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Total and partial observation–detection in linear dynamical systems with characterized sources: finite-dimensional cases

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In this work, we address the partial observation–detection problem for finite-dimensional dynamical linear systems that may not be fully observable or detectable. We introduce the concepts of 'observation–detection' and 'partial observation–detection,' which involve reconstructing either the entirety or a portion of the system state and the source reacting on the system, even when the system is not fully observable or detectable. We provide characterizations of 'observable–detectable systems' and 'observable–detectable spaces.' The reconstruction of the state and source on the observable–detectable subspace is achieved through orthogonal projection, leveraging the algebraic structure of the given finite-dimensional system. Additionally, we present examples to illustrate our approach.

Keywords: *observation; detection; dynamical systems; source detection.*

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1. Introduction

A dynamical system is a kind of multiple objects interacting with each other. Mathematically, this interaction can be represented by a model of equations and signs. The system is linked to its environment through input elements (physical elements acting on the system) and outputs (measurements or observations). The analysis of several concepts is necessary to better understand a given dynamical system and its functioning in order to optimize its use. Among the fundamental concepts constituting the analysis of systems are those of controllability, observability, stability and stabilizability [1–8].

The observation problem consists of extracting the state of the system by using the output equation and the dynamic of the system. In the case of a non-observable system, we will never be able to extract the totality of the system state; that is why we have opted for partial observability. This kind of observability consists of observing and extracting the reconstructible part of the system state from the output equation unless the system is not fully observable. In this paper, we focus just on the observability problem for finite-dimensional linear systems.

The detection notion is introduced by A. El Jai and his team [9]. It consists of reconstructing an unknown source reacting to the considered system by using a measurement tool. The application field of this notion is too large and can be applied to multiple disciplines. We can cite examples such as medicine, pollution phenomena, and the military among others.

The partial analysis of a dynamical system is necessary when the system is not observable or controllable. It is worth mentioning here the Kalman decomposition [10, 11]: Kalman has decomposed the state space into a direct sum of four vector subspaces, based on the two notions of observability and controllability. He gives the canonical form of the corresponding equations. The proof of the state space decomposition theorem was made by Kalman and L. Weiss [11]. In particular, when considering the properties of controllability and observability, several possibilities of this decomposition are indicated [11]; despite this, a complete and rigorous proof is still not produced.

Partial observation will be useful when we concentrate our attention only on a very specific parameters or a combination of state parameters. In this case, the study can be reduced and concentrated on the desired parameters or a combination of parameters. We cite as an example the work done by

D. Bichara and All [12], in which the concern is on measuring the circulating parasitemia $y_1 + y_2$ to estimate the sequestered parasitemia $y_3 + y_4 + y_5$ in a patient.

One can be interested in partial observation when dealing with a system in which many parameters act making it impossible to observe the system. As an example, it is impossible to do all medical tests and analyses in order to know the health state of the patient, that is why we limit our concentration and interest to the measurement of only some parameters, such as the temperature and pressure of the patient, to know approximately the health condition. This fact confronts us with the observation of a system that is not necessarily totally observable, which pushes us to deal with only a part of the patient's health condition.

The "partial" analysis is also necessary when the system under study does not check the standard operating conditions, i.e., when the system contains an ambiguity that prevents the behavior of the system from being known so that it can be used and controlled. We cite as examples among many others: incomplete measurement systems, complex systems and large systems. In this context, several approaches have been adopted for the partial analysis of the system. We cite the work of T. Boukhobza and his team, who proposed a method based on a graphical theoretical approach [13]. There is also a qualitative study of the two notions of observability and controllability, made by W. Kang and L. Xu [14]. In their work, they used dynamic optimization and its calculation methods as a tool to quantitatively define and measure observability and accessibility.

In our work, we considered two different problems, observation and detection, assembled and treated at the same time. Indeed, we will try to observe the state and detect the external source reaction on our system at the same time. For that, we did introduce the so-called observation–detection problem. Our approach involves combining two systems into one, where the first one describes the dynamics of the studied system. The second one describes the dynamic of the source distribution. The observation–detection problem becomes just an observational one for the final combined system.

In this work, the notion of observation–detection has been introduced, defined and characterized. A similar result has been demonstrated in the Kalman characterization of the observation of a given linear finite dimensional dynamical system for observation–detection problem. Generalization has been introduced too, by defining the so called partial observation–detection. Characterization of this notion has been proved and applied in some examples. We have determined the biggest observable–detectable part and reconstructed it. Generalization of Kalman characterization has also been given.

This paper is organized as follows. In the first part, we give our problem statement, as some preliminary results used thereafter. In the second part, we present the observation–detection notion and some characterization of it. In the third part, we give the definition of an observable–detectable subspace as well as some characterizations. In the fourth part, we explain the entire procedure and the theoretical approach followed for the partial reconstruction of the system state and the perturbation source.

2. Preliminary and problem statement

2.1. Example and problematic

Let us consider the system defined in the interval $[0, T]$ given by the following equation:

$$\begin{pmatrix} \dot{z}_1(t) \\ \dot{z}_2(t) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix} + \begin{pmatrix} f_1(t) \\ f_2(t) \end{pmatrix} \tag{1}$$

with output equation

$$y(t) = z_2(t). \tag{2}$$

General solution of this system can be written as

$$\begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix} = \begin{pmatrix} e^t z_{0,1} + \left(e^{2t} - e^t\right) z_{0,2} + \int_0^t e^{t-s} f_1(s)\, ds \\ e^{2t} z_{0,2} + \int_0^t e^{t-s} f_2(s)\, ds \end{pmatrix}, \tag{3}$$

then output function is given by the following equation:

$$y(t) = z_{0,2}\, e^{2t} + \int_0^t f_2(s)\, e^{2(t-s)}\, ds. \tag{4}$$

This system is not observable and not detector. Indeed, if we take the two following initial states,

$$z_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \tilde{z}_0 = \begin{pmatrix} -2 \\ 0 \end{pmatrix}$$

and the following two sources:

$$f(t) = \begin{pmatrix} t \\ 0 \end{pmatrix} \quad \text{and} \quad \tilde{f}(t) = \begin{pmatrix} -2t \\ 0 \end{pmatrix},$$

the corresponding output functions to (z_0, f) and $(\tilde{z}_0, \tilde{f})$ are y and $\tilde{y}$ successively with,

$$y(t) = \tilde{y}(t) = 0 \tag{5}$$

for all $t \in]0, T[$, then the system is not observable and cant detect the totality of source function.

We have

$$y(t) = e^{2t}\, z_{0,2} + \int_0^t e^{t-s} f_2(s)\, ds, \tag{6}$$

then

$$\frac{1}{2}\dot{y}(t) = e^{2t} z_{0,2} + \int_0^t e^{2(t-s)} f_2(s)\, ds + \frac{1}{2} f_2(t) \tag{7}$$

by subtracting (6) from (7), we obtain the following equation,

$$f_2(t) = \dot{y}(t) - 2y(t). \tag{8}$$

Hence we reconstructed f_2 the second component of the source f. This result lead as to think to possibility of doing the same think to all non observable linear system in the kind of (1), (2). We can then ask the following questions: If a system with the out-put equation can not detect the perturbation source, can we know some information about this source or not? If that is possible, how we can do it and characterize it? What is the relation between detection (partial detection) and observation (partial observation)?

Before trying to answer to this questions, we will try to introduce the definition of observation–detection system and characterize it. This will be the object if of section 3.

2.2. Considered problem

Let us consider the following finite dimensional linear dynamical system:

$$\begin{cases} \dot{z}(t) = A\, z(t) + f(t), \quad t_0 < t < T, \quad A \in \mathcal{M}_n(\mathbb{R}), \\ z(t_0) = z_0 \in \mathbb{R}^n, \end{cases} \tag{9}$$

augmented by the following output function

$$y = C\, z(t). \tag{10}$$

In this work, we will try to reconstruct, totally or partially, the state z and the source f, by using the output function y in case where the source f verify the following system

$$\begin{cases} \dot{f}(t) = A_1\, f(t), \quad t_0 < t < T, \quad A_1 \in \mathcal{M}_n(\mathbb{R}), \\ f(t_0) \in \mathbb{R}^n. \end{cases} \tag{11}$$

The system (9), (10) and (11) can be written in the following form:

$$\begin{cases} \dot{z} = A\, z(t) + f(t), \\ z(0) = z_0, \\ \dot{f}(t) = A_1\, f(t), \\ f(0) = f_0 \end{cases}$$

for all $t \in]t_0, T[$, or

$$\begin{cases} \begin{pmatrix} \dot{z}(t) \\ \dot{f}(t) \end{pmatrix} = \begin{pmatrix} A\, z(t) + f(t) \\ A_1\, f(t) \end{pmatrix}, \quad \forall t \in]t_0, T[, \\ \begin{pmatrix} z(0) \\ f(0) \end{pmatrix} = \begin{pmatrix} z_0 \\ f_0 \end{pmatrix}, \end{cases}$$

which is equivalent to

$$\begin{cases} \begin{pmatrix} \dot{z}(t) \\ \dot{f}(t) \end{pmatrix} = \begin{bmatrix} A & I_{n\times n} \\ 0_{n\times n} & A_1 \end{bmatrix} \begin{pmatrix} z(t) \\ f(t) \end{pmatrix}, \quad \forall t \in]t_0, T[, \\ \begin{pmatrix} z(0) \\ f(0) \end{pmatrix} = \begin{pmatrix} z_0 \\ f_0 \end{pmatrix}. \end{cases}$$

This can be also written as

$$\begin{cases} \dot{u}(t) = \mathcal{A}\, u(t), \quad \forall t \in]t_0, T[, \\ u(0) = u_0. \end{cases} \tag{12}$$

The output equation can be also written as

$$y(t) = \mathcal{C}\, u(t), \quad \forall t \in [t_0, T] \tag{13}$$

with

$$\mathcal{A} = \begin{bmatrix} A & I_{n\times n} \\ 0_{n\times n} & A_1 \end{bmatrix}, \quad \mathcal{C} = [C \quad 0_{n\times n}], \quad u(t) = \begin{pmatrix} z(t) \\ f(t) \end{pmatrix}, \quad u(0) = u_0 = \begin{pmatrix} z_0 \\ f_0 \end{pmatrix}.$$

The detection–observation of the state z and the source f is transformed, by using the system (12), (13), to a simple observation problem.

2.3. Useful results

In this subsection we present some results that will be used throughout. We introduce the observability–detectability operator given by

$$\mathcal{R}\colon u_0 \in \mathbb{R}^{2n} \longrightarrow \mathrm{L}^2\,[t_0, T; \mathbb{R}^q]\; (\mathcal{R}u_0)(t) = \mathcal{C}u(t) = [C \quad 0_{n\times n}]\, u(t),$$

with

$$(\mathcal{R}u_0)(t) = C\, z(t) = C\, e^{A(t-t_0)} z_0 + \int_{t_0}^{t} C\, e^{t-s} e^{(s-t_0)A_1} f_0\, ds.$$

$\mathcal{R}$ can also be written as

$$(\mathcal{R}u_0)(t) = \left[C\, e^{A(t-t_0)} \quad \int_{t_0}^{t} C\, e^{t-s} e^{(s-t_0)A_1}\, ds \right] u_0, \quad \forall t \in [t_0, T],$$

whose adjoint operator

$$\mathcal{R}^*\colon \mathrm{L}^2\,[t_0, T; \mathbb{R}^q] \longrightarrow \mathbb{R}^{2n}$$

can be written as

$$\mathcal{R}^*\eta = \int_{t_0}^{T} e^{(t-t_0)\mathcal{A}^{\mathrm{T}}} \mathcal{C}^{\mathrm{T}} \eta(t)\, dt, \quad \eta \in \mathrm{L}^2[t_0, T; Y].$$

We denote by $\mathcal{M}$ the following matrix

$$\mathcal{M} \equiv \mathcal{R}^*\mathcal{R} = \int_{t_0}^{T} e^{(t-t_0)\mathcal{A}^{\mathrm{T}}} \mathcal{C}^{\mathrm{T}} \mathcal{C}\, e^{(t-t_0)\mathcal{A}}\, dt \in \mathcal{M}_{2n}(\mathbb{R}).$$

Remark 1. (1) The matrix $\mathcal{M}$ is symmetric and positive semi-definite. (2) The system is observable if and only if, the matrix $\mathcal{M}$ is positive definite. (3) We have

$$\mathrm{Im}(\mathcal{R}^*) = \mathrm{Im}(\mathcal{M}), \quad \ker(\mathcal{R}) = \ker(\mathcal{M})$$

and

$$\mathbb{R}^{2n} = \mathrm{Im}(\mathcal{M}) \oplus \ker(\mathcal{M}). \tag{14}$$

We suppose for the rest of this work, without losing generality, that $\mathrm{Im}(\mathcal{M}) = \mathrm{vect}\{v_1, v_2, \ldots, v_p\}$ with $p \leqslant 2n$, and we take

$$N = \begin{pmatrix} v_1 & | & v_2 & | & \ldots & | & v_p \end{pmatrix}.$$

N is $2n \times p$ injective matrix with $\mathrm{rank}(N) = p$.

Proposition 1. The matrix $N^T \mathcal{M}\, N$ is invertible.

Remark 2. Before begin the proof of the proposition, let us mention that if $\xi \in \mathrm{Im}(N)$ (or $\xi \in \mathrm{Im}(\mathcal{M})$) and $N\xi = 0$ (or and $\mathcal{M}\xi = 0$) then $\xi = 0$.

Proof. $N^T \mathcal{M}\, N$ is a symmetric matrix, then $N^T \mathcal{M}\, N$ is surjective.

Let us show now that $N^T \mathcal{M} N$ is injective. Let us take $v \in \mathbb{R}^p$, we have

$$\begin{aligned} N^T \mathcal{M} N v = 0 &\Longrightarrow \mathcal{M} N v = 0, \quad \text{(from remark 2 because } \mathcal{M} N v \in \mathrm{Im}(N) \text{ and } \mathrm{Im}(M) = \mathrm{Im}(N)) \\ &\Longrightarrow N v = 0, \quad \text{(from remark 2 because } N v \in \mathrm{Im}(N) \text{ and } \mathrm{Im}(M) = \mathrm{Im}(N)) \\ &\Longrightarrow v = 0, \quad \text{(because } N \text{ is injective).} \end{aligned}$$

That is for all $v \in \mathbb{R}^p$, then $N^T \mathcal{M} N$ is injective. Finally we can conclude that $N^T \mathcal{M} N$ is invertible. ∎

Lemma 1. *Let R be a $m \times 2n$ matrix, $T > t_0$ and $x \in \mathbb{R}^{2n}$. Then the following properties are equivalent:*

(1) $R\, e^{(t-t_0)\mathcal{A}} x = 0$, $\forall t \in [t_0, T]$;
(2) $R\, \mathcal{A}^k x = 0$, $\forall k \in \mathbb{N}$;
(3) $R\, e^{t\mathcal{A}} x = 0$, $\forall t \in \mathbb{R}$;
(4) $R\, \mathcal{A}^{k-1} x = 0$, $1 \leqslant k \leqslant 2n$.

Proof.

– The implications (3) $\Longrightarrow$ (1) and (2) $\Longrightarrow$ (4) are equivalent.
– (1) $\Longrightarrow$ (2): If $R\, e^{(t-t_0)\mathcal{A}} x = 0$, for all $t \in [t_0, T]$ then for all $t \in [t_0, T]$ we have

$$R\, x + (t-t_0) R\, \mathcal{A}\, x + \frac{(t-t_0)^2}{2} R\, \mathcal{A}^2 x + \frac{(t-t_0)^3}{6} R\, \mathcal{A}^3 x + \ldots + \frac{(t-t_0)^j}{j!} R\, \mathcal{A}^j x + \ldots = 0.$$

The kth derivative with respect to t gives

$$R\, \mathcal{A}^k x + (t-t_0) R\, \mathcal{A}^{k+1} x + \frac{(t-t_0)^2}{2} R\, \mathcal{A}^{k+2} x + \frac{(t-t_0)^3}{6} R\, \mathcal{A}^{k+3} x + \ldots + \frac{(t-t_0)^j}{j!} R\, \mathcal{A}^{k+j} x + \ldots = 0,$$

which, for $t = t_0$, becomes $R\, \mathcal{A}^k x = 0$, for all $k \in \mathbb{N}$.
– (2) $\Longrightarrow$ (3): If $R\, \mathcal{A}^k x = 0$, for all $k \in \mathbb{N}$, then for every $t \geqslant t_0$, we have

$$R\, e^{(t-t_0)\mathcal{A}}\, x = R\, x + (t-t_0) R\, \mathcal{A}\, x + \frac{(t-t_0)^2}{2} R\, \mathcal{A}^2 x + \frac{(t-t_0)^3}{6} R\, \mathcal{A}^3 x + \ldots + \frac{(t-t_0)^j}{j!} R\, \mathcal{A}^j x + \ldots = 0.$$

– (4) $\Longrightarrow$ (2): Let us assume that $R\mathcal{A}^{k-1} x = 0$, $1 \leqslant k \leqslant 2n$, Cayley–Hamilton's theorem gives the decomposition

$$\mathcal{A}^{2n} = \sum_{j=0}^{2n-1} \beta_{2n,j} \mathcal{A}^j, \quad \beta_{2n,j} \in \mathbb{R}.$$

We deduce (by recurrence) a similar decomposition of A^k

$$\mathcal{A}^k = \sum_{j=0}^{2n-1} \beta_{kj} \mathcal{A}^j, \quad \beta_{kj} \in \mathbb{R},$$

and then

$$R\, \mathcal{A}^k x = \sum_{j=0}^{2n-1} \beta_{kj} R\, \mathcal{A}^j x = 0, \quad \forall k \geqslant 2n.$$

∎

Let us denote

$$\mathbf{O} = \begin{bmatrix} \mathcal{C} \\ \mathcal{C}\mathcal{A} \\ \vdots \\ \mathcal{C}\mathcal{A}^{2n-1} \end{bmatrix}.$$

We have the following proposition.

Proposition 2. (1) We have

$$\ker(\mathcal{M}) = \cap_{k=1}^{2n} \ker\left(\mathcal{C}\, \mathcal{A}^{k-1}\right) = \ker(\mathbf{O}), \tag{15}$$

and then $\mathrm{rg}(\mathcal{M}) = \mathrm{rg}(\mathbf{O})$.

(2) $\ker(\mathcal{M})$ is stable by $\mathcal{A}^k$ with $k = 1, 2, 3, \ldots$ and then by $e^{(t-t_0)\mathcal{A}}$, $\forall t \geqslant t_0$.

Proof.

– Let $x \in \mathbb{R}^{2n}$. $x \in \ker(\mathcal{M}) = \ker(\mathcal{R})$ equivalent to $\mathcal{C}\, e^{(t-t_0)\mathcal{A}} x = 0$, for all $t \in [t_0, T]$. Or by taking $R = \mathcal{C}$ in the Lemma (1) $\mathcal{C}\,\mathcal{A}^{k-1} x = 0$, $1 \leqslant k \leqslant 2n$, which is none other than $x \in \cap_{k=1}^{2n} \ker(\mathcal{C}\,\mathcal{A}^{k-1})$. This relation is equivalent to
$$\begin{bmatrix} \mathcal{C}\, x \\ \mathcal{C}\,\mathcal{A}\, x \\ \vdots \\ \mathcal{C}\,\mathcal{A}^{2n-1}\, x \end{bmatrix} = 0,$$
or even $x \in \ker(\mathbf{O})$ and the following properties follow. By taking the orthogonal in the relation $\ker(\mathcal{M}) = \ker(\mathbf{O})$ we obtain $\mathrm{Im}(\mathcal{M}) = \mathrm{Im}(\mathbf{O}^{\mathrm{T}})$ then $\mathrm{rg}(\mathcal{M}) = \mathrm{rg}(\mathbf{O}^{\mathrm{T}}) = \mathrm{rg}(\mathbf{O})$.

– If $x \in \ker(\mathcal{M}) = \ker(\mathcal{R})$ then $\mathcal{C}\, e^{(t-t_0)\mathcal{A}} x = 0$, for all $t \in [t_0, T]$ which, according to the Lemma 1, is equivalent to $\mathcal{C}\,\mathcal{A}^j x = 0$, for all $j \in \mathbb{N}$, then for $k \in \mathbb{N}$ we have
$$\mathcal{C}\,\mathcal{A}^j(\mathcal{A}^k x) = 0, \quad \forall j \in \mathbb{N},$$
then for $j = 0$ we have $\mathcal{A}^k x \in \ker(\mathcal{R})$. From that, since
$$e^{(t-t_0)\mathcal{A}} x = \sum_{k \geqslant 1} \frac{(t-t_0)^k}{k!} \mathcal{A}^k x \in \ker(\mathcal{R}),$$
then the stability of $\ker(\mathcal{R}) = \ker(\mathcal{M})$ follows.

■

For a subspace H of $\mathbb{R}^{2n}$, let us introduce the matrices $\mathbf{G}_H$ of order $2n$ and $\mathbf{Q}_H$ of type $(2n)^2 \times 2n$:
$$\mathbf{G}_H = \int_{t_0}^{T} e^{(t-t_0)\mathcal{A}^{\mathrm{T}}} (\mathrm{P}_H)^{\mathrm{T}} \mathrm{P}_H e^{(t-t_0)\mathcal{A}}\, dt, \quad \mathbf{Q}_H = \begin{bmatrix} \mathrm{P}_H \\ \mathrm{P}_H \mathcal{A} \\ \vdots \\ \mathrm{P}_H \mathcal{A}^{2n-1} \end{bmatrix}. \tag{16}$$

Lemma 2. *We have for every subspace H of $\mathbb{R}^{2n}$*
$$\ker(\mathbf{G}_H) = \cap\_t_0 \leqslant t \leqslant T \ker\left(\mathrm{P}_H e^{(t-t_0)\mathcal{A}}\right) = \cap_{k=1}^{2n} \ker\left(\mathrm{P}_H \mathcal{A}^{k-1}\right) = \ker(\mathbf{Q}_H). \tag{17}$$

Proof. The proof is similar to that of the Proposition 2 by taking P_H instead of $\mathcal{C}$ and by applying the Lemma 1 with $R = \mathrm{P}_H$. ■

3. Observation–detection

3.1. Definitions and characterizations

The detection notion is trying to reconstruct an unknown source that reacting (or disturbing) on a given system, the following definition is given in [9].

Definition 1. *A source is said to be detectable (or reconstructible) on $]t_0, T[$ if the knowledge of the system, together with the output, is sufficient to make the associated operator $\mathcal{Q}$ is injective with:*
$$\mathcal{Q}\colon f_0 \in \mathbb{R}^n \longrightarrow y(\cdot) \in \mathrm{L}^2\,[t_0, T; \mathbb{R}^q]\,.$$

This definition can lead us to source reconstruction. In this work, since the initial state is considered unknown, we shoos to try reconstruction of the state and the source in the same time. For that, the problem will become observation and detection in the same time. Thus the following definition.

Definition 2. *The system (9), (10) is said observable-S_f-detector during the time interval $[t_0, T]$ if for two given couples $(z_{0,1}, f_{0,1})$ and $(z_{0,2}, f_{0,2})$ that gives the same output on $[t_0, T]$, then necessary they are equal:*
$$y_1(t) = y_2(t), \quad \forall t \in [t_0, T] \Longrightarrow z_{0,1} = z_{0,2} \quad \text{and} \quad f_{0,1} = f_{0,2}. \tag{18}$$

Remark 3. (1) The previews definition is equivalent to: For all couple (z_0, f_0) that gives null output function, then they are null.

$$y(t) = 0, \quad \forall t \in [t_0, T] \Longrightarrow z_0 = 0 \quad \text{and} \quad f_0 = 0. \tag{19}$$

(2) If the system is observable-S_f-detector then we can reconstruct the initial state and the source in the same time.

Proposition 3. The system (9), (10) is observable-S_f-detector, if and only if (12), (13) is observable.

Proof. We can obtain the result simply, by taking $u = \binom{z}{f}$ and $u_0 = \binom{z_0}{f_0}$. ∎

Remark 4. (1) The system (12), (13) is observable if and only if $\mathcal{R}$ is injective

$$\ker(\mathcal{R}) = \{0\}. \tag{20}$$

(2) If $\mathcal{R}$ is injective then $\mathcal{R}^*\mathcal{R}$ is invertible then we can write

$$u_0 = [\mathcal{R}^*\mathcal{R}]^{-1}\,\mathcal{R}\,y(t). \tag{21}$$

Proposition 4. The system (9), (10) is observable-S_f-detector if and only if

$$\operatorname{rank}(O) = 2n. \tag{22}$$

Proof. The system (9), (10) is observable-S_f-detector if and only if $\mathcal{R}$ is injective, which equivalent to $\mathcal{M} = \mathcal{R}^*\mathcal{R}$ is invertible, or to $\operatorname{rank}(\mathcal{M}) = 2n$, since from (1) of the proposition 2, $\operatorname{rank}(O) = \operatorname{rank}(\mathcal{M})$, the system (9), (10) is observable-S_f-detector if and only if $\operatorname{rank}(O) = 2n$. ∎

Example 1. Let us consider the system defined in the interval $[t_0, T]$ given by the following equation:

$$\begin{pmatrix} \dot{z}_1(t) \\ \dot{z}_2(t) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix} + \begin{pmatrix} f_1(t) \\ f_2(t) \end{pmatrix} \tag{23}$$

with output equation

$$y(t) = z_2(t). \tag{24}$$

We consider that the source f is constant during the time interval $[0, T]$. Then $f(t) = f_0$ for all $t \in]0, T]$. The the matrix A is given by $A_1 = 0_{n\times n}$. We want now to verify if the system is observable-S_f-detector by using the previous proposition. We have

$$\mathcal{P} = \begin{bmatrix} \mathcal{C} \\ \mathcal{C}\mathcal{A} \\ \vdots \\ \mathcal{C}\mathcal{A}^{n-1} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 4 & 0 & 2 \\ 0 & 8 & 0 & 4 \end{bmatrix},$$

then we can simply verify that $\operatorname{rank}(\mathcal{P}) = 2$ which implies that the system is not observable-S_f-detector.

Corollary 1. *If*

$$\operatorname{rank}\left(\begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{2n-1} \end{bmatrix}\right) = 2n \quad \text{or} \quad \operatorname{rank}\left(\begin{bmatrix} C \\ C(A + A_1) \\ C(A^2 + AA_1 + A_1^2) \\ \vdots \\ C\sum_{i=0}^{2n-2} A^{2n-2-i}A_1 \end{bmatrix}\right) = 2n,$$

then the system (9), (10) is observable-S_f-detector.

Proof. From proposition 4 the system is observable-S_f-detector if and only if

$$\operatorname{rank}\left(\begin{bmatrix} \mathcal{C} \\ \mathcal{C}\mathcal{A} \\ \mathcal{C}\mathcal{A}^2 \\ \vdots \\ \mathcal{C}\mathcal{A}^{2n-1} \end{bmatrix}\right) = 2n,$$

which is equivalent to

$$\operatorname{rank}\left(\begin{bmatrix} [C\ 0_{p\times n}] \\ [C\ 0_{p\times n}]\begin{bmatrix} A & I_{n\times n} \\ 0_{n\times n} & A_1 \end{bmatrix} \\ [C\ 0_{p\times n}]\begin{bmatrix} A & I_{n\times n} \\ 0_{n\times n} & A_1 \end{bmatrix}^2 \\ \vdots \\ [C\ 0_{p\times n}]\begin{bmatrix} A & I_{n\times n} \\ 0_{n\times n} & A_1 \end{bmatrix}^{2n-1} \end{bmatrix}\right) = 2n$$

or to

$$\operatorname{rank}\left(\begin{bmatrix} C & 0_{q\times n} \\ CA & C \\ CA^2 & C(A+A_1) \\ \vdots & \vdots \\ \mathcal{C}\mathcal{A}^{2n-1} & C\sum_{i=0}^{2n-2} A^{2n-2-i}A_1 \end{bmatrix}\right) = 2n$$

we have

$$2n \geqslant \operatorname{rank}\left(\begin{bmatrix} C & 0_{q\times n} \\ CA & C \\ CA^2 & C(A+A_1) \\ \vdots & \vdots \\ \mathcal{C}\mathcal{A}^{2n-1} & C\sum_{i=0}^{2n-2} A^{2n-2-i}A_1 \end{bmatrix}\right) \geqslant \operatorname{rank}\left(\begin{bmatrix} C \\ C(A+A_1) \\ C(A^2+AA_1+A_1^2) \\ \vdots \\ C\sum_{i=0}^{2n-2} A^{2n-2-i}A_1 \end{bmatrix}\right)$$

and

$$2n \geqslant \operatorname{rank}\left(\begin{bmatrix} C & 0_{q\times n} \\ CA & C \\ CA^2 & C(A+A_1) \\ \vdots & \vdots \\ \mathcal{C}\mathcal{A}^{2n-1} & C\sum_{i=0}^{2n-2} A^{2n-2-i}A_1 \end{bmatrix}\right) \geqslant \operatorname{rank}\left(\begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{2n-1} \end{bmatrix}\right),$$

then if

$$\operatorname{rank}\left(\begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{2n-1} \end{bmatrix}\right) = 2n \quad \text{or} \quad \operatorname{rank}\left(\begin{bmatrix} C \\ C(A+A_1) \\ C(A^2+AA_1+A_1^2) \\ \vdots \\ C\sum_{i=0}^{2n-2} A^{2n-2-i}A_1 \end{bmatrix}\right) = 2n,$$

then

$$\operatorname{rank}\left(\begin{bmatrix} C & 0_{q\times n} \\ CA & C \\ CA^2 & C(A+A_1) \\ \vdots & \vdots \\ \mathcal{C}\mathcal{A}^{2n-1} & C\sum_{i=0}^{2n-2} A^{2n-2-i}A_1 \end{bmatrix}\right) = 2n, \quad \operatorname{rank}\left(\begin{bmatrix} \mathcal{C} \\ \mathcal{C}\mathcal{A} \\ \mathcal{C}\mathcal{A}^2 \\ \vdots \\ \mathcal{C}\mathcal{A}^{2n-1} \end{bmatrix}\right) = 2n$$

then, from proposition 4, the system (9), (10) is observable-S_f-detector. ∎

Lets take, for the rest of this article, H as a subspace of $\mathbb{R}^{2n}$ not reduced to $\{0\}$.

4. Partial observation–detection

4.1. Definition

In this section, we will consider partial observation-detection which consist to observe the observable part of the system state and detect the detectable part of the source in the same time even if the system is not observable-detector. For this we will introduce the following definition.

Definition 3. *The system (9), (10) is said to be H-observable-S_f-detector if for all $u_0 \in \mathbb{R}^{2n}$ we have,*

$$\mathcal{R}u_0 = 0 \Longrightarrow P_H u_0 = 0. \tag{25}$$

Remark 5. (1) The partial observation-detection notion is more general that the observation-detection one. (2) If the system (9), (10) is $\mathbb{R}^n$-observable-S_f-detector then (9), (10) is observable-S_f-detector.

Definition 4. *H is said to be observable-S_f-detectable if the system (9), (10) is H-observable-S_f-detector.*

4.2. Characterization and properties

In this section, we present some properties and characterization results for partial observation–detection.

Proposition 5. H is observable-S_f-detectable on $[t_0, T]$ if, and only if,

$$\ker(\mathcal{M}) \subseteq \bigcap_{t_0 \leqslant t \leqslant T} \ker\left(\mathrm{P}_H e^{(t-t_0)\mathcal{A}}\right). \tag{26}$$

Proof. Let $u_0 \in \mathbb{R}^{2n}$ an initial state giving the state u. On the first hand ($y(t) = 0$, $\forall t \in [t_0, T]$) is equivalent to $\mathcal{M}u_0 = 0$. On the second hand ($\mathrm{P}_H u(t) = 0$, $\forall t \in [t_0, T]$) is equivalent to

$$\mathrm{P}_H e^{(t-t_0)\mathcal{A}} u_0 = 0, \quad \forall t \in [t_0, T]$$

or to

$$u_0 \in \ker\left(\mathrm{P}_H e^{(t-t_0)\mathcal{A}}\right), \quad \forall t \in [t_0, T],$$

which reduces to $u_0 \in \cap_{t_0 \leqslant t \leqslant T} \ker\left(\mathrm{P}_H e^{(t-t_0)\mathcal{A}}\right)$. H is therefore observable-S_f-detectable if and only if, for all $u_0 \in \mathbb{R}^{2n}$,

$$\mathcal{R}u_0 = 0 \quad \Longrightarrow \quad u_0 \in \bigcap_{t_0 \leqslant t \leqslant T} \ker\left(\mathrm{P}_H e^{(t-t_0)\mathcal{A}}\right),$$

or $\ker(\mathcal{M}) \subseteq \cap_{t_0 \leqslant t \leqslant T} \ker\left(\mathrm{P}_H e^{(t-t_0)\mathcal{A}}\right)$. ■

Remark 6. The system is observable-S_f-detector if and only if every subspace of $\mathbb{R}^{2n}$ is observable-S_f-detectable.

Proposition 6. The following propositions are equivalent:
- (1) H is observable-S_f-detectable;
- (2) $\cap_{k=1}^{2n} \ker\left(\mathcal{C}\mathcal{A}^{k-1}\right) \subseteq \cap_{k=1}^{2n} \ker\left(\mathrm{P}_H \mathcal{A}^{k-1}\right)$;
- (3) $\ker(\mathcal{M}) \subseteq \ker(\mathbf{G}_H)$;
- (4) $\ker(\mathbf{O}) \subseteq \ker(\mathbf{Q}_H)$.

Proof. With the Proposition 5. H is observable-S_f-detectable if and only if,

$$\ker(\mathcal{M}) \subseteq \bigcap_{t_0 \leqslant t \leqslant T} \ker\left(\mathrm{P}_H e^{(t-t_0)\mathcal{A}}\right),$$

this one is equivalent according to the Lemma (2) to $\cap_{k=1}^{2n} \ker\left(\mathcal{C}\mathcal{A}^{k-1}\right) \subseteq \cap_{k=1}^{2n} \ker\left(\mathrm{P}_H \mathcal{A}^{k-1}\right)$ or to $\ker(\mathbf{O}) \subseteq \ker(\mathbf{Q}_H)$ which is equivalent also to $\ker(\mathcal{M}) \subseteq \ker(\mathbf{G}_H)$. ■

Remark 7. We can deduce from this proposition that a subspace H is not observable-S_f-detectable if

$$\mathrm{rg}(\mathbf{Q}_H) > \mathrm{rg}(\mathbf{O}).$$

Indeed, inclusion $\ker(\mathbf{O}) \subseteq \ker(\mathbf{Q}_H)$ gives, by taking the orthogonal, $\mathrm{Im}\left([\mathbf{Q}_H]^{\mathrm{T}}\right) \subseteq \mathrm{Im}\left(\mathbf{O}^{\mathrm{T}}\right)$. Then $\mathrm{rg}\left([\mathbf{Q}_H]^{\mathrm{T}}\right) \leqslant \mathrm{rg}\left(\mathbf{O}^{\mathrm{T}}\right)$ which gives $\mathrm{rg}\left(\mathbf{Q}_H\right) \leqslant \mathrm{rg}(\mathbf{O})$.

Proposition 7. Every observable-S_f-detectable is contained in $\mathrm{Im}(\mathcal{M})$.

Proof. For all $t \in [t_0, T]$ we have

$$\bigcap_{t_0 \leqslant s \leqslant T} \ker\left(\mathrm{P}_H e^{(s-t_0)\mathcal{A}}\right) \subseteq \ker\left(\mathrm{P}_H e^{(t-t_0)\mathcal{A}}\right).$$

By taking $t = t_0$ we get $\cap_{t_0 \leqslant t \leqslant T} \ker\left(\mathrm{P}_H e^{(t-t_0)\mathcal{A}}\right) \subseteq \ker\left(\mathrm{P}_H\right)$. If H is observable-S_f-detectable then $\ker(\mathcal{M}) \subseteq \ker\left(\mathrm{P}_H\right) = H^{\perp}$ or $H \subseteq [\ker(\mathcal{M})]^{\perp} = \mathrm{Im}(\mathcal{M})$. ■

Proposition 8. The subspace $\mathrm{Im}(\mathcal{M})$ is observable-S_f-detectable.

Proof. Let $u_0 \in \ker(\mathcal{M})$. By using Proposition (2) we have $\mathcal{C}\mathcal{A}^{k-1}u_0 = 0$, for all $1 \leqslant k \leqslant 2n$, which gives with the Lemma 1 $\mathcal{C}\mathcal{A}^k u_0 = 0$, $\forall k \in \mathbb{N}$, and $\mathcal{C}e^{s\mathcal{A}}u_0 = 0$, $\forall s \in \mathbb{R}$. Let us take $t \in [t_0, T]$. For $s = (\tau - t_0) + (t - t_0)$ we get $\mathcal{C}e^{(\tau-t_0)\mathcal{A}}\left[e^{(t-t_0)\mathcal{A}}u_0\right] = 0$, $\forall \tau \in \mathbb{R}$. In particular, for $\tau \in [t_0, T]$, we get

$$\mathcal{C}e^{(\tau-t_0)\mathcal{A}}\left[e^{(t-t_0)\mathcal{A}}u_0\right] = 0, \quad \forall \tau \in [t_0, T].$$

Then $e^{(t-t_0)\mathcal{A}}u_0 \in \ker(\mathcal{R}) = \ker(\mathcal{M})$ subsequently

$$\mathrm{P}_{\mathrm{Im}(\mathcal{M})}\left[e^{(t-t_0)\mathcal{A}}u_0\right] = 0$$

and this for all $t \in [t_0, T]$. This shows that $\mathrm{Im}(\mathcal{M})$ is observable. ■

Theorem 1. $\mathrm{Im}(\mathcal{M})$ *is the largest observable–detectable subspace (i.e. it contains all the observable-detectable subspaces).*

Proof. This is a direct result of Proposition (7) and the previous proposition. ■

Remark 8. (1) Every subspace H of $\mathbb{R}^{2n}$ is an orthogonal direct sum $H = H_0 \oplus H_1$ with H_0 observable-S_f-detectable and H_1 non observable-S_f-detectable and not containing any observable-S_f-detectable subspace. This decomposition is then unique. (2) Subspaces H_0 and H_1 are given by

$$H_0 = H \cap \mathrm{Im}(\mathcal{M}), \quad H_1 = H \cap \ker(\mathcal{M}).$$

Corollary 2. *The subspace*

$$H = \mathrm{Im}\left(C^{\mathrm{T}}\right)$$

is observable-S_f-detectable.

Proof. We have $H = \mathrm{Im}\left(C^{\mathrm{T}}\right) = \mathrm{Im}\left(\mathcal{C}^{\mathrm{T}}\right)$, then $H^{\perp} = \left[\mathrm{Im}(\mathcal{C}^{\mathrm{T}})\right]^{\perp} = \ker(\mathcal{C})$. Let $u_0 \in \ker(\mathcal{M})$ then for all $t \in [t_0, T]$, $\mathcal{C}e^{(t-t_0)\mathcal{A}}u_0 = 0$. Then $e^{(t-t_0)\mathcal{A}}u_0 \in \ker(\mathcal{C}) = H^{\perp}$ which shows that its projection on H is null, i.e., $\mathrm{P}_H e^{(t-t_0)\mathcal{A}}u_0 = 0$. Then $H = \mathrm{Im}\left(C^{\mathrm{T}}\right)$ is observable-S_f-detectable. ■

Remark 9. We deduce from these two corollaries that $\ker(\mathcal{M}) \subseteq \ker(\mathcal{C})$.

5. Partial reconstruction of the state and the source

Partial reconstruction of the system state and source reacting on the system (9), (10) can be done in the same time by reconstructing the system state of the system (12), (13). For that, in this section, we will study the problem of partial state reconstruction of the system (12), (13).

Let us consider the equations (12) and (13), with initial state $u_0 = u(t_0) \in \mathbb{R}^{2n}$ "unknown" and let $y^{\mathrm{mes}}(\cdot) \in \mathrm{L}^2\left[t_0, T; \mathbb{R}^q\right]$ be a measurement obtained on $[t_0, T]$ by a state $u(t)$ generated by the initial state $u_0 \in \mathbb{R}^{2n}$ unknown.

5.1. Reconstruction of the visible part of the state and the source

In this section, we will reconstruct the so-called visible initial state of the system (12), (13) from the output equation.

Definition 5. *We call the visible part of the initial state of the system* (12), (13) *the orthogonale projection of the initial state of the system on* $\mathrm{Im}(\mathcal{R}^*)$. *We denote it* $u^\star(t_0)$.

For all $u_0 \in \mathbb{R}^{2n}$ we have

$$\mathcal{M}u_0 = \mathcal{R}^* y, \tag{27}$$

then

$$\mathcal{M}P_{\mathrm{Im}(\mathcal{M})}u_0 = \mathcal{R}^* y. \tag{28}$$

We know that $P_{\mathrm{Im}(\mathcal{M})} = N(N^{\mathrm{T}}N)^{-1}N^{\mathrm{T}}$, then we have

$$\mathcal{M}N(N^{\mathrm{T}}N)^{-1}N^{\mathrm{T}}u_0 = \mathcal{R}^* y, \tag{29}$$

then

$$N^{\mathrm{T}}\mathcal{M}N(N^{\mathrm{T}}N)^{-1}N^{\mathrm{T}}u_0 = N^{\mathrm{T}}\mathcal{R}^* y. \tag{30}$$

Since $N^{\mathrm{T}}\mathcal{M}N$ is invertible (from Proposition 1) we have

$$(N^{\mathrm{T}}N)^{-1}N^{\mathrm{T}}u_0 = \left[N^{\mathrm{T}}\mathcal{M}N\right]^{-1}N^{\mathrm{T}}\mathcal{R}^* y, \tag{31}$$

then

$$N(N^{\mathrm{T}}N)^{-1}N^{\mathrm{T}}u_0 = \left[N^{\mathrm{T}}\mathcal{M}N\right]^{-1}N^{\mathrm{T}}\mathcal{R}^* y, \tag{32}$$

$$P_{\mathrm{Im}(\mathcal{M})}u_0 = N[N^{\mathrm{T}}\mathcal{M}N]^{-1}N^{\mathrm{T}}\mathcal{R}^* y. \tag{33}$$

Finally,

$$u^\star(t_0) = N\left[N^{\mathrm{T}}\mathcal{M}N\right]^{-1}N^{\mathrm{T}}\mathcal{R}^* y. \tag{34}$$

Proposition 9. The visible part of the initial state of the system (12), (13) has the form

$$u^\star(t_0) = N\left[N^{\mathrm{T}}\int_{t_0}^{T} e^{(\tau-t_0)\mathcal{A}^{\mathrm{T}}}\mathcal{C}^{\mathrm{T}}\mathcal{C}e^{(\tau-t_0)\mathcal{A}}\,d\tau N\right]^{-1}N^{\mathrm{T}}\int_{t_0}^{T} e^{(s-t_0)\mathcal{A}^{\mathrm{T}}}\mathcal{C}^{\mathrm{T}}y^{\mathrm{mes}}(s)\,ds. \tag{35}$$

Example 2. We will try, in this example, to reconstruct partially the stat and the source of a non observable-detector system described by the following equations:

$$\begin{pmatrix} \dot z_1(t) \\ \dot z_2(t) \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix} + \begin{pmatrix} f_1(t) \\ f_2(t) \end{pmatrix} \tag{36}$$

for all $t \in]0,1[$, with output equation

$$y(t) = z_1(t), \quad \forall t \in [0,T], \tag{37}$$

where the source function verify the following equation:

$$\begin{pmatrix} \dot f_1(t) \\ \dot f_2(t) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} f_1(t) \\ f_2(t) \end{pmatrix}, \quad \forall t \in]0,1[. \tag{38}$$

We take the initial state of the system $z_0 = \begin{pmatrix} z_{0,1} \\ z_{0,2}\end{pmatrix} = \begin{pmatrix} 1 \\ -1\end{pmatrix}$, and initial source $f_0 = \begin{pmatrix} f_{0,1} \\ f_{0,2}\end{pmatrix} = \begin{pmatrix} -2 \\ 1\end{pmatrix}$.

For that case we have:

$$A = \begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix},$$

then the system (12), (13), in this case, has the following form:

$$\begin{cases} \dot u(t) = \mathcal{A}u(t), & \forall t \in]0,1[, \\ u(0) = u_0. \end{cases} \tag{39}$$

Augmented by the following output equation:

$$y(t) = \mathcal{C}u(t), \quad \forall t \in [0,1] \tag{40}$$

with

$$\mathcal{A}=\begin{bmatrix}2&0&1&0\\-1&1&0&1\\0&0&1&1\\0&0&0&-1\end{bmatrix}\quad\text{and}\quad\mathcal{C}=\begin{bmatrix}1&0&0&0\end{bmatrix}.$$

The system (36), (37) is not observable-S_f-detector. Indeed, we have

$$\mathcal{P}=\begin{bmatrix}\mathcal{C}\\\mathcal{C}\mathcal{A}\\\vdots\\\mathcal{C}\mathcal{A}^{n-1}\end{bmatrix}=\begin{bmatrix}1&0&0&0\\2&0&1&0\\4&0&3&1\\8&0&7&2\end{bmatrix}.$$

We have $\operatorname{rank}(P)=3$, which implies that the system (39), (40) is not observable, then the system (36), (37) is not observable-S_f-detector.

We want now to reconstruct the visible part of the initial state u_0. By using a simple program in scilab application, we obtain

$$\mathcal{M}=\begin{bmatrix}\frac{e^4}{4}-\frac{1}{4}&0&\frac{e^4}{4}-\frac{e^3}{3}+\frac{1}{12}&\frac{(e-1)^3(e^1+1)}{12}\\0&0&0&0\\\frac{e^4}{4}-\frac{e^3}{3}+\frac{1}{12}&0&\frac{(3e^1+1)(e^1-1)^3}{12}&\frac{e^1}{6}+\frac{e^2}{4}-\frac{5e^3}{18}+\frac{e^4}{12}-\frac{7}{18}\\\frac{(e^1-1)^3(e^1+1)}{12}&0&\frac{e^1}{6}+\frac{e^2}{4}-\frac{5e^3}{18}+\frac{e^4}{12}-\frac{7}{18}&\frac{e^1}{9}-\frac{e^{-2}}{72}+\frac{e^2}{8}-\frac{e^3}{9}+\frac{e^4}{36}-\frac{11}{36}\end{bmatrix}$$

and

$$\operatorname{Im}(\mathcal{R}^*)=\operatorname{vect}\{e_1,e_3,e_4\}\quad\text{and}\quad\ker(\mathcal{R})=\operatorname{vect}\{e_2\}.$$

In this case,

$$N=\begin{bmatrix}1&0&0\\0&0&0\\0&1&0\\0&0&1\end{bmatrix}\quad\text{and}\quad N^{\mathrm{T}}=\begin{bmatrix}1&0&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix},$$

then

$$\begin{aligned}N^{\mathrm{T}}\mathcal{M}\,N&=\left[N^{\mathrm{T}}\int_0^1 e^{\tau\mathcal{A}^{\mathrm{T}}}\mathcal{C}^{\mathrm{T}}\mathcal{C}e^{\tau\mathcal{A}}\,d\tau\,N\right]\\&=\begin{bmatrix}\frac{e^4}{4}-\frac{1}{4}&\frac{e^4}{4}-\frac{e^3}{3}+\frac{1}{12}&\frac{(e^1-1)^3(e+1)}{12}\\\frac{e^4}{4}-\frac{e^3}{3}+\frac{1}{12}&\frac{(3e+1)(e-1)^3}{12}&\frac{e^1}{6}+\frac{e^2}{4}-\frac{5e^3}{18}+\frac{e^4}{12}-\frac{7}{18}\\\frac{(e-1)^3(e+1)}{12}&\frac{e^1}{6}+\frac{e^2}{4}-\frac{5e^3}{18}+\frac{e^4}{12}-\frac{7}{18}&\frac{e^1}{9}-\frac{e^{-2}}{72}+\frac{e^2}{8}-\frac{e^3}{9}+\frac{e^4}{36}-\frac{11}{36}\end{bmatrix}.\end{aligned}$$

We have

$$\begin{aligned}N^{\mathrm{T}}\int_0^1 e^{s\mathcal{A}^{\mathrm{T}}}\mathcal{C}^{\mathrm{T}}y^{\mathrm{mes}}(s)\,ds&=\begin{bmatrix}1&0&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\begin{pmatrix}\frac{e}{6}+\frac{e^3}{2}-\frac{e^4}{6}-\frac{1}{2}\\0\\\frac{e}{6}-\frac{3e^2}{4}+\frac{13e^3}{18}-\frac{e^4}{6}-\frac{5}{36}\\\frac{5e^3}{18}-\frac{e^{-2}}{72}-\frac{3e^2}{8}-\frac{e}{18}-\frac{e^4}{18}+\frac{7}{18}\end{pmatrix}\\&=\begin{pmatrix}\frac{e}{6}+\frac{e^3}{2}-\frac{e^4}{6}-\frac{1}{2}\\\frac{e}{6}-\frac{3e^2}{4}+\frac{13e^3}{18}-\frac{e^4}{6}-\frac{5}{36}\\\frac{5e^3}{18}-\frac{e^{-2}}{72}-\frac{3e^2}{8}-\frac{e}{18}-\frac{e^4}{18}+\frac{7}{18}\end{pmatrix},\end{aligned}$$

then, by using (35), the visible part of the initial state u_0 is given by

$$u^{\star}(t_0)=\begin{bmatrix}1&0&0\\0&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}\frac{e^4}{4}-\frac{1}{4}&\frac{e^4}{4}-\frac{e^3}{3}+\frac{1}{12}&\frac{(e^1-1)^3(e+1)}{12}\\\frac{e^4}{4}-\frac{e^3}{3}+\frac{1}{12}&\frac{(3e+1)(e-1)^3}{12}&\frac{e^1}{6}+\frac{e^2}{4}-\frac{5e^3}{18}+\frac{e^4}{12}-\frac{7}{18}\\\frac{(e-1)^3(e+1)}{12}&\frac{e^1}{6}+\frac{e^2}{4}-\frac{5e^3}{18}+\frac{e^4}{12}-\frac{7}{18}&\frac{e^1}{9}-\frac{e^{-2}}{72}+\frac{e^2}{8}-\frac{e^3}{9}+\frac{e^4}{36}-\frac{11}{36}\end{bmatrix}^{-1}$$

$$\begin{pmatrix} \frac{e}{6}+\frac{e^3}{2}-\frac{e^4}{6}-\frac{1}{2} \\ \frac{e}{6}-\frac{3e^2}{4}+\frac{13e^3}{18}-\frac{e^4}{6}-\frac{5}{36} \\ \frac{5e^3}{18}-\frac{e^{-2}}{72}-\frac{3e^2}{8}-\frac{e}{18}-\frac{e^4}{18}+\frac{7}{18} \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -2 \\ 1 \end{pmatrix}.$$

Finally, we did reconstruct a part of the initial state u_0 with $u_0 = \binom{z_0}{f_0}$, which means that we did reconstruct the initial source f_0 and the first component of the initial state z_0.

Remark 10. Its not always possible to reconstruct one or more of component of the initial state and the source. In general partial reconstruction can give as a linear combination of the component of the initial state an initial source. This have a direct relation to the algebraic decomposition of $\mathrm{Im}(\mathcal{M})$.

6. Conclusion and prospects

In this paper, we have considered the partial and total observation–detection problem for finite dimensional linear systems. In this context we have introduced the notion of an observable–detectable system and observable–detectable subspace one. We give some characterizations for this two notions. The problem of the reconstruction of the so called visible part of the state and the perturbation source was also discussed.

An important feature of partial observability–detectability concept is the possibility of using and studying a dynamical system even if the system is not fully observable–detectable. We can also reconstruct, if it possible, the most important parameters of the system without worrying about the other parameters. These ideas may allow us to do the same study for the case of infinite dimensional systems, for the case of distributed parameter systems or even for the case of semi-linear and non-linear systems.

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Повне та часткове "спостереження–виявлення" в лінійних динамічних системах з джерелом, що характеризується іншою динамічною лінійною системою: скінченновимірний випадок

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У цій статті розглядається задача часткового "спостереження–виявлення" для скінченнововимірних динамічних лінійних систем, які необов'язково повністю спостерігаються або виявляються. Введено поняття "спостереження–виявлення" та "часткове спостереження–виявлення", які передбачають відновлення або повністю, або частково, стану системи та джерела, що реагує на систему, навіть якщо система не є повністю спостережуваною або виявною. Надано деякі характеристики "спостережувано–виявної системи" та "спостережувано–виявних просторів". Відновлення стану та джерела на спостережувано-виявному підпросторі здійснюється за допомогою ортогональної проекції, використовуючи алгебраїчну структуру заданої скінченновимірної системи. Крім того, наведено приклади для ілюстрації запропонованого підходу.

Ключові слова: *спостереження; виявлення; динамічні системи; виявлення джерела.*