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A survey of the vehicle routing problem and its variants: formulations and solutions

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In the realm of industrial enterprises, enhancing logistical efficiency stands as a focal concern. The objective lies in orchestrating an optimized service with seamless flow of goods while minimizing expenses. A crucial component within any logistics framework is the administration and strategizing of distribution networks for vehicle fleets, commonly referred to as the Vehicle Routing Problem (VRP). This composition delves into an exploration of VRP and its various iterations, offering categorization and depiction of prevalent formulations and algorithms prevalent in scholarly works over the past two decades.

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1. Introduction

In 1959, Dantzig and Ramser [1] introduced the concept of the "Truck Dispatching Problem" as a means to optimize the allocation of a fleet of identical trucks for efficiently meeting the oil demand of multiple gas stations from a central hub while minimizing total distance traveled. Building upon this foundation, Clarke and Wright [2] expanded the problem into a linear optimization framework that has become a cornerstone in the fields of logistics and transportation. Their work involved effectively serving geographically dispersed customers from a central depot, accounting for trucks with different capacities.

However, the contemporary VRP models have undergone significant advancements compared to the initial formulations presented by Dantzig, Ramser, Clarke, and Wright. The objective now is to incorporate real-world complexities into the problem formulation, leading to the exploration of numerous variants. Some notable variants include the VRP with Time Windows (VRPTW), the VRP with Pickup and Delivery Problem (VRPDP), the VRP with Heterogeneous Fleets (HFVRP), the Time-Dependent VRP (TD-VRP), the Open VRP (OVRP), and the Multi-Depot VRP (MDVRP), among others.

The practical significance of these problems has attracted the attention of numerous researchers, fostering collaborations between academic institutions and companies. As a result, contemporary VRP software solutions have been widely adopted across diverse sectors, including public, private, and multinational organizations. Companies such as Coca-Cola Enterprises and Amazon have recognized the pivotal role of VRP software in optimizing their operations. According to a report by Toth and Vigo in 2002 [3], implementing computerized techniques in distribution processes can yield transportation cost savings ranging from 5% to 20%.

It is important to note that the VRP has been proven to be NP-hard by Lenstra and Rionnooy Kan [4] in 1981. This implies that no polynomial-time algorithm exists to solve the problem optimally, necessitating the adoption of heuristic and meta-heuristic methods to obtain approximate solutions for large-scale instances. Exact methods are only viable for small-scale instances and cannot provide solutions within a reasonable timeframe. Consequently, a plethora of models and algorithms have been proposed for optimal and approximate solutions to the various VRP variants.

Throughout the years, the VRP has captured the attention of researchers and practitioners alike due to its relevance and applicability in various industries. The optimization of vehicle routes and distribution networks plays a crucial role in minimizing costs, reducing environmental impact, and enhancing customer satisfaction. As a result, extensive research efforts have been dedicated to developing innovative approaches, algorithms, and mathematical models to tackle the challenges posed by the VRP and its variants. The continuous evolution and refinement of these techniques have contributed to significant advancements in the field of logistics and transportation management. By exploring and harnessing the potential of these advancements, organizations can unlock new possibilities for improving operational efficiency, resource utilization, and overall supply chain performance.

This review paper focuses on providing an overview of widely used formulations and solutions for the Capacitated Vehicle Routing Problem (CVRP) in Sections 2 and 3. It also highlights significant papers on the VRP from the past two decades in Section 4, shedding light on the recent advancements and trends in the field.

2. Formulations and variants of the VRP

The literature presents various formulations for modeling VRP, offering different perspectives. Presently, the most prevalent approaches can be categorized into two main classes: index vehicle flow formulations and set partitioning formulations. In this section, we provide an in-depth description of both approaches in the context of modeling the CVRP.

Considering an undirected graph $G = (V', E)$ with $n+1$ vertices where $V' = \{0, 1, \ldots, n\}$ and E represent the set of edges, where vertex 0 serves as the depot and $V = V'/\{0\}$ corresponds to n customers. Each edge $i, j \in E$ has a non-negative cost d_{ij}. Each customer $i \in V$ requires a supply of q_i units from the depot, where $q_0 = 0$. The problem involves m identical vehicles stationed at the depot with a capacity of Q. The objective is to utilize the depot (represented by vertex 0) to distribute goods or services to the customers. A route is defined as a cost-effective cycle in graph G that includes the depot 0, where the total demand of the visited vertices does not exceed the vehicle capacity. The goal is to design m routes, one for each vehicle, ensuring that all customers are visited exactly once and the total cost of all routes is minimized.

For a subset $F \subseteq E$, $G(F)$ denotes the subgraph $(V'(F), F)$ induced by F, where $V'(F)$ represents the set of vertices that are connected to at least one edge of F. Additionally, for a given set $S \subseteq V$, the complement of S is denoted as $\bar{S} = V' \backslash S$ and $\delta(S)$ is the cutset defined by S, where $\delta(S) = \{\{i,j\} \in E \colon i \in S,\ \ j \notin S \text{ or } i \notin S,\ j \in S\}$. Moreover, $q(S)$ is the total demand of customers in S, and $k(S)$ represents the minimum number of vehicles with a capacity of Q needed to serve all customers in S.

2.1. The vehicle flow formulations

Here we will focus only on the two index formulation which is the most popular among these formulations, it was firstly proposed by Laporte et al. [5] and is as follows:
let x_{ij} an integer variable which may take value $\{0, 1\}$, $\forall \{i, j\} \in E \backslash \{\{0, j\} \colon j \in V\}$ and value $\{0, 1, 2\}$, $\forall \{0, j\} \in E$, $j \in V$. It should be noted that if a route including a single customer j is selected in the solution, then x_{0j} is set to 2.

The formulation of the CVRP using a two-index vehicle flow approach is as follows:

$$\min \sum_{\{i,j\} \in E} d_{ij} x_{ij} \tag{1}$$

$$\text{s.t.} \sum_{\{i,j\} \in \delta(\{h\})} x_{ij} = 2, \quad \forall h \in V, \tag{2}$$

$$\sum_{\{i,j\} \in \delta(S)} x_{ij} \geqslant 2k(S), \quad \forall S \in \mathscr{S}, \tag{3}$$

$$\sum_{j \in V} x_{0j} = 2m, \tag{4}$$

$$x_{ij} \in \{0,1\}, \quad \forall \{i,j\} \in E \backslash \{\{0,j\} \colon j \in V\}, \tag{5}$$

$$x_{0j} \in \{0,1,2\}, \quad \forall \{0,j\} \in E, \quad j \in V, \tag{6}$$

where $\mathscr{S} = \{S \colon S \subseteq V, |S| \geqslant 2\}$.

The constraints (1) represent the degree constraints for each customer. The constraints (2), (3), also known as capacity constraints or generalized subtour elimination constraints, enforce that for any subset S of customers excluding the depot, a minimum of $k(S)$ vehicles must enter and leave S. Constraint (4) ensures that m vehicles depart from and come back to the depot. Additionally, constraints (5) and (6) are the integrality constraints.

2.2. The set partitioning formulations

Balinski and Quandt [6] originally introduced the set partitioning formulation of the CVRP, which assigns a binary variable to every possible route that is considered feasible.

Consider an index set $\mathscr{R}$ containing all feasible routes and a subset $\mathscr{R}_i \subset \mathscr{R}$ comprising routes that cover a given customer $i \in V$. We define a binary coefficient a_{ir}, which takes the value 1 if vertex $i \in V'$ is present in route $r \in \mathscr{R}$, and 0 otherwise (noting that $a_{0r} = 1$, $\forall r \in \mathscr{R}$). Additionally, each route $r \in \mathscr{R}$ is associated with a cost c_r.

We introduce a $(0,1)$ binary variable y_r, which takes the value 1 if and only if route $r \in \mathscr{R}$ is selected. The set partitioning formulation of the CVRP is as follows:

$$\min \sum_{r \in \mathscr{R}} c_r y_r \tag{7}$$

$$\text{s.t.} \sum_{r \in \mathscr{R}} a_{ir} y_r = 1, \quad \forall i \in V, \tag{8}$$

$$\sum_{r \in \mathscr{R}} y_r = m, \tag{9}$$

$$y_r \in \{0,1\}, \quad \forall r \in \mathscr{R}. \tag{10}$$

Constraints (8) specify that each client $i \in V$ must be covered by one route and, constraint (9) requires that m routes are selected.

It is mentioned that set partitioning formulations are often more compact and easier to solve than vehicle flow formulations. This is because set partitioning formulations involve a smaller number of decision variables and constraints compared to vehicle flow formulations. However, the vehicle flow formulations have some advantages over the set partitioning formulations. For example, it can more easily accommodate additional constraints, such as time windows. The vehicle flow formulations can also lead to tighter linear programming relaxations compared to the set partitioning formulations.

Finally, it is worth noting that numerous other formulations of the CVRP and its variants have been proposed in the literature such as the CVRP's commodity flow formulations which incorporate assignment constraints to represent vehicle routes and multicommodity flow constraints to represent the transportation of goods. Initially introduced by Garvin et al. in 1957, [7] to solve an oil delivery problem. Many of these formulations were developed to address some of the limitations associated with both the vehicle flow formulations and the set partitioning formulations. For further information on this subject, we recommend referring to [8, 9].

2.3. Variants of the VRP

The VRP exhibits a rich landscape of variants and extensions that have been studied extensively. One notable extension is the HFVRP or Mixed Fleet VRP [10], where vehicles with varying capacities are considered. This variant reflects real-world scenarios where a fleet consists of vehicles with different capabilities, such as different load capacities or fuel efficiencies [11, 12]. Another intriguing extension is the MDVRP, assuming that a company may have multiple depots from which it can serve its customers. This variant allows for more flexibility in terms of distribution strategies and can be particularly relevant in large-scale logistics networks or when serving customers across diverse geographical regions [10, 13].

The VRPTW, which assigns specific time windows for delivery to each client, is another highly investigated enhancement. Time windows can be divided into two categories: hard time windows, which rigorously forbid deliveries outside of a predetermined window, and soft time windows, which permit some flexibility by charging penalties for early or late deliveries. Realistic scheduling issues are included in this addition, which ensures that deliveries are made within predetermined time periods while accounting for temporal constraints [14–16].

The Periodic Vehicle Routing Problem (PVRP), which takes into account a planning horizon that extends over several days, expands the traditional VRP. This module takes into account situations where consumers need to return, like in recurrent service or restocking cycles. The PVRP offers a more comprehensive perspective on long-term route planning and consumer visits patterns by taking into account the time dimension across several days [17].

The responsibility of conveying items that must be picked up from a certain area and delivered to their destinations by the same vehicle is introduced by the VRPPD. When products need to be gathered from diverse origins and dispatched to certain destinations, settings like courier or delivery services sometimes use this version.

Another variation, the VRP with Simultaneous Deliveries and Pickups (VRPSPD), permits the execution of both pickups and deliveries at client locations at the same time. The VRPSPD allows a vehicle to do both tasks concurrently at a single site in contrast to the VRPPD, where pickups and deliveries happen in order. This variation accounts for scenarios in which goods must be swapped or combined at client locations before being transported elsewhere [18, 19].

The TDVRP takes into account client visit time frames or journey times that depend on the time of day. In this variation, the amount of time it takes to get from one place to another is not constant but instead changes depending on the time of day, the weather, and dynamic factors like traffic. The TDVRP attempts to produce more precise and effective route plans that adjust to real-time traffic circumstances by taking into account the temporal variations in trip times [20, 21].

There have also been many other variations and combinations of the aforementioned extensions investigated. These include combinations that incorporate many elements to provide more complex problem formulations, such as VRPTW with multi-depot (MD-VRPTW) or time-dependent (TD-VRPTW). Additionally, less commonly encountered variations include the Open VRP, Multi-compartment VRP, and Generalized VRP, among others, each addressing specific logistical challenges and considerations.

The comprehensive study of these VRP variants and their respective optimization algorithms contributes to a deeper understanding of the complexities involved in vehicle routing and provides insights into tailoring solutions to address diverse real-world scenarios [22, 23].

3. Methods of resolution

Numerous exact and metaheuristic algorithms have been proposed as a result of intensive research into the solution of the VRP and its variants. These algorithms seek to address the VRP's innate complexity and offer effective answers for practical applications.

3.1. Exact methods

Exact methods offer a range of approaches to solve the VRP and its variants. One widely used exact method is Branch and Bound, which divides the search space into branches and employs a bounding procedure to estimate the best possible solution within each branch. By using bounds, the search tree is pruned, eliminating unpromising branches and reducing the computational effort required. Another approach, Branch and Cut, combines Branch and Bound with cutting planes. Cutting planes are inequalities progressively added to the problem formulation to strengthen the bounds and improve the relaxation solution at each node of the search tree [9, 24].

Column Generation is a powerful technique specifically designed for solving large-scale optimization problems with a vast number of decision variables. It is particularly effective when a problem has a large number of potential variables, but only a subset of them are likely to be part of the optimal solution [12]. Branch and Price combines elements from Branch and Bound (bound step) and Column Generation (price step) to efficiently explore the solution space while improving the quality of the solutions obtained. This approach identifies and adds only the most promising columns (variables) to the problem formulation, resulting in a more compact and efficient representation of the problem [21].

In contrast to the Branch and X methods that treat VRP as integer linear programming (ILP) or mixed ILP (MILP), dynamic programming takes a different approach by breaking down the complex problem into simpler sub-problems. It leverages the principle of optimality, which states that an optimal solution to a problem incorporates optimal solutions to its subproblems. By recursively solving the subproblems and building up the solution step by step, dynamic programming efficiently computes the optimal solution to the VRP. This technique is particularly useful when the problem exhibits overlapping substructures that can be exploited to avoid redundant computations.

Each exact method has its own strengths and limitations, and the choice of which approach to use depends on factors such as problem size, complexity, and available computational resources. Researchers continue to refine and develop novel exact methods to improve solution quality and computational efficiency, expanding the possibilities for solving the VRP and its variants [23, 25].

3.2. Approximate methods

Most work on VRP is related to approximate methods, often called heuristics, which are designed to solve specific problems. They are satisfied with obtaining solutions as good as possible in reasonable time but do not guarantee their optimality. For instance, the savings method of Clarke and Wright (1964) [2] is one of the earliest examples of constructive heuristic used to solve the VRP. The methods of this type progressively build vehicle routes by inserting them at each stage of a customer according to several criteria of gain measures. We also find Improvement methods,which were initially implemented by Croes (1958) [26] and Lin 1965 [27] for the travelling salesman problem. They are based on the concept of k-exchanges.

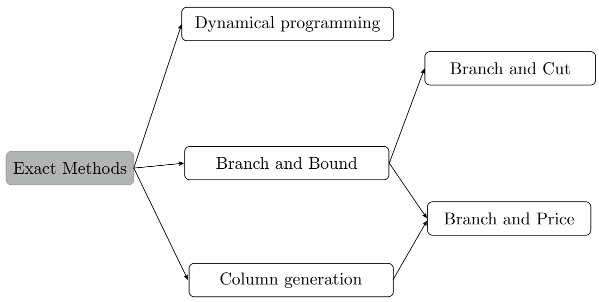

Fig. 1. Exact methods.

A meta-heuristic may be referred to as "an intelligent strategy combining the subordinate heuristics for diversification and intensification". Metaheuristics are broadly categorized into two main groups: local and population-based. Local metaheuristics focus on refining a single solution by exploring its neighborhood. These methods are efficient in finding good solutions, but they may get trapped in local optima, leading to suboptimal solutions [16, 28, 29].

In other hand, population-based metaheuristics refer to a group of optimization algorithms that utilize a population of candidate solutions to explore the search space and find the optimal solution. There are two main classes of metaheuristics: Swarm intelligence and Evolutionary computation. Swarm intelligence takes cues from the collective behavior of social organisms like bees, birds, and ants, and involves inter-actions between the population and the environment to achieve the optimal solution. Evolutionary computation, on the other hand, draws inspiration from natural evolution and applies genetic operators like selection, mutation, and crossover to allow the population to evolve over time [15]. Swarm intelligence algorithms are generally quicker and more efficient at exploring the search space, but they may face challenges in converging to a solution or getting stuck in a local optimum. In contrast, Evolutionary computation algorithms may require more time to converge, but they possess greater robustness and can tackle complex optimization problems that involve a large number of variables. Figure 1 presents the various approximatives methods to solving the VRP [30,31].

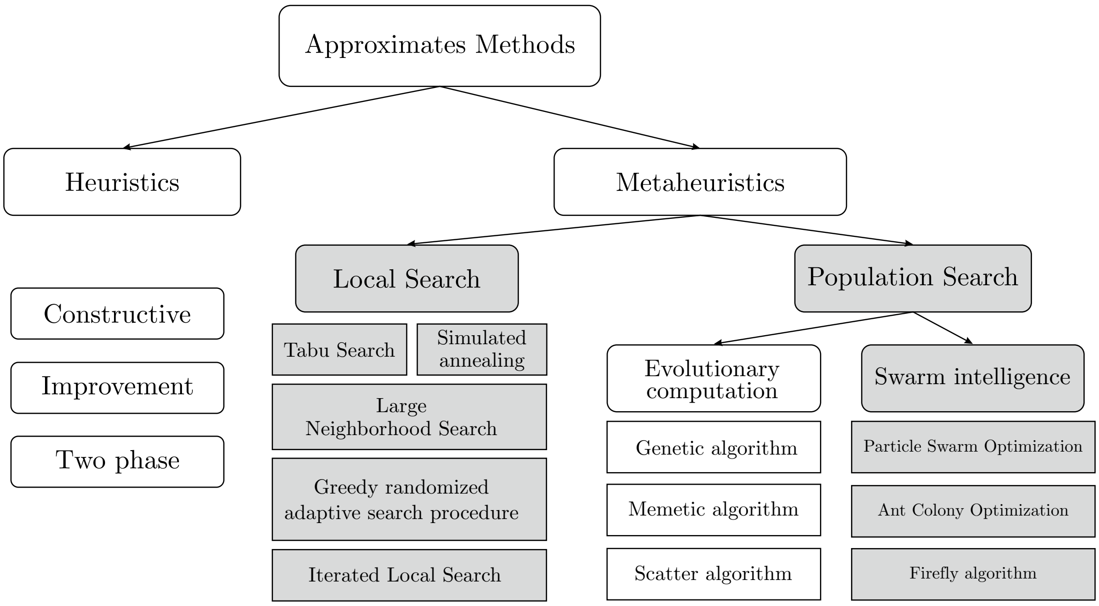

Fig. 2. Approximatives methods for VRPs.

4. Review results

4.1. Research methodology

In this paper, we will focus in most important papers published in the last twenty years related with the VRP area. Although recent surveys and reviews of available literature have primarily concentrated on specific variants of VRP and/or particular solution techniques, the aim of this current survey is to compile a list of various well-known VRP variants and the techniques used to solve them. In order to address this objective, the following research questions were devised:

RQ1: Which variants of the VRP have been explored in various papers to establish connections between logistics community goals and the field of VRP?

RQ2: How to choose the appropriate formulation far a VRP?

RQ3: What are the different approaches used to solve the VRP?

To answer these questions the search procedure has been conducted in well-known academic databases, such as Elsevier, Springer and IEEE transaction, etc. After reviewing all articles,we have identified thousands of papers including journal papers, conference proceedings, and theses. However, we have narrowed our search to focus only on papers published in high-impact journals categorized as **Q1** or **Q2**.

Table 1 and the diagram in Figure 3 summarize the sample of articles selected according to the journal of publication as well as the quartile and class of citation numbers respectively.

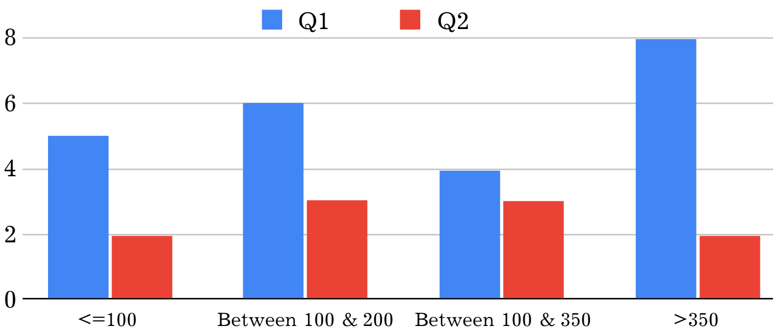

Fig. 3. Distribution of papers according to the journals quartile and citations.

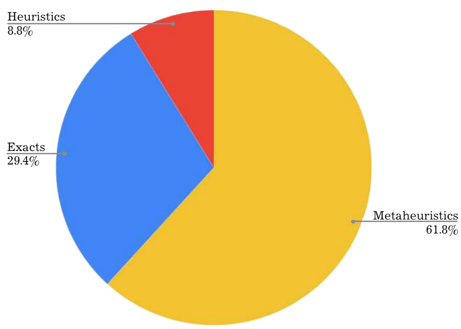

Fig. 4. Overall percentage of solutions attributes in the reviewed articles.

Table 1. List of journals associated from selected articles.

Journals	References
Computers & Operations Research	[12, 17–20, 30, 32]
European Journal of Operational Research	[11, 16, 28, 33]
Expert Systems with Applications	[14, 29, 34]
Mathematical Programming	[24, 25, 35]
Transportation Science	[21, 36]
Operations Research	[9]
Transportation Research Part C: Emerging Technologies	[23]
Journal of King Saud University-Engineering Sciences	[22]
Robotics and Computer-Integrated Manufacturing	[10]
Computers & Industrial Engineering	[37]
Applied Soft Computing	[38]
Journal of Heuristics	[39–42]
Applied Intelligence	[15]
Discrete Optimization	[13]
ScienceAsia	[43]
Journal of Zhejiang University: Science	[31]
Operations Research Letters	[44]

The overall percentages of solutions attributes in the study period are illustrated in Figure 4, we notice that the majority of articles (61.8%) use the metaheuristics. About 29.4% of the articles use the exact methods. The heuristics are used in 8.8% of the articles. We also notice that most of the authors hybridize the methods between them which is an effective way to improve the search efficiency, solution quality, robustness, and flexibility of the optimization process.

4.2. Summary of literature review

Table 2 illustrates the distribution of solution strategies, used in the articles examined by this study, according to the class of problems studied and the formulation used.

4.3. Analysis of literature

Throughout its history, the VRP has been a fundamental combinatorial issue, inspiring in-depth research into numerous problem variants. The limitations imposed and the cost functions applied determine how these versions differ. The VRPTW stands out as the most common form among the complicated real-world settings that are frequently reflected by VRP cases in current research. To ensure prompt customer service and satisfaction, the VRPTW incorporates time windows, which mirrors real-world limits and presents extra optimization issues.

A thorough analysis of the most popular VRP formulations is provided in this research. Each formulation offers a distinct view of the issue and has its own advantages and disadvantages in terms of the degree of computing complexity and the caliber of the solutions. The choice of formulation depends on the specific characteristics and requirements of the VRP instance under consideration.

While exact, heuristic, and metaheuristic approaches have been employed to solve the VRP, metaheuristics have become the preferred choice. The lack of exact solution methods that are effective for reasonably sized problems necessitates the utilization of metaheuristics. Although they cannot guarantee optimal solutions, metaheuristics have the potential to discover satisfactory solutions. Among the metaheuristic approaches, local search methods, particularly tabu search, have gained widespread acceptance as the most popular and efficient techniques. Additionally, hybrid methods that combine different algorithms have shown promise in finding excellent solutions that cannot be achieved by any single method within a reasonable timeframe.

However, despite the growing inclusion of real-life constraints in recent literature, a majority of authors still primarily propose problem-specific methods that lack applicability to other variants of the problem. These methods often involve manipulating parameters to achieve favorable performance for specific instances or benchmark scenarios. Consequently, many of the proposed solution methods

Table 2. Comparison of the model and solution between the different papers chosen.

Problem Studied	**Formulation**	**Solution**	**Authors**
CVRP	Index vehicle flow	Branch-and-Cut	Lysgaard et al. (2004)
		Branch-and-Cut	Achuthan et al. (2003)
		Firefly algorithm	Altabeeb et al. (2019)
		Hybrid Simulated Annealing with Tabu Search	Lin et al. (2009)
		Particle swarm optimization	Chen et al. (2006)
		Clarke and Wright savings	Pichpibul et al. (2012)
	Set partitioning	Branch-cut-and-price	Pecin et al. (2017)
		An exact algorithm	Baldacci et al. (2008)
	Commodity flow	Branch-and-Cut	Baldacci et al. (2004)
MDVRP	Index vehicle flow	Three hybrid heuristics	Mirabi et al. (2010)
		Cooperative coevolutionary algorithm	De Oliveira et al. (2016)
	Commodity flow	Variable Neighborhood Search	Reyes-Rubiano et al. (2020)
	Set partitioning	Combining statistical learning with metaheuristics	Calvet et al. (2016)
		Ant Colony Optimization	Yu et al. (2009)
		Granular Tabu Search	Escobar et al. (2014)
	Combining Index vehicle flow and set partitioning	Cutting Planes and Column Generation	Contardo et al. (2014)
HFVRP	Set partitioning	Iterated Local Search	Subramanian et al. (2012)
		Column Generation	CHOI et al. (2007)
		Iterated Local Search	Penna et al. (2013)
VRPTW	Index vehicle flow	Multiple Temperature Pareto Simulated Annealing	Baños et al. (2013)
		Genetic algorithm	Ombuki et al. (2006)
		Tabu search	Lau et al. (2003)
VRPSPD	Commodity flow	Tabu Search	Montané et al. (2006)
		Particle Swarm Optimization	TJ Ai et al. (2009)
		Large neighborhood search	Hornstra et al. (2020)
TD-VRPTW	Commodity flow	A Multiple Ant Colony System algorithm hybridized with Insertion Heuristic	Balseiro el al. (2011)
	Set partitioning Set covering	Branch and Price	Dabia et al. (2013)
		Tabu Search	Gmira et al. (2021)
MD-VRPTW	Index vehicle flow	Variable Neighborhood Search	Polacek et al. (2004)
MD-HFVRPTW	Set partitioning	Branch-and-Cut-and-Price	Bettinelli et al. (2011)
MD-TD-HFVRPTW	Index vehicle flow	Constructive heuristic	Afshar-Nadjafi et al. (2017)
HFPVRP	Three formulations	Kernel Search	Huerta-Munoz et al. (2022)
A class of VRP	Set partitioning	Iterated Local Search	Subramanian et al. (2013)
		Branch-Cut-and-Price	Pessoa et al. (2020)

are not easily adaptable for real-life applications. In our classification, there is a scarcity of articles that propose broad algorithms capable of solving multiple variants. Finally, it has been noted that there is a lack of benchmarks for the more realistic versions of VRP. This presents an opportunity for further research in this area, motivating researchers to create publicly available data sets to compare and assess the effectiveness of different VRP algorithms.

5. Conclusion

In conclusion, this paper provides an overview of recent research on the VRP and its common variants, with a particular focus on mathematical formulation and solution strategies. Our analysis reveals that the primary objective in these studies is to minimize the total cost associated with routing operations. The classification tables generated from the results highlight the diverse formulations and algorithms employed, tailored to address specific problem characteristics.

Looking ahead, with advancements in technology such as GPS and mobile communication, we anticipate a growing interest in dynamic VRP, allowing for real-time adjustments to routing decisions as drivers are on the move. This dynamic approach holds promise for enhancing the efficiency and responsiveness of routing operations in the future.

It is important to recognize that the publications reviewed in this work presume that all problem data will be available in advance. However, in real-world situations, there can be ambiguity over trip timings or precise consumer requirements. Therefore, it is crucial to create models and approaches that can handle unknown variations of the situation, taking into account variables like trip time unpredictability and shifting client expectations.

Overall, this survey offers a comprehensive overview of the present state of research on the VRP and its variations, illuminating the prevalent goals, hypotheses, and approaches to solving problems. The knowledge gathered from this work can serve as a roadmap for future research, highlighting the necessity of addressing dynamic factors and uncertainties to improve the applicability and efficacy of VRP models in real-world scenarios.

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Огляд задачі маршрутизації транспортного засобу та її варіантів: постановки та розв'язування

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У сфері промислових підприємств підвищення ефективності логістики є першорядним завданням. Мета полягає в організації оптимізованого обслуговування з безперебійним потоком товарів за умови мінімізації витрат. Ключовим компонентом будь-якої системи логістики є адміністрування та розроблення стратегії розподільних мереж для парків транспортних засобів, що зазвичай називають проблемою маршрутизації транспортних засобів (VRP). Ця композиція присвячена дослідженню VRP та його різноманітних ітерацій, пропонуючи класифікацію та зображення прийнятих формулювань і алгоритмів, поширених у наукових роботах за останні два десятиліття.

Ключові слова: *VRP; опитування; точні методи; математичне формулювання; мета-евристики.*