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On the use of the spectral element method for the modeling of fluid–structure interaction problems

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This study addresses a fluid–structure interaction problem that models flow in a channel. Simulations were conducted to investigate the method's effectiveness when applied to real obstacle scenarios, where the obstacle is explicitly represented within the channel. To tackle the Navier–Stokes equations, we utilized the spectral–Fourier–asymptotic approach, which is a mesh-free method that combines Chebyshev polynomials and Fourier series with the asymptotic method based on power series.

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1. Introduction

In recent years, there have been significant advancements in various numerical and modeling approaches used in fluid flow simulations. These advancements have been driven by the need for more accurate physical space descriptions, with the rise of more intricate configurations that involve the coupling of multiple physical and scale factors, there has been a growing need to employ various numerical techniques to explore fluid behavior under different external and internal forces [5]. To achieve this goal, researchers have turned to a range of numerical methods such as finite element methods and finite difference methods, or finite volume methods [2–4] are commonly used to describe the fluid domain in an Eulerian reference framework [1], producing numerical solutions for fixed spatial meshes. However, this approach faces challenges when obstacles move within the domain, as in flows around hydraulic turbines, wind turbines, or mixers. Moreover, studying the interaction between an incompressible fluid and a structure through numerical simulations has been an active research topic for the past decade, with numerous studies conducted in this field.

This study offers a comprehensive account of the method, as outlined in sections 3 and 4. Section 2 briefly outlines the theoretical formulation used to derive the Navier–Stokes equations, which are then solved using the ANM-SM method. The numerical aspects of the resolution of the Navier–Stokes equations are discussed, along with the techniques employed. A resolution strategy is also presented, incorporating a mixed formulation that integrates two domains (fluid and solid), along with a fine coupling at the interface. To evaluate the method's performance and reliability, validation tests are conducted, and the results are presented in section 5. The accuracy of the proposed approach is established by comparing the obtained results with those computed by Ansys, thus validating the approach's validity.

2. Geometry of the problem: an investigation into FSI

The primary objective of this study is to analyze the properties of a viscous flow that involves an incompressible and Newtonian fluid around two types of obstacles: a half disk and a rectangle. Figure 1 provides a visual illustration of the geometry and boundary conditions employed in this investigation.

The following equations regulate the problem of fluid-structure interaction,

$$\begin{cases} \dfrac{\partial V}{\partial t} + \rho \cdot c_j \dfrac{\partial V_i}{\partial x_i} = \dfrac{\partial \sigma_i}{\partial x_i} + f_i \quad \text{in} \quad \Omega_f, & (a) \\ \dfrac{\partial V}{\partial t} + \dfrac{\partial \rho V_i}{\partial x_i} = 0 \quad \text{in} \quad \Omega_f, & (b) \\ V = \lambda V_{imp} \quad \text{over} \quad \Gamma_f, & (c) \\ V_i = \dot{y}\, d_i \quad \text{over} \quad \Gamma_c, & (d) \\ \boldsymbol{m} \cdot \ddot{y} \cdot \boldsymbol{D} + \boldsymbol{k} \cdot \boldsymbol{y} \cdot \boldsymbol{D} = -\displaystyle\int_{\Gamma_c} \boldsymbol{\sigma} \cdot \boldsymbol{n} \cdot \boldsymbol{dl}. & (e) \end{cases} \tag{1}$$

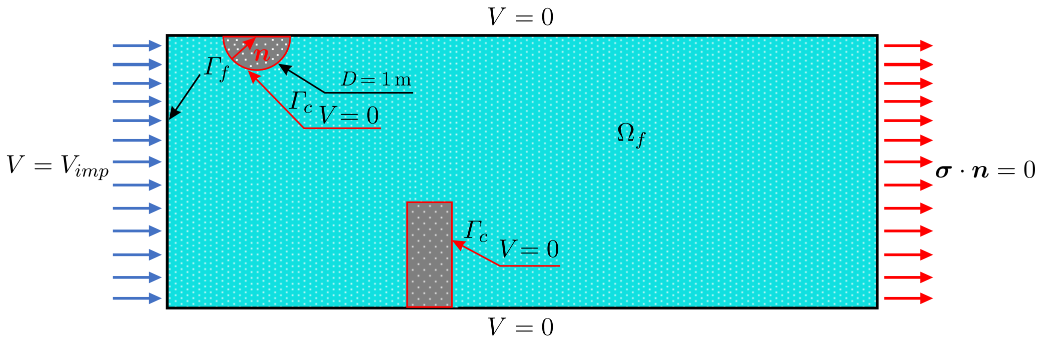


Fig. 1. Boundary conditions and interaction-fluid problem geometry.

The stress tensor is represented by σ, while the flow velocity vector is denoted by V and the imposed velocity is denoted by V_{imp}. The body force is indicated by f_i, and the fluid density by ρ. Additionally, the obstacle's mass per unit length and spring stiffness are represented by m and k, respectively.

The equations presented in (1.a) and (1.b) pertain to the continuity of an incompressible fluid flow that exhibits viscosity. On the other hand, equation (1.c) sets out the velocity condition applied to the boundary (Γ_f), while equation (1.d) specifies the compatibility prerequisite between the velocity of the fluid and that of the obstacle at the boundary (Γ_c). Additionally, equation (1.e) provides the motion equation of the obstacle when it experiences fluid force F that acts perpendicularly to its outer surface (Γ_c).

According to the impact of the fluid on the obstruction

$$\boldsymbol{F} = \int_{\Gamma_c} \boldsymbol{\sigma} \cdot \boldsymbol{n} \cdot \boldsymbol{dl}. \tag{2}$$

The formulation of the stationary issue makes two assumptions: a geometry velocity of zero ($\tilde{u}$) and a convection velocity of c_j equal to the fluid component velocity in the case of an obstruction in motion while the fluid flow around it is stationary. As a result, the governing equations for the stationary problem are as follows:

$$\begin{cases} V_j \dfrac{\partial V_i}{\partial x_i} = \dfrac{1}{\rho}\left(\dfrac{\partial \sigma_i}{\partial x_i} + f_i\right) \quad \text{in} \quad \Omega_f, & (a) \\ \dfrac{\partial V_i}{\partial x_i} = 0 \quad \text{in} \quad \Omega_f, & (b) \\ V = \lambda V_{imp} \quad \text{over} \quad \Gamma_f, & (c) \\ V_i = 0 \quad \text{over} \quad \Gamma_c, & (d) \\ \boldsymbol{k} \cdot \boldsymbol{y} \cdot \boldsymbol{D} = -\displaystyle\int_{\Gamma_c} \boldsymbol{\sigma} \cdot \boldsymbol{n} \cdot \boldsymbol{dl}. & (e) \end{cases} \tag{3}$$

In the continuity equation (3.a), we insert a big parameter G known as the penality parameter to satisfy the impressibility requirement as follows:

$$\frac{\partial V_i}{\partial x_i} - \frac{1}{G} \cdot p = 0. \tag{4}$$

3. Quadratic term solver for statements

ANM is a solver that uses a high-order Taylor series expansion with respect to a scalar parameter to solve nonlinear problems. This method enables the user to track the solution curves and solve nonlinear problems in a step-by-step manner [7, 8]. Essentially, the nonlinear problems are linearized, and the resulting linear equations are solved using a discretization technique, such as the spectral method used in this study. It is worth noting that the proposed algorithm is designed to solve the nonlinear terms of equation systems (3),

$$\begin{cases} \boldsymbol{L}(\boldsymbol{U}) + \boldsymbol{Q}(\boldsymbol{U}, \boldsymbol{U}) = 0 \quad \text{in} \quad \Omega, & (a) \\ V = \lambda V_{imp} \quad \text{over} \quad \partial\Omega_V. & (b) \end{cases} \tag{5}$$

To account for ANM, the nonlinear equations expressed in (3) can be reconfigured using linear operators represented by $\boldsymbol{L}(\boldsymbol{U})$, quadratic operators indicated as $\boldsymbol{Q}(\boldsymbol{U}, \boldsymbol{U})$, and a mixed unknown vector $\boldsymbol{U}$, which comprises various unknowns of the problem, including pressure p and velocity V.

If a mixed vector $\boldsymbol{U}$ is shown:

$$\boldsymbol{U} = \left\{ \begin{array}{c} V \\ p \end{array} \right\}. \tag{6}$$

A non-linear issue with the Reynolds number-corresponding configuration is the system (3).

4. Approach spectral

A set of discretization approaches for solving partial differential equation systems is known as spectral techniques. These techniques often employ polynomial bases to approximation the solution. We shall employ the Chebyshev polynomial of collocation-based spectrum approach in this fashion, sometimes referred to as the pseudo-spectral method due to its simplicity and great precision [6, 7]. In our investigation, collocation points were employed, and they are based on Gauss–Lobatto [8]. The collocation points are characterized by a periodical function using the Chebyshev polynomial as shown in the citations below:

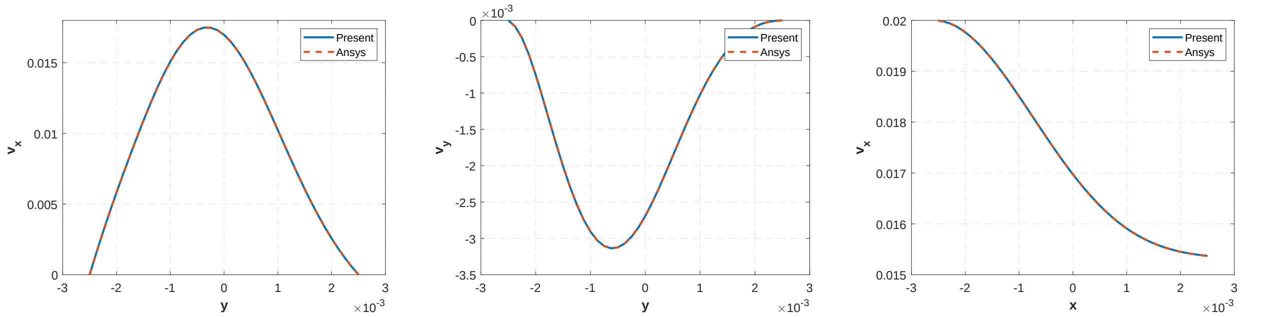
$$x_j = \cos\left(\pi \frac{j}{N}\right) \quad \text{for} \quad j = 0, \dots N-1. \tag{7}$$

Using spectral theory a way to approximate a function that is born between -1 and 1. There are other forms for this, as stated in the references on the Chebyshev $D_{ij}^{(1)}$ differenciation matrix. The current investigation primarily focuses on a specific version, denoted as $D_{ij}^{(1)}$. This matrix is considered the most practical and is defined as:

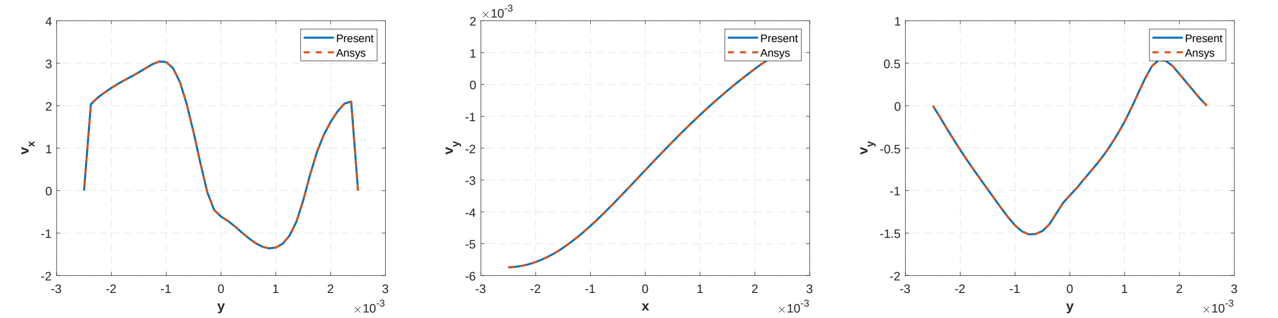
$$\begin{cases} (D_N)_{ij} = \dfrac{\xi_i}{\xi_j} \dfrac{(-1)^{i+j}}{x_i - x_j}, & i \neq j, \quad (a) \\ (D_N)_{ii} = -\displaystyle\sum_{j=0,\, j\neq i}^{N} (D_N)_{ij}, & i = j. \quad (b) \end{cases} \tag{8}$$

5. Discussion and numerical findings

This study investigates the behavior of obstacles under a transverse flow, as depicted in Figure 1. To enforce the boundary conditions, we set the outlet condition to zero and the obstacle condition to Dirichlet. Additionally, we impose a velocity u_0 and zero condition on the upper and lower boundaries. The Chebyshev spectral grid with $N = 83$ is employed, and the adhesion condition is imposed on the boundaries of the obstacle domain. To investigate the effect of velocity on the stationary Navier–Stokes problem, we solve the problem for different Reynolds numbers and focus relating location to the velocity distribution x for $y = 0$, as shown in Figure 2. Specifically, we examine the cases where $\text{Re} = 10$ and $\text{Re} = 100$. We also compare our results with those obtained using Ansys and demonstrate that our proposed method converges with increasing point distribution.



a (Re $=10$, ($x=0$)) ***b*** (Re $=10$, ($x=0$)) ***c*** (Re $=10$, ($y=0$))



d (Re $=100$, ($x=0$)) ***e*** (Re $=100$, ($y=0$)) ***f*** (Re $=100$, ($x=0$))

Fig. 2. Comparison of the outcomes with those of Ansys: evolution of velocity as a function of vertical displacement for different Reynolds number values and fixed x or y.

According to Figure 2, the streamlines behind the barriers and the recirculation zone are used to study the velocity development for various low Re values (Re $=10$, Re $=100$).

a (Re $=10$) ***b*** (Re $=10$)

c (Re $=100$) ***d*** (Re $=100$)

Fig. 3. For various Reynolds numbers, the flow of streamlines and contours.

Figure 3 demonstrates that there is no apparent velocity vector inside the barriers, providing evidence that the adhesion condition is valid there. The figure also shows the fluid flow patterns and the impact of the barriers on the flow. When the Re is 10 or 100, obstructions in the flow's opposite direction exert a force that creates vortices behind them. However, as the Reynolds number increases, two symmetrical vortex areas begin to form behind the barriers (as shown in Figure 3), and their size also increases.

6. Conclusion

In this study, we used the ANM-SM method to explore the fluid-structure interaction (FSI). To achieve this, we devised a novel method capable of solving nonlinear equilibrium equations. By using Chebyshev collocation points, we integrated irregular multiple domains. By utilizing this method, we were able to take into account various boundary conditions, effectively solve the Navier–Stokes equation. Following several iterations of the ANM, we observed the emergence of well-formed vortices at the back of the obstacle, which further confirms the reliability and effectiveness of the proposed approach.

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Про використання методу спектральних елементів для моделювання задач взаємодії рідини та структури

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У цьому дослідженні розглядається задача взаємодії рідини та структури, яка моделює течію в каналі. Моделювання проведено, щоб дослідити ефективність методу при застосуванні до реальних сценаріїв з перешкодами, де перешкода явно представлена в каналі. Для розв'язання рівнянь Нав'є–Стокса використано спектральний асимптотичний підхід Фур'є, який є безсітковим методом, що поєднує поліноми Чебишева та ряди Фур'є з асимптотичним методом, який базується на степеневих рядах.

Ключові слова: *рідина–структура; асимптотичний метод; спектральний безсітковий метод.*