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Algorithmic implementation of an exact three-point difference scheme for a certain class of singular Sturm–Liouville problems

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In this article, we present a new algorithmic implementation of exact three-point difference schemes for a certain class of singular Sturm–Liouville problems. We demonstrate that computing the coefficients of the exact scheme at any grid node x_j requires solving two auxiliary Cauchy problems for the second-order linear ordinary differential equations: one problem on the interval $[x_{j-1}, x_j]$ (forward) and one problem on the interval $[x_j, x_{j+1}]$ (backward). We have also proven the coefficient stability theorem for the exact three-point difference scheme.

Keywords: *singular Sturm–Liouville problem; exact three-point difference scheme; coefficient stability.*

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1. Introduction

Exact difference schemes for linear boundary value problems have been first introduced in [1,2]. These schemes allow for the construction of truncated difference schemes of any order of accuracy. In [3], these results were applied to the Sturm–Liouville problem, and in [4], to the singular Sturm–Liouville problem with coefficients of a special form.

However, the practical use of such truncated schemes in the case of variable coefficients of a differential equation requires the calculation of multiple integrals at each grid node x_j, posing computational challenges. Addressing the need for high accuracy in practical calculations for nonlinear boundary value problems, truncated difference schemes of high order were developed in [5].

In [6, 7], building upon the ideas presented in [8, 9], a new algorithmic realization of the exact three-point difference scheme (ETDS) via truncated three-point difference schemes (TDS) of any order of accuracy was developed and justified for the Sturm–Liouville problem. These articles demonstrate that the coefficients of the ETDS and the right-hand side at any grid node can be expressed through the solutions of two auxiliary Cauchy problems, each of which can be numerically solved using any one-step method, such as the Taylor series expansion or the Runge–Kutta methods.

In this paper, we extend these findings by demonstrating that coefficients of the ETDS for the Sturm–Liouville problem with a singularity at the ends of the segment $[-1, 1]$ can also be expressed through solutions of the auxiliary Cauchy problems for second-order linear ordinary differential equations.

2. Exact three-point difference scheme for singular Sturm–Liouville problem

In the present article we consider the following singular Sturm–Liouville problem

$$\frac{d}{dx}\left[k(x)\frac{du}{dx}\right] - q(x)u(x) = -\lambda r(x)u(x), \quad x \in (-1, 1), \tag{1}$$

$$u(-1) \neq \infty, \quad u(1) \neq \infty, \tag{2}$$

where

$$k(x) = (1-x^2)k_1(x), \quad 0 < C_1 \leqslant k_1(x) \leqslant C_2, \quad 0 < C_3 \leqslant q(x) \leqslant C_4, \quad 0 < C_5 \leqslant r(x) \leqslant C_6, \tag{3}$$

C_i, $i = 1, 2, \ldots, 6$ are constants. We introduce the regular grid

$$\bar{\omega}_h = \{x_j = -1 + (j-0,5)h,\ h = 2/N,\ j = 1, 2, \ldots, N,\ x_0 = -1,\ x_{N+1} = 1\}$$

and take the pattern functions $v_\alpha^j(x,\lambda)$, $\alpha = 1,2$, $j = 1, 2, \ldots, N$ as the solutions of the following Cauchy problems

$$\begin{aligned} &\frac{d}{dx}\left[k(x)\frac{dv_1^1}{dx}\right] - q(x)v_1^1(x,\lambda) + \lambda r(x)v_1^1(x,\lambda) = 0, \quad x \in (x_0, x_2), \\ &v_1^1(x_0,\lambda) = 1, \quad k(x)\frac{dv_1^1(x,\lambda)}{dx}\bigg|_{x=x_0} = 0, \end{aligned} \tag{4}$$

$$\begin{aligned} &\frac{d}{dx}\left[k(x)\frac{dv_\alpha^j}{dx}\right] - q(x)v_\alpha^j(x,\lambda) + \lambda r(x)v_\alpha^j(x,\lambda) = 0, \quad x \in (x_{j-1}, x_{j+1}), \\ &v_\alpha^j(x_{j+(-1)^\alpha},\lambda) = 0, \quad k(x)\frac{dv_\alpha^j(x,\lambda)}{dx}\bigg|_{x=x_{j+(-1)^\alpha}} = (-1)^{\alpha+1}, \\ &\alpha = 1, 2, \quad j = 3-\alpha, 4-\alpha, \ldots, N+1-\alpha, \end{aligned} \tag{5}$$

$$\begin{aligned} &\frac{d}{dx}\left[k(x)\frac{dv_2^N}{dx}\right] - q(x)v_2^N(x,\lambda) + \lambda r(x)v_2^N(x,\lambda) = 0, \quad x \in (x_{N-1}, x_{N+1}), \\ &v_2^N(x_{N+1},\lambda) = 1, \quad k(x)\frac{dv_2^N(x,\lambda)}{dx}\bigg|_{x=x_{N+1}} = 0. \end{aligned} \tag{6}$$

Similarly to [4], we establish the properties of the pattern functions.

Lemma 1. *The functions $v_\alpha^j(x,\lambda) > 0$, $\alpha = 1,2$ have the following properties:*

1) *$v_\alpha^j(x,\lambda) > 0$, $\alpha = 1,2$ for all $x \in (x_{j-1}, x_{j+1})$ $j = 1, 2, \ldots, N$ and are linearly independent at each of these intervals;*
2) *these functions satisfy the next relation*

$$v_1^j(x_{j+1},\lambda) = v_2^j(x_{j-1},\lambda), \quad j = 2, 3, \ldots, N-1, \quad v_2^j(x_j,\lambda) = v_1^{j+1}(x_{j+1},\lambda), \quad j = 1, 2, \ldots, N-1,$$

$$\begin{aligned} v_1^1(x_2,\lambda) &= v_1^1(x_1,\lambda) + v_2^1(x_1,\lambda)\int_{x_0}^{x_1} v_1^1(\xi,\lambda)[q(\xi) - \lambda r(\xi)]\,d\xi \\ &+ v_1^1(x_1,\lambda)\int_{x_1}^{x_2} v_2^1(\xi,\lambda)[q(\xi) - \lambda r(\xi)]\,d\xi, \\ v_1^j(x_{j+1},\lambda) &= v_1^j(x_j,\lambda) + v_2^j(x_j,\lambda) + v_2^j(x_j,\lambda)\int_{x_{j-1}}^{x_j} v_1^j(\xi,\lambda)[q(\xi) - \lambda r(\xi)]\,d\xi \\ &+ v_1^j(x_j,\lambda)\int_{x_j}^{x_{j+1}} v_2^j(\xi,\lambda)[q(\xi) - \lambda r(\xi)]\,d\xi, \quad j = 2, \ldots, N-1, \\ v_2^N(x_{N-1},\lambda) &= v_2^N(x_N,\lambda) + v_2^N(x_N,\lambda)\int_{x_{N-1}}^{x_N} v_1^N(\xi,\lambda)[q(\xi) - \lambda r(\xi)]\,d\xi \\ &+ v_1^N(x_N,\lambda)\int_{x_N}^{x_{N+1}} v_2^N(\xi,\lambda)[q(\xi) - \lambda r(\xi)]\,d\xi. \end{aligned}$$

Proof.

1) We now prove that the functions $v_\alpha^j(x,\lambda)$, $\alpha = 1,2$, are linearly independent. As is known, for the linear independence of solutions of problem (4), (5) it is necessary and sufficient, that the Wronskian should be different from zero if at least in one point of the interval $[x_{j-1}, x_{j+1}]$. Let us assume the contrary for $j = 2, 3, \ldots, N-1$. Then the Wronskian $W[v_1^j(x,\lambda), v_2^j(x,\lambda)]$ is identically equal to zero on the interval $[x_{j-1}, x_{j+1}]$. Calculating the Wronskian at the points $x_{j+(-1)^\alpha}$, $\alpha = 1,2$, and taking the fact that $v_2^j(x_{j-1},\lambda) = v_1^j(x_{j+1},\lambda)$ into account, we obtain

$$W[v_1^j(x,\lambda), v_2^j(x,\lambda)]_{x=x_{j+(-1)^\alpha}} = -\frac{v_1^j(x_{j+1},\lambda)}{k(x_{j+(-1)^\alpha})}.$$

It follows that $v_1^j(x_{j+1},\lambda)=0$, i.e., $v_1^j(x,\lambda)$ is the solution of boundary-value problem

$$\begin{gathered}\frac{d}{dx}\left[k(x)\frac{dv_1^j}{dx}\right] - q(x)v_1^j(x,\lambda) + \lambda r(x)v_1^j(x,\lambda) = 0, \quad x\in(x_{j-1},x_{j+1}),\\ v_1^j(x_{j-1},\lambda) = v_1^j(x_{j+1},\lambda) = 0, \quad j=2,3,\dots,N-1.\end{gathered} \tag{7}$$

We now show that for sufficiently small $h<h_0$ and for $\lambda=\lambda_m$, $1\leqslant m\leqslant k$, $k\ll N$, problem (7) has only the trivial solution. For this purpose, it is sufficient to show that for $h<h_0$ the following inequality is satisfied:

$$-[q(x)-\lambda r(x)] \leqslant \lambda r(x) < \underline{\mu}_1 \quad \forall x\in[x_{j-1},x_{j+1}],$$

where $\underline{\mu}_1$ is the lower estimate of the smallest eigenvalue of the problem

$$\frac{d}{dx}\left[k(x)\frac{dv}{dx}\right] + \mu v(x) = 0, \quad x\in(x_{j-1},x_{j+1}), \quad v(x_{j-1}) = v(x_{j+1}) = 0.$$

This problem is known to be equivalent to the variational problem of finding the minimum of the functional

$$\min\int_{x_{j-1}}^{x_{j+1}} k(\xi)[v'(\xi)]^2\,d\xi$$

under condition

$$\|v\|^2 = \int_{x_{j-1}}^{x_{j+1}} v^2(x)\,dx = 1.$$

Considering that $h<1$, $x_{j-1}\leqslant\xi\leqslant x_{j+1}$ for $j=2,3,\dots,N-1$,

$$k(\xi)\geqslant C_1(1-\xi^2)\geqslant C_1[1-(-1+0,5h)^2] = C_1\left[h-\frac{h^2}{4}\right] > \frac{3}{4}hC_1$$

and $\min\int_{x_{j-1}}^{x_{j+1}}[v'(\xi)]^2\,d\xi = \frac{\pi^2}{4h^2}$, we obtain $\mu_1 > \frac{3C_1\pi^2}{16h} = \underline{\mu}_1$. Hence, there exists h_0 such that for all $h<h_0$ the inequality

$$h\lambda r(x) < \frac{3C_1\pi^2}{16} \quad \forall x\in[x_{j-1},x_{j+1}], \quad j=2,3,\dots,N-1$$

is satisfied. Consequently, it follows that $v_1^j(x,\lambda)\equiv 0$, $x\in[x_{j-1},x_{j+1}]$ for $h<h_0=\frac{3C_1\pi^2}{16\lambda C_6}$, which contradicts the condition $k(x)\frac{dv_1^j(x,\lambda)}{dx}\Big|_{x=x_{j-1}} = 1$.

Similar considerations show that $v_1^j(x,\lambda)\neq 0$ holds in any point of the interval $(x_{j-1},x_{j+1}]$, i.e., our function is of constant-sign on this interval. Thus, $v_1^j(x,\lambda)$, $v_2^j(x,\lambda)$, $j=2,3,\dots,N-1$ are linearly independent on the interval $[x_{j-1},x_{j+1}]$.

For $j=1$ at $\lambda=\lambda_m$, $m=1,2,\dots,k$, $k\ll N$, we have $v_1^1(x,\lambda)=c_m u_m(x)$, where $u_m(x)$ is the eigenfunction of the problem (1), (2) which corresponds to the eigenvalue λ_m. We denote by $x_{\min}^m$ the minimum, and by $x_{\max}^m$ the maximum zero of the function $u_m(x)$ on the interval $(-1,1)$. If we choose $h<h_1=\frac{2}{3}(1+x_{\min}^m)$, then $x_2=-1+1.5h<-1+(1+x_{\min}^m)=x_{\min}^m$ is obtained. Hence, $v_1^1(x,\lambda)\neq 0$, $x\in[-1,x_2]$. From the fact that $v_1^1(x_2,\lambda)\neq 0$, follows the linear independence of the functions $v_1^1(x,\lambda)$, $v_2^1(x,\lambda)$. Similarly, for $j=N$ we get that for $h<h_2=\frac{2}{3}(1-x_{\max}^m)$ the inequality $v_2^N(x,\lambda)\neq 0$, $x\in[x_{N-1},1]$ is satisfied (then $x_{N-1}=-1+(N-1.5)h=1-1.5h>1-(1-x_{\max}^m)=x_{\max}^m$) and therefore $v_1^N(x,\lambda)$ and $v_2^N(x,\lambda)$ are linearly independent.

Since from (4), (5)

$$\begin{gathered}v_1^1(x,\lambda) = 1+\int_{x_0}^{x}\frac{1}{k(t)}\int_{x_0}^{t}(q(\xi)-\lambda r(\xi))v_1^1(\xi,\lambda)\,d\xi\,dt,\\ v_1^j(x,\lambda) = \int_{x_{j-1}}^{x}\frac{1}{k(t)}\left(1+\int_{x_{j-1}}^{t}(q(\xi)-\lambda r(\xi))v_1^j(\xi,\lambda)\,d\xi\right)dt, \quad j=2,\dots,N,\end{gathered} \tag{8}$$

then according to (3) and to the mean value theorem, there exists a point $\bar{x} \in (x_{j-1}, x)$ such that

$$v_1^1(x,\lambda) \geqslant 1 - \int_{x_0}^{x} \frac{1}{k(t)} \int_{x_0}^{t} |q(\xi) - \lambda r(\xi)| \left|v_1^1(\xi,\lambda)\right| d\xi\, dt \geqslant 1 - \frac{C_4 + \lambda C_6}{C_1(1-\bar{x}^2)}(x - x_0) \int_{x_0}^{\bar{x}} \left|v_1^1(\xi,\lambda)\right| d\xi,$$

$$\begin{aligned} v_1^j(x,\lambda) &\geqslant \int_{x_{j-1}}^{x} \frac{1}{k(t)} \left(1 - \int_{x_{j-1}}^{t} |q(\xi) - \lambda r(\xi)| \left|v_1^j(\xi,\lambda)\right| d\xi\right) dt \\ &\geqslant \int_{x_{j-1}}^{x} \frac{dt}{k(t)} \left(1 - (C_4 + \lambda C_6) \int_{x_{j-1}}^{x} \left|v_1^j(\xi,\lambda)\right| d\xi\right) \\ &= \int_{x_{j-1}}^{x} \frac{dt}{k(t)} \left(1 - (C_4 + \lambda C_6)(x - x_{j-1}) \left|v_1^j(\bar{x},\lambda)\right|\right). \end{aligned}$$

From these inequalities, it follows that $v_1^j(x,\lambda) > 0$, $j = 1, 2, \ldots, N$ on the interval $(x_{j-1}, x_{j-1} + \delta)$ for any small $\delta > 0$. Since the functions $v_1^j(x,\lambda)$, $j = 1, 2, \ldots, N$ are of constant-sign, they are positive on the entire interval (x_{j-1}, x_{j+1}).

2) Proof is carried out by analogy with the proof of the corresponding properties from [10, p. 141]. ■

The following assertion is valid.

Lemma 2. *Suppose that the assumptions* (3) *is satisfied. Then, for*

$$h \leqslant h_0 = \frac{1}{2}\sqrt{\frac{C_1(1 - x_{j+1}^2)}{C_4 + \lambda C_6}} \tag{9}$$

the following assertions are valid:

(i) *The pattern functions have the properties:* $v_1^j(x,\lambda)$, $j = 2, 3, \ldots, N$ *increase monotonically on* $(x_{j-1}, x_{j+1}]$, *and the functions* $v_2^j(x,\lambda)$, $j = 1, 2, \ldots, N-1$ *decrease monotonically on* $[x_{j-1}, x_{j+1})$;

(ii) *For all* $j = 3 - \alpha, 4 - \alpha, \ldots, N + 1 - \alpha$, $\alpha = 1, 2$, *it holds that*

$$\frac{2}{3C_2(1+x)(1 - x_{j+(-1)^\alpha})} \leqslant \frac{v_\alpha^j(x,\lambda)}{\left|x - x_{j+(-1)^\alpha}\right|} \leqslant \frac{2}{C_1(1-x)(1 + x_{j+(-1)^\alpha})}. \tag{10}$$

Proof. We only prove the assertions for the pattern function $v_1^j(x,\lambda)$ since those for the $v_2^j(x,\lambda)$ follow analogously.

Note that equation (8) in connection with assumptions (3) leads, for any bounded λ, to the inequality

$$\begin{aligned} v_1^j(x,\lambda) &\leqslant \int_{x_{j-1}}^{x} \frac{dt}{1-t^2} \left[\frac{1}{C_1} + \frac{C_4 + \lambda C_6}{C_1} \int_{x_{j-1}}^{x} v_1^j(t,\lambda)\, dt\right] \\ &= \frac{1}{2} \ln\left(1 + \frac{2(x - x_{j-1})}{(1-x)(1+x_{j-1})}\right) \left[\frac{1}{C_1} + \frac{C_4 + \lambda C_6}{C_1} \int_{x_{j-1}}^{x} v_1^j(t,\lambda)\, dt\right]. \end{aligned}$$

Using the well-known inequality

$$\frac{r}{r+1} \leqslant \ln(1+r) \leqslant r, \tag{11}$$

which is true for $r \geqslant 0$, we thus obtain

$$v_1^j(x,\lambda) \leqslant \frac{x - x_{j-1}}{(1-x)(1+x_{j-1})} \left[\frac{1}{C_1} + \frac{C_4 + \lambda C_6}{C_1} \int_{x_{j-1}}^{x} v_1^j(t,\lambda)\, dt\right].$$

We now make the substitution

$$\bar{v}_1^j(x,\lambda) = \frac{v_1^j(t,\lambda)(1-x)(1+x_{j-1})}{x - x_{j-1}}$$

in order to obtain

$$\bar{v}_1^j(x,\lambda) \leqslant \frac{1}{C_1} + \frac{C_4 + \lambda C_6}{C_1(1+x_{j-1})} \int_{x_{j-1}}^{x} \frac{t - x_{j-1}}{1-t} \bar{v}_1^j(t,\lambda)\, dt.$$

Applying the Gronwall inequality (see, e.g., [11, p. 37]), we obtain

$$\bar{v}_1^j(x,\lambda) \leqslant \frac{1}{C_1}\exp\left\{\frac{C_4+\lambda C_6}{C_1(1+x_{j-1})}\int_{x_{j-1}}^{x}\frac{t-x_{j-1}}{1-t}\,dt\right\}$$

or, equivalently,

$$v_1^j(t,\lambda) \leqslant \frac{1}{C_1}\frac{x-x_{j-1}}{(1-x)(1+x_{j-1})}\exp\left\{\frac{C_4+\lambda C_6}{C_1(1+x_{j-1})}\int_{x_{j-1}}^{x}\frac{t-x_{j-1}}{1-t}\,dt\right\} \quad \forall x\in[x_{j-1},x_{j+1}].$$

From the last estimate and the equality

$$k(x)\frac{dv_1^j(x,\lambda)}{dx} = 1+\int_{x_{j-1}}^{x}(q(\xi)-\lambda r(\xi))v_1^j(\xi,\lambda)\,d\xi,$$

follows the inequality

$$\begin{aligned}
k(x)\frac{dv_1^j(x,\lambda)}{dx} &\geqslant 1-\int_{x_{j-1}}^{x}|q(\xi)-\lambda r(\xi)|\,v_1^j(\xi,\lambda)\,d\xi\\
&\geqslant 1-\frac{C_4+\lambda C_6}{C_1(1+x_{j-1})}\int_{x_{j-1}}^{x}\frac{t-x_{j-1}}{1-t}\exp\left\{\frac{C_4+\lambda C_6}{C_1(1+x_{j-1})}\int_{x_{j-1}}^{t}\frac{\xi-x_{j-1}}{1-\xi}\,d\xi\right\}dt\\
&\geqslant 2-\exp\left\{\frac{(C_4+\lambda C_6)}{C_1(1+x_{j-1})}\int_{x_{j-1}}^{x}\frac{t-x_{j-1}}{1-t}\,dt\right\}.
\end{aligned}$$

Since

$$\int_{x_{j-1}}^{x}\frac{t-x_{j-1}}{1-t}\,dt = -(x-x_{j-1})+(1-x_{j-1})\ln\left(1+\frac{x-x_{j-1}}{1-x}\right),$$

and due to inequality (11), we get

$$\int_{x_{j-1}}^{x}\frac{t-x_{j-1}}{1-t}\,dt \leqslant -(x-x_{j-1})+\frac{(x-x_{j-1})(1-x_{j-1})}{1-x} = \frac{(x-x_{j-1})^2}{1-x}.$$

Hence,

$$k(x)\frac{dv_1^j(x,\lambda)}{dx} \geqslant 2-e^{\frac{(C_4+\lambda C_6)(x-x_{j-1})^2}{C_1(1+x_{j-1})(1-x)}} \geqslant 2-e^{\frac{4(C_4+\lambda C_6)h^2}{C_1(1-x_{j+1}^2)}}$$

holds which proves that $v_1^j(x,\lambda)$ grows monotonically on $(x_{j-1},x_{j+1}]$ given that the function $g(t)=2-e^t$ decreases monotonically, $g(1/2)>0$ and the validity of condition (9).

Returning to equality (8), we obtain with the help of the proved assertion (***i***) that

$$\begin{aligned}
v_1^j(x,\lambda) &\leqslant \frac{1}{C_1}\int_{x_{j-1}}^{x}\frac{dt}{1-t^2}+\frac{C_4+\lambda C_6}{2C_1}v_1^j(x,\lambda)\int_{x_{j-1}}^{x}\frac{t-x_{j-1}}{1-t^2}\,dt\\
&= \frac{1}{2C_1}\ln\left(1+\frac{2(x-x_{j-1})}{(1-x)(1+x_{j-1})}\right)+\frac{C_4+\lambda C_6}{4C_1}v_1^j(x,\lambda)\\
&\quad\times(x-x_{j-1})\left[\ln\left(1+\frac{x-x_{j-1}}{1-x}\right)-\ln\left(1+\frac{x-x_{j-1}}{1-x_{j-1}}\right)\right].
\end{aligned}$$

Using the inequality (11), we get

$$v_1^j(x,\lambda) \leqslant \frac{x-x_{j-1}}{C_1(1-x)(1+x_{j-1})}+\frac{C_4+\lambda C_6}{2C_1}v_1^j(x,\lambda)\frac{(x-x_{j-1})^2}{1-x^2},$$

and

$$\begin{aligned}
v_1^j(x,\lambda) &\geqslant \frac{1}{2C_2}\ln\left(1+\frac{2(x-x_{j-1})}{(1-x)(1+x_{j-1})}\right)-\frac{C_4+\lambda C_6}{2C_1}v_1^j(x,\lambda)\frac{(x-x_{j-1})^2}{1-x^2}\\
&\geqslant \frac{x-x_{j-1}}{C_2(1+x)(1-x_{j-1})}-\frac{C_4+\lambda C_6}{2C_1}v_1^j(x,\lambda)\frac{(x-x_{j-1})^2}{1-x^2}.
\end{aligned}$$

Hence,

$$\frac{v_1^j(x,\lambda)}{x-x_{j-1}}\left(1-\frac{2(C_4+\lambda C_6)h^2}{C_1(1-x_{j+1}^2)}\right)\leqslant\frac{1}{C_1(1-x)(1+x_{j-1})},$$
$$\frac{v_1^j(x,\lambda)}{x-x_{j-1}}\left(1+\frac{2(C_4+\lambda C_6)h^2}{C_1(1-x_{j+1}^2)}\right)\geqslant\frac{1}{C_2(1+x)(1-x_{j-1})},$$

which, taking the condition (9) into account, proves the estimate (10). ∎

Lemma 3. *Suppose that the assumptions of Lemma 2 are satisfied. Then, for the problem (1)–(3) there exists ETDS of the form*

$$\Lambda y_j+\lambda\rho_j y_j\equiv(ay_{\bar{x}})_{x,j}-d_jy_j+\lambda\rho_jy_j=0,\quad j=1,2,\ldots,N,\quad y_0\neq\infty,\quad y_{N+1}\neq\infty,\tag{12}$$

where

$$\begin{aligned}&y_{\bar{x},j}=\frac{y_j-y_{j-1}}{h},\quad y_{x,j}=\frac{y_{j+1}-y_j}{h},\quad a_j=\left[\frac{1}{h}v_1^j(x_j,\lambda)\right]^{-1},\quad j=2,3,\ldots,N,\\&a_1=a_{N+1}=0,\quad d_j=T^{x_j}(q,\lambda),\quad \rho_j=T^{x_j}(r,\lambda),\quad j=1,2,\ldots,N,\\&T^{x_j}(w(\xi),\lambda)=\frac{1}{hv_1^j(x_j,\lambda)}\int_{x_{j-1}}^{x_j}v_1^j(\xi,\lambda)\,w(\xi)\,d\xi+\frac{1}{hv_2^j(x_j,\lambda)}\int_{x_j}^{x_{j+1}}v_2^j(\xi,\lambda)\,w(\xi)\,d\xi,\end{aligned}\tag{13}$$

and

$$0<\left(1-x_{j-1/2}^2\right)C_1'\leqslant a_j\leqslant\left(1-x_{j-1/2}^2\right)C_2',\quad C_1'=\frac{C_1}{2},\quad C_2'=\frac{3C_2}{2},\quad x_{j-1/2}=x_j-\frac{h}{2},\tag{14}$$

$$0<C_3'\leqslant d_j\leqslant C_4',\quad C_4'=2C_4,\quad 0<C_5'\leqslant\rho_j\leqslant C_6',\quad C_6'=2C_6.\tag{15}$$

The solution $y(x)$ of problem (12) coincides with the solution $u(x)$ of the original problem (1), (2) at nodes of the grid ω_h up to a constant multiplier.

Proof. First of all, we note that the problem (1), (2) is equivalent to the sequence of problems

$$\begin{aligned}&\frac{d}{dx}\left[k(x)\frac{du}{dx}\right]-q(x)u(x)=-\lambda r(x)u(x),\quad x\in(x_0,x_2),\\&k(x)\frac{du}{dx}\bigg|_{x=x_0}=0,\quad u(x_2)=u_2,\end{aligned}\tag{16}$$

$$\begin{aligned}&\frac{d}{dx}\left[k(x)\frac{du}{dx}\right]-q(x)u(x)=-\lambda r(x)u(x),\quad x\in(x_{j-1},x_{j+1}),\\&u(x_{j-1})=u_{j-1},\quad u(x_{j+1})=u_{j+1},\quad j=2,3,\ldots,N-1,\end{aligned}\tag{17}$$

$$\begin{aligned}&\frac{d}{dx}\left[k(x)\frac{du}{dx}\right]-q(x)u(x)=-\lambda r(x)u(x),\quad x\in(x_{N-1},x_{N+1}),\\&u(x_{N-1})=u_{N-1},\quad k(x)\frac{du}{dx}\bigg|_{x=x_{N+1}}=0,\end{aligned}\tag{18}$$

whose Green's functions have the form

$$G^j(x,\xi)=\frac{1}{v_1^j(x_{j+1},\lambda)}\begin{cases}v_1^j(x,\lambda)\,v_2^j(\xi,\lambda), & x_{j-1}\leqslant x\leqslant\xi,\\ v_1^j(\xi,\lambda)\,v_2^j(x,\lambda), & \xi\leqslant x\leqslant x_{j+1},\end{cases}\qquad j=1,2,\ldots,N.$$

We construct an exact three-point difference scheme. For this purpose, we write the obvious integral representation of (16)–(18). Then, we have

$$\int_{x_{j-1}}^{x_{j+1}}G^j(x,\xi)\frac{d}{d\xi}\left[k(\xi)\frac{du}{d\xi}\right]d\xi-\int_{x_{j-1}}^{x_{j+1}}G^j(x,\xi)[q(\xi)-\lambda r(\xi)]u(\xi)\,d\xi=0,\quad j=1,2,\ldots,N.\tag{19}$$

Calculating the integral in the left-hand side of (19) by integration by parts and using (4)–(6), we get

$$-\frac{v_1^j(x,\lambda)}{v_1^j(x_{j+1},\lambda)}\,k(\xi)\frac{dv_2^j(\xi,\lambda)}{d\xi}u(\xi)\bigg|_x^{x_{j+1}}-\frac{v_2^j(x,\lambda)}{v_1^j(x_{j+1},\lambda)}\,k(\xi)\frac{dv_1^j(\xi,\lambda)}{d\xi}u(\xi)\bigg|_{x_{j-1}}^x=0.\tag{20}$$

For $j=1$, we have
$$\frac{v_1^1(x,\lambda)}{v_1^1(x_2,\lambda)}\left[u_2+k(x)\frac{dv_2^1(x,\lambda)}{dx}u(x)\right]-\frac{v_2^1(x,\lambda)}{v_1^1(x_2,\lambda)}k(x)\frac{dv_1^1(x,\lambda)}{dx}u(x)=0.$$
From (4) and (5), it follows that
$$\begin{aligned}k(x)\frac{dv_1^1(x,\lambda)}{dx}&=\int_{x_0}^{x}[q(\xi)-\lambda r(\xi)]\,v_1^1(\xi,\lambda)\,d\xi,\\ k(x)\frac{dv_2^1(x,\lambda)}{dx}&=-1-\int_{x}^{x_2}[q(\xi)-\lambda r(\xi)]\,v_2^1(\xi,\lambda)\,d\xi.\end{aligned}\tag{21}$$
Thus,
$$\frac{v_1^1(x,\lambda)}{v_1^1(x_2,\lambda)}\left[u_2-u(x)\left(1+\int_{x}^{x_2}[q(\xi)-\lambda r(\xi)]v_2^1(\xi,\lambda)\,d\xi\right)\right]\\ -\frac{v_2^1(x,\lambda)}{v_1^1(x_2,\lambda)}u(x)\int_{x_0}^{x}[q(\xi)-\lambda r(\xi)]v_1^1(\xi,\lambda)\,d\xi=0.\tag{22}$$
For $x=x_1$, let us multiply equality (22) by $\frac{v_1^1(x_2,\lambda)}{hv_1^1(x_1,\lambda)v_2^1(x_1,\lambda)}$. Note that due to $v_2^1(x_1,\lambda)=v_1^2(x_2,\lambda)$, we have
$$\frac{u_2-u_1}{hv_1^2(x_2,\lambda)}-u_1\left[\frac{1}{hv_2^1(x_1,\lambda)}\int_{x_1}^{x_2}[q(\xi)-\lambda r(\xi)]\,v_2^1(\xi,\lambda)\,d\xi\right.\\ \left.+\frac{1}{hv_1^1(x_1,\lambda)}\int_{x_0}^{x_1}[q(\xi)-\lambda r(\xi)]\,v_1^1(\xi,\lambda)d\xi\right]=0,$$
or, equivalently, in view of $a_1=0$,
$$\frac{1}{h}\left(a_2u_{x,1}-a_1u_{\bar{x},1}\right)-d_1u_1+\lambda\rho_1u_1=0.$$
For $j=2,3,\dots,N-1$, equality (20) has the form
$$\frac{v_1^j(x,\lambda)}{v_1^j(x_{j+1},\lambda)}\left[u_{j+1}+k(x)\frac{dv_2^j(x,\lambda)}{dx}u(x)\right]+\frac{v_2^j(x,\lambda)}{v_1^j(x_{j+1},\lambda)}\left[u_{j-1}-k(x)\frac{dv_1^j(x,\lambda)}{dx}u(x)\right]=0.$$
Since it follows from (5) that
$$k(x)\frac{dv_\alpha^j(x,\lambda)}{dx}=(-1)^{\alpha+1}+\int_{x_{j+(-1)^\alpha}}^{x}[q(\xi)-\lambda r(\xi)]\,v_\alpha^j(\xi,\lambda)\,d\xi,\quad \alpha=1,2,\tag{23}$$
we have
$$\frac{v_1^j(x,\lambda)}{v_1^j(x_{j+1},\lambda)}\left[u_{j+1}-\left(1+\int_{x}^{x_{j+1}}[q(\xi)-\lambda r(\xi)]\,v_2^j(\xi,\lambda)\,d\xi\right)u(x)\right]\\ +\frac{v_2^j(x,\lambda)}{v_1^j(x_{j+1},\lambda)}\left[u_{j-1}-\left(1+\int_{x_{j-1}}^{x}[q(\xi)-\lambda r(\xi)]\,v_1^j(\xi,\lambda)\,d\xi\right)u(x)\right]=0.\tag{24}$$

Taking in (24) $x=x_j$, multiplying the obtained equality by $\frac{v_1^j(x_{j+1},\lambda)}{hv_1^j(x_j,\lambda)v_2^j(x_j,\lambda)}$, and using the properties of the pattern functions $v_1^j(x_{j+1},\lambda)=v_2^j(x_{j-1},\lambda)$, $v_2^j(x_j,\lambda)=v_1^{j+1}(x_{j+1},\lambda)$, we arrive at the exact three-point difference scheme (12) for $j=2,3,\dots,N-1$.

Let us rewrite (20) for $j=N$ by
$$\frac{v_1^N(x,\lambda)}{v_1^N(x_{N+1},\lambda)}\,k(x)\,\frac{dv_2^N(x,\lambda)}{dx}\,u(x)+\frac{v_2^N(x,\lambda)}{v_1^N(x_{N+1},\lambda)}\left[u_{N-1}-k(x)\,\frac{dv_1^N(x,\lambda)}{dx}\,u(x)\right]=0.$$
Then, considering the equalities
$$\begin{aligned}k(x)\frac{dv_1^N(x,\lambda)}{dx}&=1+\int_{x_{N-1}}^{x}[q(\xi)-\lambda r(\xi)]\,v_1^N(\xi,\lambda)\,d\xi,\\ k(x)\frac{dv_2^N(x,\lambda)}{dx}&=-\int_{x}^{x_{N+1}}[q(\xi)-\lambda r(\xi)]\,v_2^N(\xi,\lambda)\,d\xi,\end{aligned}\tag{25}$$

which follow from (5) and (6), we obtain

$$-\frac{v_1^N(x,\lambda)}{v_1^N(x_{N+1},\lambda)}\int_x^{x_{N+1}}[q(\xi)-\lambda r(\xi)]\,v_2^N(\xi,\lambda)\,d\xi\cdot u(x)$$
$$+\frac{v_2^N(x,\lambda)}{v_1^N(x_{N+1},\lambda)}\left[u_{N-1}-\left(1+\int_{x_{N-1}}^{x}[q(\xi)-\lambda r(\xi)]\,v_1^N(\xi,\lambda)\,d\xi\right)u(x)\right]=0. \quad (26)$$

Taking $x=x_N$ and multiplying the obtained equality by $\frac{v_1^N(x_{N+1},\lambda)}{hv_1^N(x_N,\lambda)v_2^j(x_N,\lambda)}$, we obtain

$$-\frac{u_N-u_{N-1}}{hv_1^N(x_N,\lambda)}-u_N\left[\frac{1}{hv_2^N(x_N,\lambda)}\int_{x_N}^{x_{N+1}}[q(\xi)-\lambda r(\xi)]\,v_2^N(\xi,\lambda)\,d\xi\right.$$
$$\left.+\frac{1}{hv_1^N(x_N,\lambda)}\int_{x_{N-1}}^{x_N}[q(\xi)-\lambda r(\xi)]\,v_1^N(\xi,\lambda)\,d\xi\right]=0,$$

which due to $a_{N+1}=0$, can be written as

$$\frac{1}{h}\left(a_{N+1}u_{x,N}-a_Nu_{\bar{x},N}\right)-d_Nu_N+\lambda\rho_Nu_N=0.$$

Inequality (14) follows from (10). Indeed,

$$a_j=\frac{h}{v_1^j(x_j,\lambda)}\leqslant\frac{3}{2}C_2(1+x_j)(1-x_{j-1})\leqslant\frac{3}{2}C_2(1-x_{j-1/2}^2),$$
$$a_j\geqslant\frac{1}{2}C_2(1-x_j)(1+x_{j-1})\geqslant\frac{1}{2}C_2(1-x_{j-1/2}^2),\quad j=2,3,\dots,N.$$

We now prove estimate (15). Since

$$d_j=\frac{1}{hv_1^j(x_j,\lambda)}\int_{x_{j-1}}^{x_j}v_1^j(\xi,\lambda)\,q(\xi)\,d\xi+\frac{1}{hv_2^j(x_j,\lambda)}\int_{x_j}^{x_{j+1}}v_2^j(\xi,\lambda)q(\xi)\,d\xi,$$

in view of the positivity and monotonicity of functions $v_1^j(x,\lambda)$, $v_2^j(x,\lambda)$ we have

$$d_j\leqslant\frac{C_4}{h}\left[\int_{x_{j-1}}^{x_j}\frac{v_1^j(\xi,\lambda)}{v_1^j(x_j,\lambda)}\,d\xi+\int_{x_j}^{x_{j+1}}\frac{v_2^j(\xi,\lambda)}{v_2^j(x_j,\lambda)}\,d\xi\right]\leqslant 2C_4.$$

In addition, using estimates (10), we obtain

$$d_j\geqslant\frac{C_3}{h}\left[\frac{1}{v_1^j(x_j,\lambda)}\int_{x_{j-1}}^{x_j}v_1^j(\xi,\lambda)\,d\xi+\frac{1}{v_2^j(x_j,\lambda)}\int_{x_j}^{x_{j+1}}v_2^j(\xi,\lambda)\,d\xi\right]\geqslant C_3'.$$

Analogously, the inequality $0<C_5'\leqslant\rho_j\leqslant 2C_6$ can be proven. ■

Note that if the solution of problem (1) is normalized by the condition

$$\int_0^1 r(x)\,u^2(x)\,dx=1,$$

then, for the exact normalization on the grid, we have

$$\sum_{j=2}^{N}\int_{x_{j-1}}^{x_j}r(x)\left[\frac{v_1^j(x,\lambda)}{v_1^j(x_j,\lambda)}y_j+\frac{v_2^{j-1}(x,\lambda)}{v_2^{j-1}(x_{j-1},\lambda)}y_{j-1}\right]^2dx$$
$$+\int_{-1}^{x_1}r(x)\left[\frac{v_1^1(x,\lambda)}{v_1^1(x_j,\lambda)}y_1\right]^2dx+\int_{x_N}^{1}r(x)\left[\frac{v_2^N(x,\lambda)}{v_2^N(x_N,\lambda)}y_N\right]^2dx=1.$$

3. Coefficient stability of ETDS

When calculating the coefficients of difference schemes, errors are inevitable. Therefore, it is natural to require coefficient stability of difference schemes, i.e., stability to perturbations of the coefficients (see [1]). In the following, we prove the coefficient stability of the constructed difference schemes. The property of coefficient stability of a difference scheme allows us to prove the convergence of truncated three-point difference schemes.

We consider the difference problem (12) in the space H_h of grid functions y with the following scalar product and norms:

$$(y,v)=\sum_{\xi\in\omega_h} h\,y(\xi)\,v(\xi),\quad \|y\|=(y,y)^{1/2},\quad \|y\|_C=\max_{\xi\in\omega_h}|y(\xi)|.$$

Suppose that $\lambda^h=\lambda^h_n$ is the nth eigenvalue of this problem, and that $y=y_n$ is the normalized eigenfunction. There exist N real eigenvalues $\lambda^h_1,\lambda^h_2,\ldots,\lambda^h_N$, to which the appropriate eigenfunctions $y_1,y_2,\ldots,y_N$ correspond. The eigenfunctions are orthonormalized with weight ρ, such that $(\rho y_n,y_m)=0$ holds for $n\neq m$ and $(\rho y_n,y_n)=1$.

Multiplying (12) scalarwise by y and taking the difference Green formula (see [10, p. 47]) and the equalities $a_1=a_{N+1}=0$ into account, we find

$$\lambda^h=R_N(y)=\frac{(a,y^2_{\bar{x}})+(d,y^2)}{(\rho,y^2)}.$$

It is easy to see that the difference problem (12) is equivalent to the variational problem

$$\lambda^h_1=\min_y R_N(y),\quad \lambda^h_n=\max_{y_m}\min_{(\rho y,y_m)=0} R_N(y),\quad m=1,2,\ldots,n-1,\quad n=2,3,\ldots,N.$$

The following assertion is valid (see [4]):

Lemma 4. *For the eigenvalues and the eigenfunctions of problem (12)–(15) the following estimates are satisfied:*

$$M_1n^2\leqslant\lambda^h_n\leqslant M_2n^2, \tag{27}$$

$$\left\|\sqrt{a}y_n\right\|_C\leqslant M_3\sqrt{n},\quad \|a(y_n)_{\bar{x}}\|_C\leqslant M_4n^{3/2}, \tag{28}$$

where the constants M_1, M_2, M_3, M_4 do not depend on h and n, $n=1,2,\ldots,N$.

Together with the ETDS (12)–(15), we consider the perturbed three-point difference scheme

$$\tilde{\Lambda}\tilde{y}+\tilde{\lambda}^h\tilde{\rho}\tilde{y}=0,\quad x\in\omega_h,\quad \tilde{y}_0\neq\infty,\quad \tilde{y}_{N+1}\neq\infty, \tag{29}$$

where

$$\tilde{\Lambda}\tilde{y}=(\tilde{a}\tilde{y}_{\bar{x}})_x-\tilde{d}\tilde{y},\quad x\in\omega_h,\quad \tilde{a}_1=\tilde{a}_{N+1}=0.$$

Introducing a function $z=y-\tilde{y}$, we obtain the boundary value problem

$$\Lambda z+\lambda^h\rho z=-\Psi(x),\quad x\in\omega_h,\quad z_0\neq\infty,\quad z_{N+1}\neq\infty, \tag{30}$$

where

$$\Psi(x)=\Lambda\tilde{y}+\lambda^h\rho\tilde{y}.$$

Using the equation (29), we can rewrite the function $\Psi(x)$ into

$$\begin{aligned}\Psi(x)&=\Lambda\tilde{y}+\lambda^h\rho\tilde{y}-\tilde{\Lambda}\tilde{y}-\tilde{\lambda}^h\tilde{\rho}\tilde{y}\\&=\big((a-\tilde{a})\tilde{y}_{\bar{x}}\big)_x-(d-\tilde{d})\tilde{y}+\tilde{\lambda}^h(\rho-\tilde{\rho})\tilde{y}+\big(\lambda^h-\tilde{\lambda}^h\big)\rho\tilde{y}\\&=\psi(x)+(\lambda^h-\tilde{\lambda}^h)\rho\tilde{y},\end{aligned}$$

where

$$\psi(x)=\eta_x+\psi^*(x),\quad \eta=(a-\tilde{a})\tilde{y}_{\bar{x}},\quad \psi^*=-(d-\tilde{d})\tilde{y}+\tilde{\lambda}^h(\rho-\tilde{\rho})\tilde{y}. \tag{31}$$

The parameter λ^h is an eigenvalue for the difference operator of problem (30). Thus, the inhomogeneous equation (30) is solvable only if the eigenfunction $y(x)$ is orthogonal to the right-hand side of equation (30), or, more precisely, if the equality

$$(\Psi,y)=(\psi,y)+(\lambda^h-\tilde{\lambda}^h)(\rho\tilde{y},y)=0 \tag{32}$$

is satisfied.

Only a single eigenfunction, determined accurately up to an arbitrary multiplier C_0, corresponds to the eigenvalue λ^h. We choose this multiplier in a way such that the function $\bar{y}=C_0y$ is orthogonal to the difference $\bar{z}=\bar{y}-\tilde{y}$:

$$(\rho\bar{y},\bar{z})=0. \tag{33}$$

Due to the normalization condition $(\rho y, y) = 1$, we thus obtain

$$(\rho\tilde{y}, y) = (\rho y, \bar{y} - \bar{z}) = (\rho y, \bar{y}) - (\rho y, \bar{z}) = C_0(\rho y, y) = C_0.$$

If $\tilde{y} \to y$ as $h \to 0$, we can assume that $C_0 > 0$.

Further,

$$\left(\rho, \tilde{y}^2\right) = \left(\rho, (\bar{y} - \bar{z})^2\right) = \left(\rho, \bar{y}^2\right) - 2(\rho, \bar{z}\bar{y}) + \left(\rho, \bar{z}^2\right) = C_0^2(\rho y, y) + \left(\rho, \bar{z}^2\right) = C_0^2 - (\rho, \bar{z}\tilde{y}),$$

is valid and, hence,

$$1 - C_0^2 = -(\rho, \bar{z}\tilde{y}) - \left[\left(\rho, \tilde{y}^2\right) - \left(\tilde{\rho}, \tilde{y}^2\right)\right]. \tag{34}$$

We use equality (32) for determining $\lambda^h - \tilde{\lambda}^h$:

$$\lambda^h - \tilde{\lambda}^h = -\frac{(\psi, y)}{(\rho\tilde{y}, y)} = -\frac{(\psi, \bar{y})}{C_0^2}. \tag{35}$$

We transform the right-hand side of equation (35) by taking (31), the summation by parts formula (see, e.g., [10, p. 47]), and the equalities $a_1 = a_{N+1} = 0$ into account

$$(\psi, \bar{y}) = -(\eta, \bar{y}_{\bar{x}}) + (\psi^*, \bar{y}).$$

From this and the estimates (28) for $\bar{y}$, $\bar{y}_{\bar{x}}$, we find

$$\begin{aligned} \left|\lambda^h - \tilde{\lambda}^h\right| &\leqslant \frac{|(\psi, \bar{y})|}{C_0^2} \leqslant \frac{|(\eta, \bar{y}_{\bar{x}})| + |(\psi^*, \bar{y})|}{C_0^2} \\ &\leqslant \frac{\|\bar{y}_{\bar{x}}\|_C(1, |\eta|) + \|\bar{y}\|_C(1, |\psi^*|)}{C_0^2} \leqslant Mn^{3/2}[(1, |\eta|) + (1, |\psi^*|)]. \end{aligned}$$

We arrive at the following assertion.

Lemma 5. *Suppose that the conditions (14), (15) for the difference Sturm–Liouville problem (12) are satisfied. Then, the estimate*

$$\left|\lambda_n^h - \tilde{\lambda}_n^h\right| \leqslant Mn^{3/2}[(1, |\eta|) + (1, |\psi^*|)] \tag{36}$$

is valid, where the constant $M > 0$ depends on C_i', $i = 1, 2, \ldots, 6$, and C_0.

We now find an estimate for $\bar{z}$. Since $\bar{y} = C_0 y$, we see that $\bar{y}$ satisfies equation (12) and $\left(\rho, \bar{y}^2\right) = C_0^2$, and for $\bar{z} = \bar{y} - \tilde{y}$ we arrive at problem (30).

This problem is reduced to a discrete analogue of the integral equation

$$\bar{z}(x) = \lambda^h(G(x, \xi), \rho(\xi)\bar{z}(\xi)) + (G(x, \xi), \Psi(\xi)), \tag{37}$$

where $G(x, \xi) = G^h(x, \xi)$ is the difference Green function of the operator $\Lambda y = (ay_{\bar{x}})_x - dy$ with boundary conditions $y_0 \neq \infty$, $y_{N+1} \neq \infty$ (see [4]).

The eigenfunction $\bar{y}$ of problem (12) satisfies the equation

$$\bar{y}(x) = \lambda^h\left(G(x, \xi), \rho(\xi)\bar{y}(\xi)\right). \tag{38}$$

We transform equations (37) and (38) into such a form such that the corresponding kernels become symmetric. For this purpose, we use the substitutions

$$v(x) = \sqrt{\rho(x)}\bar{z}(x), \quad \varphi(x) = \sqrt{\rho(x)}\bar{y}(x), \quad K(x, \xi) = \sqrt{\rho(x)\rho(\xi)}G(x, \xi).$$

Then equations (37) and (38) take the form

$$v_n(x) = \lambda_n^h(K(x, \xi), v_n(\xi)) + f(x), \quad f(x) = (K(x, \xi), \bar{\Psi}(\xi)), \quad \bar{\Psi}(\xi) = \frac{\Psi(\xi)}{\sqrt{\rho(\xi)}}, \tag{39}$$

$$\varphi_n(x) = \lambda_n^h(K(x, \xi), \varphi_n(\xi)). \tag{40}$$

The condition of orthogonality of the function $f(x)$ to functions $\varphi_n(x)$ is satisfied in view of condition (32):

$$\begin{aligned} (\varphi_n(x), f(x)) &= \left(\varphi_n(x), (K(x, \xi), \bar{\Psi}(\xi))\right) = \left(\bar{\Psi}(\xi), (K(x, \xi), \varphi_n(x))\right) \\ &= \frac{1}{\lambda_n^h}\left(\bar{\Psi}(\xi), \varphi_n(\xi)\right) = \frac{1}{\lambda_n^h}\left(\frac{\Psi}{\sqrt{\rho}}, \sqrt{\rho}\bar{y}\right) = \frac{1}{\lambda_n^h}(\Psi, \bar{y}) = 0. \end{aligned}$$

We rewrite condition (33) as

$$(\varphi_n, v_n) = 0. \tag{41}$$

Searching for the solution $v(x) = v_n(x)$ of equation (39) of the form

$$v_n(x) = f(x) + \sum_{\substack{k=1,\\ k\neq n}}^{N-1} c_k \varphi_k(x) \tag{42}$$

under the additional condition (41), we substitute this expression to the right-hand side of equation (39) to obtain

$$v_n(x) = f(x) + \lambda_n^h \sum_{\substack{k=1,\\ k\neq n}}^{N-1} c_k (K(x,\xi), \varphi_k(\xi)) + \lambda_n^h (K(x,\xi), f(\xi)). \tag{43}$$

Expanding $f(x)$ by the eigenfunctions $\{\varphi_k(x)\}$

$$f(x) = \sum_{\substack{k=1,\\ k\neq n}}^{N-1} f_k \varphi_k(x), \quad f_k = (f, \varphi_k),$$

it follows that

$$(K(x,\xi), f(\xi)) = \sum_{\substack{k=1,\\ k\neq n}}^{N-1} \frac{f_k}{\lambda_k^h} \varphi_k(x).$$

Thus, in view of (40), we can rewrite equality (43) into

$$v_n(x) = f(x) + \lambda_n^h \sum_{\substack{k=1,\\ k\neq n}}^{N-1} \left[\frac{c_k}{\lambda_k^h} + \frac{f_k}{\lambda_k^h} \right] \varphi_k(x). \tag{44}$$

Due to equality (44), we have

$$c_k = (v_n - f, \varphi_k) = \frac{\lambda_n^h}{\lambda_k^h} c_k + \frac{\lambda_n^h}{\lambda_k^h} (f, \varphi_k),$$

and substituting $c_k = \lambda_n^h (f, \varphi_k)/(\lambda_k^h - \lambda_n^h)$ into (42), we obtain

$$v_n(x) = f(x) + \sum_{\substack{k=1,\\ k\neq n}}^{N-1} \frac{\lambda_n^h (f, \varphi_k)}{\lambda_k^h - \lambda_n^h} \varphi_k(x). \tag{45}$$

Multiplying the equation (45) by $a^\mu(x)$, $0 < \mu \leqslant 1$, we can estimate the second term on the right-hand side of this equation by

$$\left| \sum_{\substack{k=1,\\ k\neq n}}^{N-1} \frac{\lambda_n^h (f, \varphi_k)}{\lambda_k^h - \lambda_n^h} a^\mu(x) \varphi_k(x) \right| \leqslant \|f\| \, \|a^\mu \varphi_k\| \, \lambda_n^h \sum_{\substack{k=1,\\ k\neq n}}^{N-1} \frac{|\varphi_k|}{|\lambda_k^h - \lambda_n^h|} \leqslant M \|f\| \lambda_n^h \sum_{\substack{k=1,\\ k\neq n}}^{N-1} \frac{(\lambda_k^h)^{1/4}}{|\lambda_k^h - \lambda_n^h|}.$$

Let $\varepsilon > 0$ be an arbitrary number independent of h. We choose the number n_0 in a way such that $\lambda_{n_0}^h \geqslant (1+\varepsilon)\lambda_n^h$. Then,

$$\sum_{k=n_0}^{N-1} \frac{(\lambda_k^h)^{1/4}}{|\lambda_k^h - \lambda_n^h|} \leqslant \frac{1+\varepsilon}{\varepsilon} \sum_{k=n_0}^{N-1} \frac{(\lambda_k^h)^{1/4}}{\lambda_k^h} \leqslant \frac{M'}{\varepsilon} \sum_{k=n_0}^{N-1} \frac{1}{(\lambda_k^h)^{3/4}} \leqslant M,$$

where the constant $M > 0$ is independent of h.

Since $\lambda_k^h \to \lambda_k$ for $k \leqslant n_0$ as $h \to 0$ (see [4], Theorem 1), we have for a sufficiently small $h \leqslant h_0$, that

$$\sum_{k=1}^{n_0-1} \frac{(\lambda_k^h)^{1/4}}{|\lambda_k^h - \lambda_n^h|} \leqslant M,$$

where M does not depend on h.

Hence, the estimate

$$\|a^\mu v_n\|_C \leqslant M \, \|a^\mu f\|_C \tag{46}$$

is satisfied.

We transform the expression for $f(x)$ into

$$f(x) = (K(x,\xi), \bar{\Psi}(\xi)) = \left(\sqrt{\rho(x)}\sqrt{\rho(\xi)}G(x,\xi), \frac{\Psi(\xi)}{\sqrt{\rho(\xi)}}\right) = \sqrt{\rho(x)}(G(x,\xi), \Psi(\xi))$$
$$= (\lambda^h - \tilde{\lambda}^h)\sqrt{\rho(x)}(G(x,\xi), \rho(\xi)\tilde{y}(\xi)) + \sqrt{\rho(x)}(G(x,\xi), \eta_{\xi}(\xi) + \psi^*(\xi))$$
$$= (\lambda^h - \tilde{\lambda}^h)\sqrt{\rho(x)}(G(x,\xi), \rho(\xi)\tilde{y}(\xi)) + \sqrt{\rho(x)}\{-(\tilde{a}(\xi)G_{\bar{\xi}}(x,\xi), \eta(\xi)) + (G(x,\xi), \psi^*(\xi))\}.$$

Hence, taking the estimates (see [4])

$$\|a^{\mu}(\xi)G(x,\xi)\|_C \leqslant C_7, \quad \left\|\tilde{a}(\xi)G_{\bar{\xi}}(x,\xi)\right\|_C \leqslant C_8,$$

into account where the constants C_7, C_8 do not depend on h and n, we obtain

$$\|a^{\mu}f\|_C \leqslant \left\|\left(a^{\mu}(x)\tilde{a}(\xi)G_{\bar{\xi}}(x,\xi), \frac{\eta(\xi)}{\tilde{a}(\xi)}\right)\right\|_C + \left\|\sqrt{\rho(x)}\left(a^{\mu}(x)G(x,\xi), \psi^*(\xi)\right)\right\|_C$$
$$+ \left\|\sqrt{\rho(x)}\left(a^{\mu}G(x,\xi), \rho(\xi)\tilde{y}(\xi)\right)\right\|_C|\lambda^h - \tilde{\lambda}^h| \leqslant M_1\left\{\left(1, \left|\frac{\eta}{\tilde{a}}\right|\right) + (1, |\psi^*|)\right\} + M_2|\lambda^h - \tilde{\lambda}^h|.$$

Substituting this estimate into (46), returning back to the function $\bar{z}(x) = v(x)/\sqrt{\rho(x)}$ and taking the inequality (28) as well as Lemma 4 into account, yields

$$\|a^{\mu}\bar{z}\|_C \leqslant M\left\{\left(1, \left|\frac{\eta}{\tilde{a}}\right|\right) + (1, |\psi^*|)\right\}.$$

We are interested in the difference $z = y - \tilde{y}$ which is expressed by

$$z = \frac{\bar{z}}{C_0} + \frac{1 - C_0}{C_0}\tilde{y} = \frac{\bar{z}}{C_0} + \frac{1 - C_0^2}{C_0(1 + C_0)}\tilde{y}.$$

Since $\|a^{\mu}\tilde{y}\|_C$ is bounded, it follows that for a sufficiently small h, we have

$$\|a^{\mu}z\|_C \leqslant \frac{\|a^{\mu}\bar{z}\|_C}{C_0} + \left|\frac{1 - C_0^2}{C_0(1 + C_0)}\right|\|a^{\mu}\tilde{y}\|_C \leqslant M(C_0)\left(\|a^{\mu}\bar{z}\|_C + \left|1 - C_0^2\right|\right).$$

As it is apparent from formula (34),

$$\left|1 - C_0^2\right| \leqslant \left(\rho, \bar{z}^2\right)^{1/2}\left(\rho, \tilde{y}^2\right)^{1/2} + \left|\left(\rho, \tilde{y}^2\right) - \left(\tilde{\rho}, \tilde{y}^2\right)\right| \leqslant M_1\|a^{\mu}\bar{z}\|_C + \left|\left(\rho, \tilde{y}^2\right) - \left(\tilde{\rho}, \tilde{y}^2\right)\right|.$$

If μ is chosen as $\mu = 0.5 + \varepsilon$, where $0 < \varepsilon \leqslant 0.5$, we get

$$\left\|a^{0.5+\varepsilon}z\right\|_C \leqslant M\left(\left\|a^{0.5+\varepsilon}\bar{z}\right\|_C + \left|\left(\rho, \tilde{y}^2\right) - \left(\tilde{\rho}, \tilde{y}^2\right)\right|\right).$$

Inserting the estimate for $\left\|a^{0.5+\varepsilon}\bar{z}\right\|_C$, we make sure that by $\varepsilon \to 0$ the following proposition is true:

Theorem 1. *Suppose that the assumptions of Lemma 4 are satisfied. Then, for sufficiently small $h \leqslant h_0$, we have the following estimates:*

$$\left\|\sqrt{a}(y_n - \tilde{y}_n)\right\|_C \leqslant M_1\left\{\left(1, \left|\frac{\eta}{\tilde{a}}\right|\right) + (1, |\psi^*|)\right\} + M_2\left|\left(\rho, \tilde{y}^2\right) - \left(\tilde{\rho}, \tilde{y}^2\right)\right|,$$
$$\left|\lambda_n^h - \tilde{\lambda}_n^h\right| \leqslant M_3\left\{(1, |\eta|) + (1, |\psi^*|)\right\},$$

where the constants M_i, $i = 1,2,3$ depend only on C_i', $i = 1,2,\ldots,6$, and C_0.

This theorem proves the continuous dependence of the solution of problem (12) on the coefficients, that is, the coefficient stability.

4. Algorithmic realization of ETDS

We pass to the algorithmic realization of ETDS (12). First of all, note that this scheme can be written in the form

$$(ay_{\bar{x}})_{x,j} - (d_j - \lambda\rho_j)y_j = 0, \quad j = 1,2,\ldots,N, \quad y_0 \neq \infty, \quad y_{N+1} \neq \infty, \tag{47}$$

where

$$a_1 = a_{N+1} = 0, \quad a_j = \left[\frac{1}{h}v_1^j(x_j,\lambda)\right]^{-1}, \quad j = 2,3,\ldots,N, \tag{48}$$

$$d_j - \lambda\rho_j = \frac{1}{hv_1^j(x_j,\lambda)} \int_{x_{j-1}}^{x_j} v_1^j(\xi,\lambda)[q(\xi) - \lambda r(\xi)]\, d\xi + \frac{1}{hv_2^j(x_j,\lambda)} \int_{x_j}^{x_{j+1}} v_2^j(\xi,\lambda)[q(\xi) - \lambda r(\xi)]\, d\xi, \quad j = 1, 2, \ldots, N. \tag{49}$$

We express the coefficients a_j, $d_j - \lambda\rho_j$ of the difference scheme via the solutions of the Cauchy problems (4)–(6). Due to (48), we already have the necessary representation for a_j.

In view of (21), (23) and (25), we rewrite equalities (49) as

$$\begin{aligned}
d_j - \lambda\rho_j &= \frac{1}{hv_1^j(x_j,\lambda)} \left[k(x_j)\frac{dv_1^j(x_j,\lambda)}{dx} - 1\right] + \frac{1}{hv_2^j(x_j,\lambda)} \left[-k(x_j)\frac{dv_2^j(x_j,\lambda)}{dx} - 1\right] \\
&= h^{-1}\sum_{\alpha=1}^{2}(-1)^{\alpha+1}\left[v_\alpha^j(x_j,\lambda)\right]^{-1}\left[m_\alpha^j(x_j,\lambda) + (-1)^\alpha\right], \quad j = 2, 3, \ldots, N-1, \\
d_1 - \lambda\rho_1 &= \frac{1}{hv_1^1(x_1,\lambda)} k(x_1)\frac{dv_1^1(x_1,\lambda)}{dx} + \frac{1}{hv_2^1(x_1,\lambda)} \left[-k(x_1)\frac{dv_2^1(x_1,\lambda)}{dx} - 1\right] \\
&= h^{-1}\sum_{\alpha=1}^{2}(-1)^{\alpha+1}\left[v_\alpha^1(x_1,\lambda)\right]^{-1}\left[m_\alpha^1(x_1,\lambda) + \alpha - 1\right], \\
d_N - \lambda\rho_N &= \frac{1}{hv_1^N(x_N,\lambda)} \left[k(x_N)\frac{dv_1^N(x_N,\lambda)}{dx} - 1\right] + \frac{1}{hv_2^N(x_N,\lambda)} \left[-k(x_N)\frac{dv_2^N(x_N,\lambda)}{dx}\right] \\
&= h^{-1}\sum_{\alpha=1}^{2}(-1)^{\alpha+1}\left[v_\alpha^N(x_N,\lambda)\right]^{-1}\left[m_\alpha^N(x_N,\lambda) + \alpha - 2\right],
\end{aligned}$$

where

$$m_\alpha^j(x,\lambda) = k(x)\frac{dv_\alpha^j(x,\lambda)}{dx}.$$

Thus, the ETDS (47)–(49) can be written in the form

$$(ay_{\bar{x}})_{x,j} - (d_j - \lambda\rho_j)y_j = 0, \quad j = 2, 3, \ldots, N-1,$$

$$\frac{1}{h}a_2 y_{x,1} - (d_1 - \lambda\rho_1)y_1 = 0, \quad -\frac{1}{h}a_N y_{\bar{x},N} - (d_N - \lambda\rho_N)y_N = 0,$$

where

$$\begin{aligned}
a_j &= \left[\frac{1}{h}v_1^j(x_j,\lambda)\right]^{-1}, \quad j = 2, 3, \ldots, N, \\
d_j - \lambda\rho_j &= h^{-1}\sum_{\alpha=1}^{2}(-1)^{\alpha+1}\left[v_\alpha^j(x_j,\lambda)\right]^{-1}\left[m_\alpha^j(x_j,\lambda) + (-1)^\alpha\right], \quad j = 2, 3, \ldots, N-1, \\
d_1 - \lambda\rho_1 &= h^{-1}\sum_{\alpha=1}^{2}(-1)^{\alpha+1}\left[v_\alpha^1(x_1,\lambda)\right]^{-1}\left[m_\alpha^1(x_1,\lambda) + \alpha - 1\right], \\
d_N - \lambda\rho_N &= h^{-1}\sum_{\alpha=1}^{2}(-1)^{\alpha+1}\left[v_\alpha^N(x_N,\lambda)\right]^{-1}\left[m_\alpha^N(x_N,\lambda) + \alpha - 2\right].
\end{aligned}$$

Thus, for calculating the coefficients a_j, $d_j - \lambda\rho_j$ of ETDS for any node x_j of the grid ω_h, it is necessary to solve two Cauchy problems (4)–(6) with smooth coefficients: at $\alpha = 1$ on the interval $[x_{j-1}, x_j]$ (forward) and at $\alpha = 2$ on the interval $[x_j, x_{j+1}]$ (backward). If each of the specified Cauchy problems is solved numerically by any one-step method (Taylor series expansion or Runge–Kutta method), we will obtain a truncated difference scheme. The investigation of accuracy and development of the algorithm of finding the solution of such scheme will be performed in further work.

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Алгоритмічна реалізація точної триточкової різницевої схеми для деякого класу сингулярних задач Штурма–Ліувілля

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У цій статті розроблено нову алгоритмічну реалізацію точних триточкових різницевих схем на нерівномірній сітці для деякого класу сингулярних задач Штурма–Ліувілля. Показано, що для обчислення коефіцієнтів точної схеми в довільному вузлі сітки x_j потрібно розв'язати дві допоміжні задачі Коші для лінійних звичайних диференціальних рівнянь другого порядку: одну на відрізку $[x_{j-1}, x_j]$ (вперед) і одну на відрізку $[x_j, x_{j+1}]$ (назад). Доведено теорему про коефіцієнтну стійкість точної триточкової різницевої схеми.

Ключові слова: *сингулярна задача Штурма–Ліувілля; точна триточкова різницева схема; коефіцієнтна стійкість.*