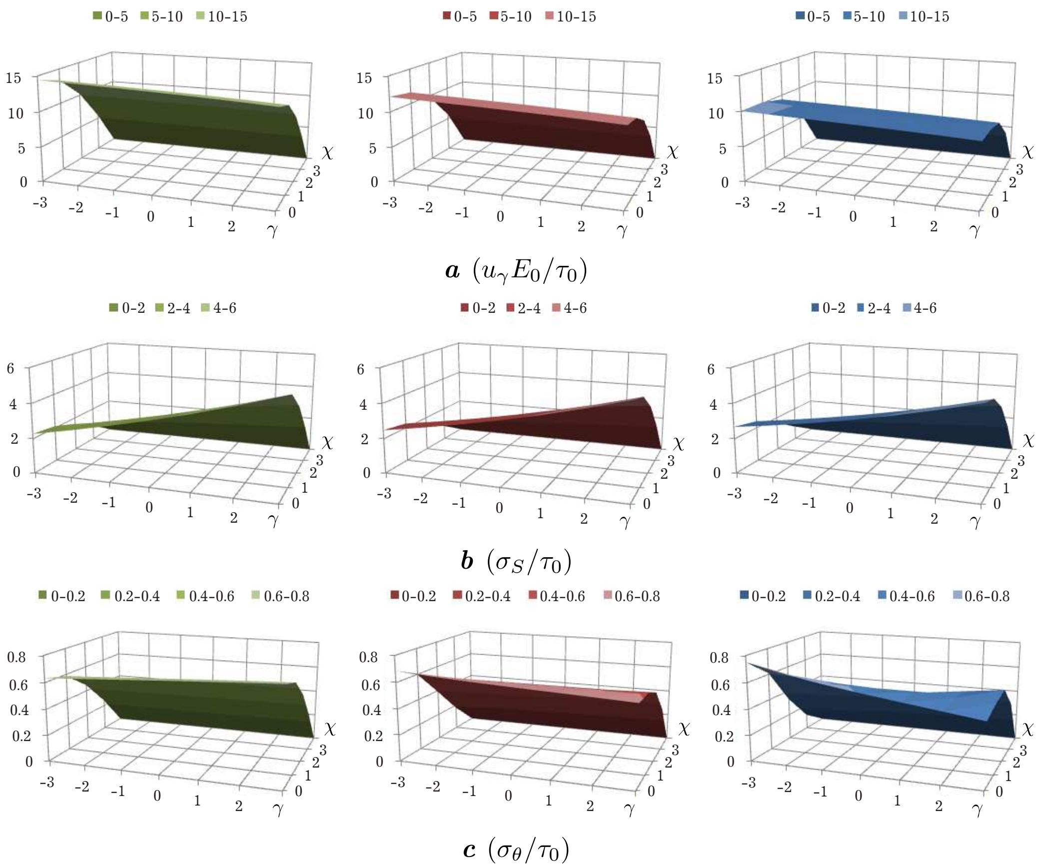


Fig. 2. Characteristics of the stress state of cylinders for loading $n = 1$, $k = 1$.

The effect of ellipticity on the distribution of axial stress fields is noticeable in the case of uniform tangential stress ($n = 1$, $k = 0$ (Figure 1b)). At the same time, when increasing the parameter Δ, the maximum stress amplitude values increase by 7% for $\Delta = 0.05$ and by 13% for $\Delta = 0.1$, compared to circular cylinders.

The distribution of axial stresses along the guide is nonlinear in the case of tangential stresses at $n = 1$, $k = 0, 1, 3$ (Figures 1, 2, 5b) and $n = 2$, $k = 0$ (Figure 3b). For tangential stress $n = 5$, $k = 0$, axial stresses do not change in the direction of the guide.

For the distribution of circular stresses, we have the following picture. They reach their maximum value on the outer surface at the cross-section $\theta = \pi/2$ for $n = 1, 2$, $k = 0$ (Figure 1, 3c) and at the cross-section $\theta = 0$ for $n = 5$, $k = 0$ (Figure 4c); on the inner surface of cross-section $\theta = 0$ for $n = 1$, $k = 1, 3$ (Figures 2, 5c). The effect of ellipticity occurs in the case of loading for $n = 1$, $k = 0, 1$ (Figures 1, 2c).

At the same time, the maximum amplitude values of the circular stresses increase for $n = 1$, $k = 0$ (Figure 1c) and decrease for $n = 1$, $k = 1$ (Figure 2c) about 1.2 times at $\Delta = 0.05$ and, accordingly, 1.5 times at $\Delta = 0.1$, compared to circular cylinders at $\Delta = 0$.

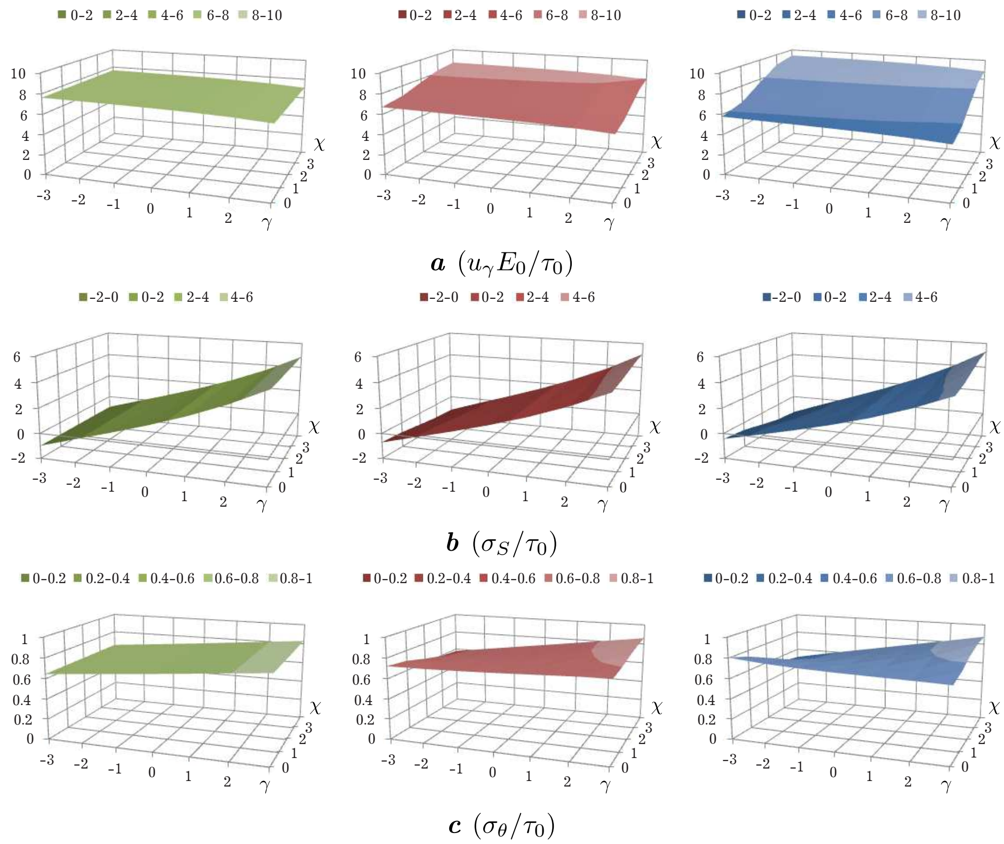


Fig. 3. Characteristics of the stress state of cylinders for loading $n = 2$, $k = 0$.

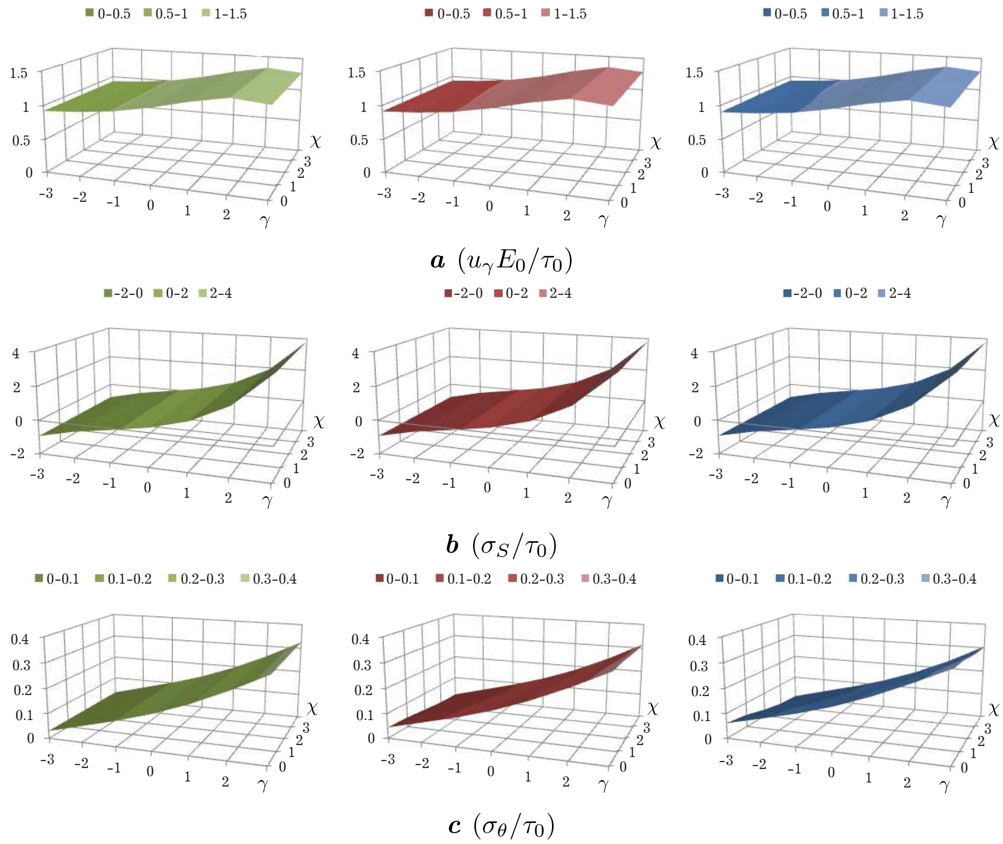
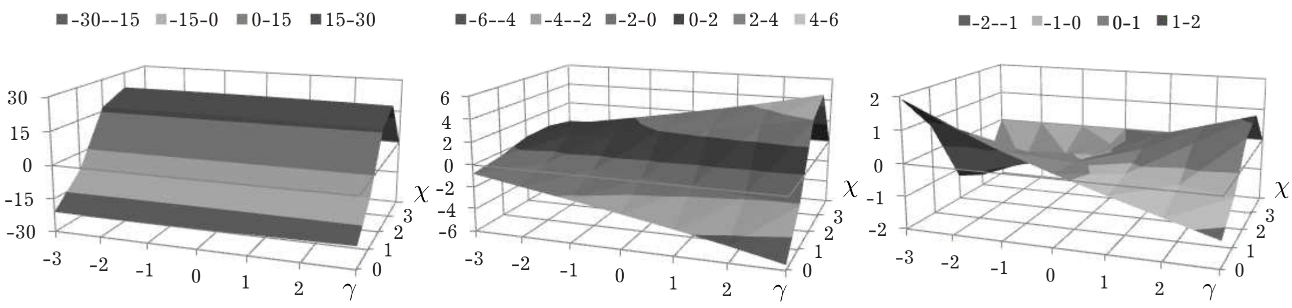


Fig. 4. Characteristics of the stress state of cylinders for loading $n = 5$, $k = 0$.



a $(u_\gamma E_0/\tau_0)$ ***b*** (σ_S/τ_0) ***c*** (σ_θ/τ_0)

Fig. 5. Characteristics of the stress state of cylinders for loading $n = 1$, $k = 3$.

The distribution of circular stresses along the guide for all considered variants of tangential stresses has a pronounced nonlinear character.

4. Conclusions

Based on a spatial model, employing analytical methods of variable separation combined with the approximation of functions by discrete Fourier series and the numerical method of discrete orthogonalization, the problem of the stress state of hollow orthotropic cylinders with an elliptical cross-section under the influence of tangential stresses on the outer surface is solved under specific boundary conditions. Different regularities of tangential stress variation are considered, and the effect of ellipticity on the distribution of displacement and stress fields is clarified.

In particular, it is shown that in the case of tangential stresses with parameters $n = 1$, $k = 3$, the values of both displacements and stresses do not depend on the change in the degree of ellipticity. In the case of tangential stresses with parameters $n = 1$, $k = 0, 1$,displacements and stresses are most sensitive to changes in the parameter Δ. The results obtained in the article can be used for strength calculations of structural elements of a similar shape under similar laws of external load distribution, or as theoretical studies of the mechanics of a deformable solid body.

[1] Musii R. S., Zhydyk U. V., Turchyn Ya. B., Svidrak I. H., Baibakova I. M. Stressed and strained state of layered cylindrical shell under local convective heating. Mathematical Modeling and Computing. **9** (1), 143–151 (2022).

[2] Lugovyi P. Z., Orlenko S. P. Effect of the Asymmetry of Cylindrical Sandwich Shells on their Stress–Strain State under Non-Stationary Loading. International Applied Mechanics. **57** (5), 543–553 (2021).

[3] Wang B., Hao P., Ma X., Tian K. Knockdown factor of buckling load for axially compressed cylindrical shells: state of the art and new perspectives. Acta Mechanica Sinica. **38**, 421440 (2022).

[4] Chekurin V. F., Postolaki L. I. Axially symmetric elasticity problems for the hollow cylinder with the stress-free ends. Analytical solving via a variational method of homogeneous solutions. Mathematical Modeling and Computing. **7** (1), 48–63 (2020).

[5] Zhang X., He Y., Li Z., Zhai Z., Yan R., Chen X. Static and dynamic analysis of cylindrical shell by different kinds of B-spline wavelet finite elements on the interval. Engineering with Computers. **36**, 1903–1914 (2020).

[6] Chekurin V. F., Postolaki L. I. Application of the Variational Method of Homogeneous Solutions for the Determination of Axisymmetric Residual Stresses in a Finite Cylinder. Journal of Mathematical Sciences. **249**, 539–552 (2020).

[7] Levchuk S. A., Khmel'nyts'kyi A. A. The Use of One of the Potential Theory Methods to Study the Static Deformation of Composite Cylindrical Shells. Strength Mater. **53**, 258–264 (2021).

[8] Pabyrivskyi V. V., Pabyrivska N. V., Pukach P. Ya. The study of mathematical models of the linear theory of elasticity by presenting the fundamental solution in harmonic potentials. Mathematical Modeling and Computing. **7** (2), 259–268 (2020).

[9] Grigorenko Ya. M., Rozhok L. S. Stress Analysis of Orthotropic Hollow Noncircular Cylinders. International Applied Mechanics. **40**, 679–685 (2004).

[10] Grigorenko Ya. M., Vasilenko A. T., Emel'yanov N. G., et al. Statics of Structural Members. Vol. 8 of the 12-volume series Mechanics of Composites. Kyiv, A.S.K. (1999).

[11] Lekhnitskii S. G. Theory of elasticity of an anisotropic elastic body. San Francisco, Holden-Day Inc. (1963).

[12] Godunov S. K. Numerical solution of boundary-value problems for a system of linear ordinary differential equations. Usp. Mat. Nauk. **16** (3), 171–174 (1961).

[13] Korn G. A., Korn T. M. Mathematical Handbook for Scientists and Engineers. McGraw-Hill, New York (1961).

[14] Hamming R. W. Numerical Methods for Scientists and Engineers. MG Graw-Hill, New York (1962).

[15] Grigorenko Ya. M., Rozhok L. S. Stress Analysis of Hollow Orthotropic Cylinders with Oval Cross-Section. International Applied Mechanics. **57** (2), 160–171 (2021).

[16] Grigorenko Ya. M., Vlaikov G. G., Grigorenko P. Ya. Numerical-analytical solution of shell mechanics problems based on various models. Kyiv, Publishing house "Academperiodika" (2006).

Моделювання напруженого стану некругових ортотропних порожнистих циліндрів за різних видів навантаження

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З використанням просторової моделі лінійної теорії пружності на основі нетрадиційного підходу, що базується на редукції вихідної тривимірної крайової задачі, яка описується системою диференціальних рівнянь в частинних похідних зі змінними коефіцієнтами, до одновимірної крайової задачі для системи звичайних диференціальних рівнянь зі сталими коефіцієнтами, розв'язано задачу про напружений стан порожнистих еліптичних ортотропних циліндрів, що знаходяться під дією різних видів навантаження, за певних граничних умов на торцях. Зниження вимірності вихідної задачі здійснюється за допомогою аналітичних методів відокремлення змінних в двох координатних напрямках в поєднанні з методом апроксимації функцій дискретними рядами Фур'є. Одномірна крайова задача розв'язується стійким чисельним методом дискретної ортогоналізації.

Ключові слова: *дискретні ряди Фур'є; метод дискретної ортогоналізації; просторовий напружений стан; ортотропний матеріал; плоский торець; порожнисті еліптичні циліндри.*